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LOOSENESS AND INDEPENDENCE NUMBER OF TRIANGULATIONS ON CLOSED SURFACES

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Abstract

The *looseness* of a triangulation G on a closed surface F^2, denoted by $\xi(G)$, is defined as the minimum number k such that for any surjection $c : V(G) \to \{1, 2, \ldots, k+3\}$, there is a face uvw of G with $c(u), c(v)$ and $c(w)$ all distinct. We shall bound $\xi(G)$ for triangulations G on closed surfaces by the independence number of G denoted by $\alpha(G)$. In particular, for a triangulation G on the sphere, we have

$$\xi(G) \leq \frac{11\alpha(G) - 10}{6}$$

and this bound is sharp. For a triangulation G on a non-spherical surface F^2, we have

$$\xi(G) \leq 2\alpha(G) + l(F^2) - 2,$$

where $l(F^2) = \lfloor (2 - \chi(F^2))/2 \rfloor$ with Euler characteristic $\chi(F^2)$.

Keywords: triangulations, closed surfaces, looseness, k-loosely tight, independence number.

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1. Introduction

A *triangulation* G on a closed surface F^2 is a fixed embedding of a simple graph embedded on F^2 with only triangular faces, except K_3 on the sphere. (In this paper, F^2 will be used to represent a closed surface.) Thus, each face can be identified as a triplet uvw consisting of its three corners. Negami and Midorikawa [5] have defined a combinatorial invariant for a triangulation G called the "looseness" of G, as follows.

Let G be a triangulation on F^2 and $c : V(G) \to \{1, 2, \ldots, n\}$ a surjection, which is regarded as an assignment of *colors* $1, 2, \ldots, n$. If c assigns three distinct colors to the three corners u, v and w of a face uvw, then the face uvw is said to be *heterochromatic* for c. We say that G is *k-loosely tight* if every surjection $f : V(G) \to \{1, 2, \ldots, k+3\}$ admits a heterochromatic face. The *looseness* of G is defined as the minimum number k such that G is k-loosely tight and is denoted by $\xi(G)$. It is obvious that $\xi(G) \leq |V(G)| - 3$, since all faces are heterochromatic if all vertices have distinct colors.

In particular, if $\xi(G) = 0$, then G is said to be *tight*. It is not so difficult to see that a tight triangulation must be a complete graph. However, there have been constructed many untight triangulations with complete graphs [2, 3]. This suggests that the value of $\xi(G)$ depends on the embedding of G on F^2.

Negami and Midorikawa [5] have shown various facts on the looseness of triangulations and given us some lower bounds for $\xi(G)$ involving combinatorial invariants of G. (We can find a study on a relation between the looseness of G and some graph invariants of the dual of G in [4].) For example, they have considered the *independence number* $\alpha(G)$, which is the maximum size of independent sets. Take an independent set S of size $\alpha = \alpha(G)$ and assign α different colors to the vertices in S one by one and another color to the remaining vertices. It is clear that there is no heterochromatic face for this assignment. This implies that $\alpha + 1 \leq \xi(G) + 2$ by the definition of $\xi(G)$ and hence we have $\alpha(G) - 1 \leq \xi(G)$.

In this paper, we shall give a nontrivial upper bound for $\xi(G)$ using the independence number $\alpha(G)$, as in the following.

Theorem 1. *For any triangulation G on the sphere, we have*

$$\xi(G) \leq \frac{11\alpha(G) - 10}{6}.$$

In fact, this bound is the best possible.

The following is the result for triangulations on surfaces with high genera. Throughout the paper, $\chi(F^2)$ stands for Euler characteristic of F^2 and we further put $l(F^2) = \left\lfloor \frac{2-\chi(F^2)}{2} \right\rfloor$.

Theorem 2. *Let G be a triangulation on F^2 with $\chi(F^2) < 2$. Then*

$$\xi(G) \leq 2\alpha(G) + l(F^2) - 2.$$

Let us consider the value $l(F^2)$ defined for F^2. In particular, if F^2 is orientable and of genus g, then $\chi(F^2) = 2 - 2g$. Thus the value of $l(F^2)$ in the theorem coincides with the genus g of F^2 and can be regarded as the maximum number of mutually disjoint *nonseparating* simple closed curves on F^2 (i.e., one whose removal does not disconnect the surface).

