Discussiones Mathematicae
Graph Theory 36 (2016) 439–453
doi:10.7151/dmgt.1867

CHARACTERIZATIONS OF GRAPHS HAVING LARGE PROPER CONNECTION NUMBERS

Chira Lumduanhom[1]

Department of Mathematics
Srinakharinwirot University
Sukhumvit Soi 23, Bangkok, 10110, Thailand

Elliot Laforge and Ping Zhang

Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008, USA

e-mail: ping.zhang@wmich.edu

Abstract

Let G be an edge-colored connected graph. A path P is a proper path in G if no two adjacent edges of P are colored the same. If P is a proper $u-v$ path of length $d(u,v)$, then P is a proper $u-v$ geodesic. An edge coloring c is a proper-path coloring of a connected graph G if every pair u,v of distinct vertices of G are connected by a proper $u-v$ path in G, and c is a strong proper-path coloring if every two vertices u and v are connected by a proper $u-v$ geodesic in G. The minimum number of colors required for a proper-path coloring or strong proper-path coloring of G is called the proper connection number $\mathrm{pc}(G)$ or strong proper connection number $\mathrm{spc}(G)$ of G, respectively. If G is a nontrivial connected graph of size m, then $\mathrm{pc}(G) \leq \mathrm{spc}(G) \leq m$ and $\mathrm{pc}(G) = m$ or $\mathrm{spc}(G) = m$ if and only if G is the star of size m. In this paper, we determine all connected graphs G of size m for which $\mathrm{pc}(G)$ or $\mathrm{spc}(G)$ is $m-1, m-2$ or $m-3$.

Keywords: edge coloring, proper-path coloring, strong proper-path coloring.

2010 Mathematics Subject Classification: 05C15, 05C38, 05C75.

[1]Research supported by a New Researcher Grants sponsored by Ministry of Science & Technology, Thailand.

1. Introduction

One of the most fundamental properties that a graph G can possess is that of being connected. In a connected graph G, there is at least one path connecting every two vertices. Often we are also interested in paths possessing some prescribed property – such as a path of minimum length or a path of maximum length. Should the graph G be edge-colored (that is, every edge is assigned a color from some prescribed set of colors), there are properties of interest that have been studied and others that can be studied, including various ways in which G can be connected. There are many instances when one is interested in subgraphs H of an edge-colored graph G possessing a specific coloring—often resulting in a monochromatic H (in which every two edges of H are colored the same), a rainbow H (in which no two edges of H are colored the same) or a properly colored H (in which no two adjacent edges of H are colored the same).

If G is a monochromatic graph, then every two vertices are connected by at least one monochromatic path since G is connected. Similar statements can be made if G is a rainbow graph or if G is a properly colored graph. In particular, if G is a rainbow graph, then every two vertices of G are connected by a rainbow path. On the other hand, if our major interest is whether every two vertices of G are connected by at least one rainbow path, then it is typically unnecessary for the edges of G to be colored with distinct colors. That is, if G has size m, it is quite likely that there exists an edge coloring of G using fewer than m colors and having the property that every two vertices are connected by at least one rainbow path. An edge-colored graph with this property is said to be rainbow-connected. Such a coloring is called a rainbow coloring of G.

Formally, a *rainbow coloring* of a connected graph G is an edge coloring c of G with the property that for every two vertices u and v of G, there exists a $u-v$ *rainbow path* (no two edges of the path are colored the same). In this case, G is *rainbow-connected* (*with respect to* c). The minimum number of colors needed for a rainbow coloring of G is referred to as the *rainbow connection number* of G and is denoted by $\text{rc}(G)$. There is a related concept concerning rainbow colorings. Let c be a rainbow coloring of a connected graph G. For two vertices u and v of G, a *rainbow* $u-v$ *geodesic* in G is a rainbow $u-v$ path of length $d(u,v)$, where $d(u,v)$ is the distance between u and v (the length of a shortest $u-v$ path in G). The graph G is called *strongly rainbow-connected* if G contains a rainbow $u-v$ geodesic for every two vertices u and v of G. In this case, the coloring c is called a *strong rainbow coloring* of G. The minimum number of colors needed for a strong rainbow coloring of G is referred to as the *strong rainbow connection number* $\text{src}(G)$ of G. Thus $\text{rc}(G) \leq \text{src}(G)$ for every connected graph G. These concepts were introduced and studied by Chartrand, Johns, McKeon and Zhang [2] in 2008. In recent years, this topic has been studied by many and there is now a book [6] on rainbow colorings published in 2012.

While this concept was introduced for the purpose of studying connected graphs by means of rainbow paths in edge-colored graphs, additional motivation occurred in a paper by Anne Ericksen. The Department of Homeland Security in the United States was created in 2003 in response to weaknesses discovered in the transfer of classified information after the September 11, 2001 terrorist attacks. In [5] Ericksen made the following observation:

> *An unanticipated aftermath of those deadly attacks was the realization that law enforcement and intelligence agencies couldn't communicate with each other through their regular channels from radio systems to databases. The technologies utilized were separate entities and prohibited shared access, meaning there was no way for officers and agents to cross check information between various organizations.*

While the information needs to be protected since it relates to national security, there must also be procedures that permit access between appropriate parties. This two-fold issue can be addressed by assigning information transfer paths between agencies which may have other agencies as intermediaries while requiring a large enough number of passwords and firewalls that is prohibitive to intruders, yet small enough to manage (that is, enough so that one or more paths between every pair of agencies have no password repeated). An immediate question arises: What is the minimum number of passwords or firewalls needed that allows one or more secure paths between every two agencies so that the passwords along each path are distinct? This situation can be represented (modeled) by a graph and studied by means of rainbow colorings.

The most-studied edge colorings of a graph G are proper edge colorings in which every two adjacent edges of G are assigned distinct colors. The minimum number of colors needed in a proper coloring of G is referred to as the *chromatic index* of G, denoted by $\chi'(G)$. One property that a properly edge-colored graph G has is that for every two vertices u and v, each $u-v$ path of G is properly colored. However, if we are primarily concerned with a graph G containing a properly colored $u-v$ path for every two vertices u and v of G, then it is possible that this can be accomplished using fewer than $\chi'(G)$ colors.

Inspired by rainbow colorings and proper colorings in graphs, the concepts of proper-path colorings and strong proper-path colorings were introduced and studied in [1]. Let G be an edge-colored connected graph, where adjacent edges may be colored the same. A path P in G is *properly colored* or, more simply, P is a *proper path* in G if no two adjacent edges of P are colored the same. An edge coloring c is a *proper-path coloring* of a connected graph G if every pair u, v of distinct vertices of G are connected by a proper $u-v$ path in G. If k colors are used, then c is referred to as a *proper-path k-coloring.* The minimum k for which G has a proper-path k-coloring is called the *proper connection number* $\mathrm{pc}(G)$ of G.

A proper-path coloring using $\text{pc}(G)$ colors is referred to as a *minimum proper-path coloring*. Since every rainbow coloring or every proper edge coloring is a proper-path coloring, it follows that $\text{pc}(G)$ exists. If G is a nontrivial connected graph of order n and size m, then

$$1 \le \text{pc}(G) \le \min\{\chi'(G), \text{rc}(G)\} \le m. \tag{1}$$

Furthermore, $\text{pc}(G) = 1$ if and only if $G = K_n$, and $\text{pc}(G) = m$ if and only if $G = K_{1,m}$ is a star of size m.

While these concepts were introduced to parallel corresponding concepts with rainbow colorings for the purpose of studying connected graphs by means of properly colored paths in edge-colored graphs, there is a corresponding motivation to what was introduced for rainbow colorings of graphs. With regard to the national security discussion, we are then interested in the answer to the following question.