On the other hand, if F^2 is nonorientable and of genus k, then $\chi(F^2) = 2 - k$. However, we should notice that there are two types of simple closed curves on a nonorientable closed surface. A simple closed curve locally separates the surface into two sides. A simple closed curve is said to be 1-*sided* if its tubular neighborhood is homeomorphic to a Möbius band, and to be 2-*sided* otherwise. (Every simple closed curve on an orientable closed surface is 2-sided.) Considering the fact that every nonorientable closed surface can be obtained from an orientable one by adding one or two crosscaps, we can find that $l(F^2)$ is equal to the maximum number of mutually disjoint nonseparating 2-sided simple closed curves on F^2.

As is mentioned above, the looseness $\xi(G)$ depends on the embedding of G on F^2. That is, if $f : G \to F^2$ is another embedding of G on F^2, the value of $\xi(f(G))$ might be different from $\xi(G)$. Let $\xi_{\max}(G)$ and $\xi_{\min}(G)$ denote the maximum and the minimum values of $\xi(f(G))$ taken over all embeddings $f : G \to F^2$, respectively. Then these can be regarded as invariants for an abstract graph G which can be embedded in F^2 as a triangulation. The above theorem implies that $\alpha(G) - 1 \leq \xi_{\min}(G)$ and $\xi_{\max}(G) \leq 2\alpha(G) + l(F^2) - 2$, and hence we obtained the following corollary.

Corollary 3. *Let G be a graph which can be embedded on F^2 as a triangulation. Then we have*

$$\xi_{\max}(G) - \xi_{\min}(G) \leq \alpha(G) + l(F^2) - 1.$$

Recently, Negami [6] has shown that $\xi_{\max}(G) - \xi_{\min}(G) \leq 2l(F^2)$. If $\alpha(G)$ is enough small relatively to $l(F^2)$, then our bound is superior to his bound. In particular, if $\alpha(G) = 1$, that is, if G is isomorphic to the complete graph K_n with $n = |V(G)|$ as a graph, then we have

$$\xi_{\max}(K_n) - \xi_{\min}(K_n) \leq l(F^2).$$

However, this estimation is not so good, since it is known $\xi_{\max}(K_n) \leq O(l^{\frac{1}{4}})$ for $l = l(F^2)$ in [5].

In Section 2, we introduce the notion of "dividing systems" for color assignments with no heterochromatic face. Using this, we shall give the upper bounds in Theorems 1 and 2 in Section 3, and consider how good those estimations are in Section 4.

2. Dividing Systems of Cycles

Here, we shall prepare a useful notion called "the dividing system" to investigate the looseness of triangulations. This has been introduced in [5] and a similar notion can be found in [2, 3]. Also it has been used to characterize 1-loosely tight triangulations on the sphere, the projective plane, the torus and the Klein bottle [8].

Let G be a triangulation on F^2 and $c : V(G) \to \{1, 2, \dots, h\}$ an assignment of colors. The assignment c is said to be *hetero-free* (or a *hetero-free h-assignment*) if there is no heterochromatic face for c. The *dividing system* $\Lambda_c(G)$ of G for c is the subgraph in the dual G^* of G induced by all edges dual to those edges in G that join two vertices having distinct colors for c. It is clear that each vertex in $\Lambda_c(G)$ has degree either 0, 2 or 3 and that it has degree 3 if and only if it corresponds to a heterochromatic face. Thus, if c is hetero-free, then $\Lambda_c(G)$ consists of several disjoint cycles, each of which can be regarded as a simple closed curve on F^2. Such a cycle is 2-sided, since it locally separates two colors into both sides.

Let $C_1, \dots, C_n$ be mutually disjoint 2-sided cycles in G^* such that both sides of C_i meet different components of $F^2 - C_1 \cup \cdots \cup C_n$. Such a set $\{C_1, \dots, C_n\}$ of cycles in G^* is called a *dividing system* of cycles for G and each component of $F^2 - C_1 \cup \cdots \cup C_n$ a *region* of the dividing system. The looseness of triangulations can be characterized in terms of these notions, as follows.