> *What is the minimum number of passwords or firewalls that allow one or more secure paths between every two agencies where as we progress from one step to another along such a path, we are required to change passwords?*

As with rainbow colorings and strong rainbow colorings, there is an analogous concept of proper-path colorings (see [1]). Let c be a proper-path coloring of a nontrivial connected graph G. For two vertices u and v of G, a *proper* $u - v$ *geodesic* in G is a proper $u - v$ path of length $d(u, v)$. If there is a proper $u - v$ geodesic for every two vertices u and v of G, then c is called a *strong proper-path coloring* of G or a *strong proper-path k-coloring* if k colors are used. The minimum number of colors needed to produce a strong proper-path coloring of G is called the *strong proper connection number* or simply the *strong connection number* $\text{spc}(G)$ of G. A strong proper-path coloring using $\text{spc}(G)$ colors is a *minimum strong proper-path coloring*. In general, if G is a nontrivial connected graph, then $1 \le \text{pc}(G) \le \text{spc}(G) \le \chi'(G)$. Since every strong rainbow coloring of G is a strong proper-path coloring of G, it follows that $\text{spc}(G) \le \text{src}(G)$. Therefore, if G is a nontrivial connected graph of order n and size m, then

$$1 \le \text{spc}(G) \le \min\{\chi'(G), \text{src}(G)\} \le m. \tag{2}$$

Similarly, $\text{spc}(G) = 1$ if and only if $G = K_n$, and $\text{spc}(G) = m$ if and only if $G = K_{1,m}$ is the star of size m.

To illustrate these concepts, consider the two proper-path colorings of the 5-cycle C_5 and the proper-path coloring of the 3-regular graph G shown in Figure 1. The coloring in Figure 1(a) is a minimum proper-path coloring of C_5 and so $\text{pc}(C_5) = 2$. The coloring in Figure 1(b) is a minimum strong proper-path coloring of C_5 and so $\mathrm{spc}(C_5) = 3$. The coloring in Figure 1(c) is both a minimum proper-path coloring and a minimum strong proper-path coloring of G and so $\mathrm{pc}(G) = \mathrm{spc}(G) = 3$. Note that this 3-regular graph G is not 1-factorable and so $\chi'(G) = 4$.

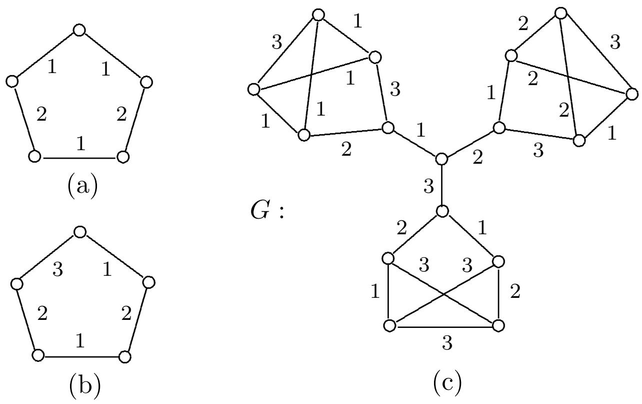


Figure 1. Illustrating proper-path colorings and strong proper-path colorings.

In [1] the numbers $\mathrm{pc}(G)$ and $\mathrm{spc}(G)$ were determined for several well-known classes of graphs G and relationships among these five edge colorings (namely, proper-path colorings, strong proper-path colorings, rainbow colorings, strong rainbow colorings and proper edge colorings) were investigated. Furthermore, several realization theorems were established for the five edge coloring parameters (namely $\mathrm{pc}(G)$, $\mathrm{spc}(G)$, $\mathrm{rc}(G)$, $\mathrm{src}(G)$ and $\chi'(G)$) of a connected graph G. All graphs of size m having rainbow connection numbers $m-2$ and $m-3$ have been characterized by Li, Sun and Zhao in [7]. By (1) and (2), if G is a nontrivial connected graph of size m, then $\mathrm{pc}(G) \leq m$ and $\mathrm{spc}(G) \leq m$. As we mentioned above, the star $K_{1,m}$ of size m is the only nontrivial connected graph of size m having proper connection number and strong proper connection number m. In this paper, we present characterizations of those connected graphs of size m having proper connection number or strong proper connection number $m-1$, $m-2$ or $m-3$. The following preliminary results will be useful to us.

Proposition 1.1 [1]. *Let G be a nontrivial connected graph containing bridges. If b is the maximum number of bridges incident with a single vertex in G, then* $\mathrm{pc}(G) \geq b$ *and* $\mathrm{spc}(G) \geq b$.