Lemma 4. *Let G be a triangulation on F^2. Then $\xi(G) + 2$ is equal to the maximum number of regions taken over all dividing systems of cycles for G.*

Proof. Let $R_{\max}$ denote the maximum number of regions in the lemma. Let $c : V(G) \to \{1, 2, \dots, \xi(G) + 2\}$ be a hetero-free $(\xi(G) + 2)$-assignment and consider $\Lambda_c(G)$. Then $\Lambda_c(G)$ consists of mutually disjoint cycles, which form a dividing system of cycles for G. Each region contains only one color and each of $\xi(G) + 2$ colors appears in some region. Thus, this dividing system has at least $\xi(G) + 2$ regions and hence $\xi(G) + 2 \leq R_{\max}$.

On the other hand, let $\{C_1, \dots, C_n\}$ be a dividing system of cycles for G which attains the maximum $R_{\max}$. Assign colors $1, 2, \dots, R_{\max}$ to all regions one by one and define a color assignment to vertices so that each vertex gets the same color as the region containing it has. Then this color assignment with $R_{\max}$ colors is hetero-free. This implies that $\xi(G) + 2 \geq R_{\max}$. Hence we have $\xi(G) + 2 = R_{\max}$. ∎

Note that the number of regions of any dividing system on the sphere or the projective plane is equal to the number of cycles plus one, since any 2-sided simple closed curve separates these surfaces. However, the number of regions cannot be determined uniquely only by the number of cycles in a dividing system on any closed surface except the sphere and the projective plane.

Since any 2-sided simple closed curve on the sphere and the projective plane separates the surface, any set of mutually disjoint 2-sided cycles in the dual G^* becomes a dividing system for G automatically. Thus, the following corollary is an immediate consequence of the above lemma.

Corollary 5. *Let G be a triangulation on the sphere or the projective plane. Then $\xi(G) + 1$ is equal to the maximum number of mutually disjoint 2-sided cycles in the dual G^* of G.*

Let G be a triangulation on F^2 with a hetero-free h-assignment, and let $\Lambda = \Lambda_c(G)$ denote a dividing system for G. Then the *division graph* Γ_Λ can be constructed as follows. Prepare a vertex for each region of Λ and join two vertices by an edge whenever the two regions corresponding to them meet each other along a cycle in Λ. Clearly, since each edge of Γ_Λ corresponds to a cycle of Λ, and since any 2-sided simple closed curve on the sphere and the projective plane separates the surface, we have the following.

Lemma 6. *Let G be a triangulation on the sphere or the projective plane and let Λ be a dividing system for G with h regions. Then the division graph Γ_Λ is a tree of h vertices.*

3. Upper Bounds

In this section, we shall give upper bounds of the estimations in Theorems 1 and 2.

Lemma 7. *If a triangulation G on the sphere admits a hetero-free h-assignment, then $\alpha(G) \geq \frac{6h-2}{11}$.*

Proof. Consider a dividing system $\Lambda_c(G)$ of G for c. Then the number of regions of $\Lambda_c(G)$ is at least h, where we note $h \geq 3$ by the definition. Now let T be the division graph of $\Lambda_c(G)$. By Lemma 6, T is a tree with at least h vertices. Let $V_i = |\{v \in V(T) : \deg_T(v) = i\}|$. Then, since T is a tree, we have

$$\sum_i V_i = |V(T)| \geq h, \qquad \sum_i iV_i \geq 2h - 2. \tag{1}$$

Case 1. $V_1 \geq \frac{6h-2}{11}$.
We have

$$\alpha \geq V_1 \geq \frac{6h-2}{11},$$

since any two vertices of degree one are not adjacent in a tree with at least three vertices.

Case 2. $V_1 < \frac{6h-2}{11}$.
We focus on each region R_v of $\Lambda_c(G)$ corresponding to a vertex $v \in V(T)$ of degree exactly i in T, for $i \geq 2$. Let H be the subgraph of G induced by the vertices belonging to R_v. Since R_v has exactly i neighboring regions, H is a plane graph with at least i faces; note that $|V(H)| \geq 3$ here. By Euler's formula and the fact that $|E(H)| \leq 3|V(H)| - 6$, we have $|V(H)| \geq \frac{F+4}{2}$, where F is the number of faces of H. Since $F \geq i$, we have

$$|V(H)| \geq \frac{i+4}{2}. \tag{2}$$

Now we shall estimate $|V(G)|$. By (1), (2) and the assumption in Case 2, we have

$$\begin{aligned}
|V(G)| &\geq V_1 + \sum_{i\geq 2} \frac{i+4}{2} V_i = \frac{1}{2}\left(\sum_{i\geq 1}(i+4)V_i - 3V_1\right) \\
&= \frac{1}{2}\left(\sum_{i\geq 1} iV_i + 4\sum_{i\geq 1} V_i - 3V_1\right) \geq \frac{1}{2}(6h - 2 - 3V_1) \\
&> \frac{1}{2}\left(6h - 2 - 3 \times \frac{6h-2}{11}\right) = \frac{4(6h-2)}{11}.
\end{aligned}$$

Applying Four Color Theorem [1] to G, we have

$$\alpha(G) \geq \frac{|V(G)|}{4} > \frac{6h-2}{11}.$$

This completes a proof. ■

Now we shall deal with the nonspherical case.

Lemma 8. *Let G be a triangulation on F^2. If G admits a hetero-free h-assignment, then $h \leq 2\alpha(G) + l(F^2)$.*

Proof. Let $c : V(G) \to \{1, 2, \ldots, h\}$ be any hetero-free h-assignment, and consider the dividing system $\Lambda = \Lambda_c(G)$ for c, which consists of disjoint cycles in G^*. We estimate the number of regions of Λ. Now, construct a division graph Γ_Λ associated with Λ. Take a spanning tree T in Γ_Λ and let $E = \{e_1, \ldots, e_{l'}\}$ be the set of edges in Γ_Λ not belonging to T. Then we have $l' \le l(F^2)$.

Since T is a tree, we can color its vertices with black and white so that any two adjacent vertices get different colors. Call a region "black" (respectively, "white") if its corresponding vertex in Γ_Λ gets black (respectively, white). This vertex coloring of T cannot extend to a proper vertex coloring of the whole Γ_Λ in general, since Γ_Λ might contain an odd cycle. Let B (respectively, W) be the set of edges in E which join two black (respectively, white) vertices in Γ_Λ and let W be for "white". Then $B \cap W = \emptyset$ and $|B \cup W| = |B| + |W| \le l' \le l(F^2)$. We call each edge "a black edge" if it belongs to B and "a white edge" if to W.

Choose a set U_B of black vertices in Γ_Λ as small as possible so that U_B *covers* B, that is, at least one of the ends of each edge in B belongs to U_B, and take one vertex of G arbitrarily from each black region corresponding to a black vertex in $V(\Gamma_\Lambda) - U_B$. Let S_B be the set of such chosen vertices. Then S_B forms an independent set in G. Define U_W and S_W for "white" similarly. Also S_W is an independent set in G.

It is clear that the number of regions of Λ is equal to $|S_B|+|S_W|+|U_B|+|U_W|$. Since both S_B and S_W are independent sets, we have $|S_B| + |S_W| \le 2\alpha(G)$ and also have $|U_B| + |U_W| \le l' \le l(F^2)$. Therefore, the number of regions of Λ is at most $2\alpha(G) + l(F^2)$ and hence h does not exceed this. ■

4. TRIANGULATIONS WITH HIGH LOOSENESS

In this section, for a fixed independence number, we construct triangulations on closed surfaces with high looseness, which give lower bounds for the estimations in Theorems 1 and 2.

Lemma 9. *For any positive integer α, there exists a triangulation G on the sphere with $\alpha(G) > \alpha$ and looseness $\xi(G) = \frac{11\alpha(G)-10}{6}$.*

Proof. Let O be a plane triangulation isomorphic to an octahedron with disjoint 3-cycles $a_1a_2a_3$ and $b_1b_2b_3$, where a_i is joined to b_j, b_k for any distinct $i, j, k \in \{1, 2, 3\}$. Let A be an annulus triangulation obtained from O by removing the interior of the two faces $a_1a_2a_3$ and $b_1b_2b_3$, adding a vertex of degree 3 labeled x (respectively, y) to $\{b_1, a_2, a_3\}$ (respectively, $\{a_1, b_2, b_3\}$), and *adding* K_4 to other four faces. (See Figure 1. Adding K_4 to a face $v_1v_2v_3$ is to put a plane triangulation isomorphic to K_4 with boundary $u_1u_2u_3$ and add edges v_iu_j, v_iu_k for any distinct $i, j, k \in \{1, 2, 3\}$.) Since $V(A)$ can be decomposed into six vertex-disjoint K_4's, A has independence number 6.

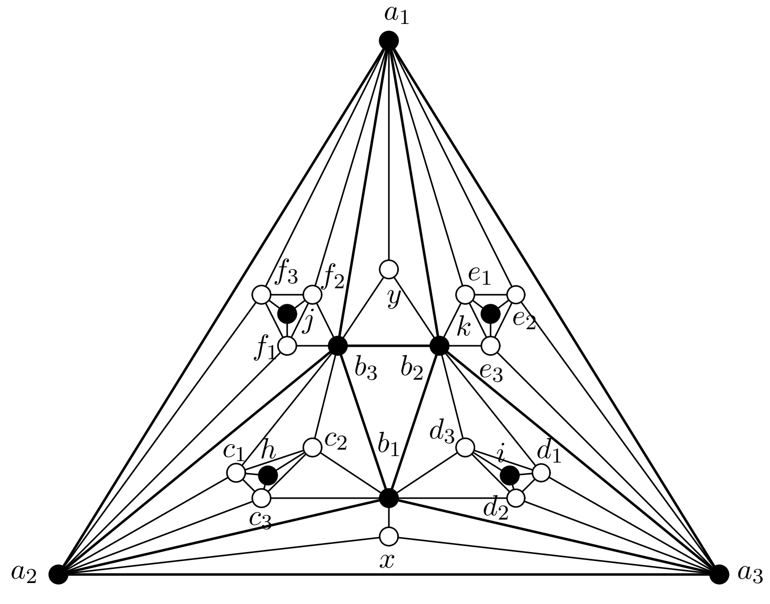

Figure 1. Annulus triangulation A with 24 vertices.

Let G_1 be the plane triangulation obtained from A by pasting a 2-cell along each of the two boundary components of the annulus and adding K_4 to each of the two new faces. Then G_1 is a plane triangulation such that $V(G_1)$ can be decomposed into eight vertex-disjoint K_4's, and hence G_1 has independence number $6 + 2 = 8$.

Let us construct a hetero-free assignment of G_1, as follows. Color $V(O) = \{a_1, a_2, a_3, b_1, b_2, b_3\}$ by a single color. Color each of the six K_4's added by two colors so that an *central* vertex (i.e., one of degree 3 in G_1) and the other three are colored by distinct colors. Use a single color for each of the two vertices x, y of degree 3 added. Consequently, using all distinct colors to them, we get a hetero-free 15-assignment of G_1, and hence we have $\xi(G_1) \geq 15 - 2 = 13$. This coincides with the upper bound in Theorem 1: $\frac{11\times 8-10}{6} = 13$.

This can be generalized to an infinite sequence of plane triangulations attaining the equality in Theorem 1, as follows. Let $\bar{A}_1, \dots, \bar{A}_t$ be t copies of A. Connect $\bar{A}_i$ and $\bar{A}_{i+1}$ by inserting an annulus to join the boundary $a_1a_2a_3$ of $\bar{A}_i$ and the boundary $b_1b_2b_3$ of $\bar{A}_{i+1}$, and add edges a_ib_j, a_ib_k in the added annulus for any distinct $i, j, k \in \{1, 2, 3\}$. Let A_t be the resulting annulus triangulation. Let G_t be the plane triangulation obtained from A_t by adding K_4 to each of the two boundary components after capping it off by a 2-cell, and color G_t similarly to that for G_1 by using all distinct colors. Since each $\bar{A}_i$ is colored by eleven colors, we get a hetero-free assignment of G_t by $11t + 4$ colors. On the other hand, since each $\bar{A}_i$ has independence number six, G_t has independence number $6t+2$, that is, $\alpha(G_t) = 6t+2$. Therefore, we have

$$\xi(G_t) \geq (11t+4) - 2 = 11 \times \frac{\alpha(G_t)-2}{6} + 2 = \frac{11\alpha(G_t)-10}{6}.$$

So the Lemma follows, since $\alpha(G_t)$ can be taken to be arbitrarily large. ■

Remark 10. Actually, using the appropriate number of K_4's added in the above argument, we can construct a triangulation G on the sphere with $\alpha = \alpha(G)$ and $\xi(G) = \left\lfloor \frac{11\alpha(G)-10}{6} \right\rfloor$ for any integer $\alpha \geq 1$. However, the construction is so complicated that we omit it.

For some nonspherical surfaces F^2, we can construct an example of a triangulation G on F^2 such that $\alpha(G) \geq \alpha$ for a fixed $\alpha > 0$ and

$$\xi(G) \geq \frac{11\alpha(G)-10}{6},$$

as follows. Let H be the triangulation constructed in Lemma 9, and let c be the hetero-free $(\xi(H)+2)$-assignment for H. We remove a vertex y from H(see Figure 1 again), and let H' be the resulting triangulation. Let K be a triangulation on F^2 by a complete graph K_n. (It is known that a complete graph K_n triangulates F^2 if and only if $\frac{(n-3)(n-4)}{6} = 2 - \chi(F^2)$ [7].) Now, let G be a triangulation obtained from K and H' by identifying a face of K and the face $a_1b_2b_3$ of H'. Since the graph of K is complete, we have $\alpha(G) = \alpha(H) \geq \alpha$. On the other hand, we color the vertices of G so that the vertices in $V(H')$ are colored by c in G and the vertices in $V(G) - V(H')$ by $c(y)$. Then we obtain a hetero-free assignment of G, and hence we have $\xi(G) \geq \xi(H) \geq \frac{11\alpha(H)-10}{6} = \frac{11\alpha(G)-10}{6}$.

5. Proof of Theorems

***Proof of Theorems* 1 *and* 2.** Let G be a triangulation on the sphere. By the definition, G admits a hetero-free $(\xi(G)+2)$-assignment. By Lemma 7, we have $\alpha(G) \geq \frac{6(\xi(G)+2)-2}{11}$ and hence we obtain the upper bound of Theorem 1. On the other hand, Lemma 9 gives an example attaining the upper bound of the theorem. Similarly, the proof of Theorem 2 follows from Lemma 8. ■

We focused on a relation between looseness and independence number of triangulations. Can we bound looseness of triangulations by using another invariant of graphs, for example, diameter of graphs? In a triangulation G, from a fixed vertex v of G, color each vertex of G according to its distance from v. Then we can get a hetero-free assignment. It seems to be interesting to bound looseness of triangulations in various ways.

We considered an upper bound for looseness of triangulations on closed surface by independence number. In the spherical case, we gave a best possible bound, but we did not in the nonspherical case. So can we improve Theorem 2? Moreover, as we can see, 3-cuts of graphs play an important role for the estimation of the loosen ess in our theorems. So what happens when we assume the 4-connectivity of triangulations?

In the spherical case, Four Color Theorem is essential for bounding independence number of planar graphs. Hence the argument in Theorem 1 does not seem to apply to even the projective-planar case, though Lemma 6 holds on the projective plane. So it will be interesting to extend our theorem to the projective plane or other fixed closed surfaces.

References

[1] K. Appel, W. Haken and J. Koch, *Every planar map is four colorable*, Illinois J. Math. **21** (1977) 429–567.

[2] J.L. Arocha, J. Bracho and V. Neumann-Lara, *On the minimum size of tight hypergraphs*, J. Graph Theory **16** (1992) 319–326.
doi:10.1002/jgt.3190160405

[3] J.L. Arocha, J. Bracho and V. Neumann-Lara, *Tight and untight triangulations surfaces by complete graphs*, J. Combin. Theory Ser. B **63** (1995) 185–199.
doi:10.1006/jctb.1995.1015

[4] J. Czap, S. Jendrol', F. Kardoš and J. Miškuf, *Looseness of plane graphs*, Graphs Combin. **27** (2011) 73–85.
doi:10.1007/s00373-010-0961-6

[5] S. Negami and T. Midorikawa, *Loosely-tightness of triangulations of closed surfaces*, Sci. Rep. Yokohama Nat. Univ., Sec. I **43** (1996) 25–41.

[6] S. Negami, *Looseness ranges of traingulations on closed surfaces*, Discrete Math. **303** (2005) 167–174.
doi:10.1016/j.disc.2005.01.010

[7] G. Ringel, Map Color Theorem (Springer-Verlag, Berlin Heidelberg, 1974).
doi:10.1007/978-3-642-65759-7

[8] T. Tanuma, *One-loosely tight triangulations on closed surfaces*, Yokohama Math. J. **47** (1999) 203–211.

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