Proposition 1.2 [1]. *If T is a nontrivial tree, then* $\mathrm{pc}(T) = \mathrm{spc}(T) = \chi'(T) = \Delta(T)$, *where $\Delta(T)$ is the maximum degree of T.*

Proposition 1.3 [1]. *If G is a nontrivial connected graph and H is a connected spanning subgraph of G, then* $\mathrm{pc}(G) \leq \mathrm{pc}(H)$. *Furthermore,*

$$\mathrm{pc}(G) \leq \min\{\Delta(T) : T \text{ is a spanning tree of } G\}.$$

A *Hamiltonian path* in a graph G is a path containing every vertex of G and a graph having a Hamiltonian path is often called a *traceable graph.* The following is an immediate consequence of Proposition 1.3.

Corollary 1.4 [1]**.** *If G is a traceable graph that is not complete, then* $\text{pc}(G) = 2$.

Lemma 1.5. *If G is a connected graph of size m_G and H is a proper connected subgraph of size m_H in G, then* $\text{pc}(G) \leq m_G - m_H + \text{pc}(H)$ *and* $\text{spc}(G) \leq m_G - m_H + \text{spc}(H)$.

Proof. Let $\text{pc}(H) = a$ and $b = m_G - m_H + a$. Suppose that c_H is a minimum proper-path coloring of H using the colors $1, 2, \ldots, a$. Then c_H can be extended to a proper-path b-coloring c of G by assigning the $m_G - m_H$ colors $a+1, a+2, \ldots, b = m_G - m_H + a$ to the $m_G - m_H$ edges in $E(G) - E(H)$, which implies that $\text{pc}(G) \leq m_G - m_H + \text{pc}(H)$.

A similar argument shows that $\text{spc}(G) \leq m_G - m_H + \text{spc}(H)$. ∎

We refer to the books [3, 4] for graph theory notation and terminology not described in this paper.

2. Characterizing Graphs of Size m Having Proper Connection Number $m-1, m-2$ or $m-3$

We have seen in (1) that if G is a connected graph of size m, then $\text{pc}(G) \leq m$ and $\text{pc}(G) = m$ if and only if $G = K_{1,m}$. We now characterize all connected graphs G of size m for which $\text{pc}(G) \in \{m-1, m-2, m-3\}$, beginning with those connected graphs G of size $m \geq 3$ with $\text{pc}(G) = m-1$. A *double star* is a tree of diameter 3. Thus each double star has exactly two non-end-vertices called the *central vertices* of the double star. If the central vertices of a double star have degrees a and b, respectively, then it is denoted by $S_{a,b}$, where the order of $S_{a,b}$ is $a + b$. By Proposition 1.2, if T is a nontrivial tree, then $\text{pc}(T) = \Delta(T)$ and so $\text{pc}(S_{a,b}) = \max\{a, b\}$.

Proposition 2.1. *Let G be a connected graph of size $m \geq 3$. Then* $\text{pc}(G) = m-1$ *if and only if $G = S_{2,m-1}$.*

Proof. Let $G = S_{2,m-1}$. Since $\text{pc}(G) = \Delta(G) = m-1$ by Proposition 1.2, it remains to verify the converse. Let G be a connected graph of size $m \geq 3$ such that $\text{pc}(G) = m-1$. We claim that G is a tree. If this is not the case, then G contains a cycle. Let $C = (v_1, v_2, \ldots, v_\ell, v_1)$, where $\ell \geq 3$, be a cycle of G. If $\ell = 3$, then the coloring that assigns the color 1 to each edge of C and distinct colors from the set $\{2, 3, \ldots, m-2\}$ to the remaining $m-3$ edges is a proper-path coloring of G and so $\text{pc}(G) \leq m-2$, which is a contradiction. If $\ell \geq 4$, then the coloring that assigns (i) the color 1 to $v_\ell v_1$ and v_2v_3, (ii) the color 2 to v_1v_2 and v_3v_4 and (iii) distinct colors from the set $\{3,4,\dots,m-2\}$ to the remaining $m-4$ edges is a proper-path coloring of G and so $\mathrm{pc}(G) \le m-2$, which is again impossible. Thus G is a tree and so $G = S_{2,m-1}$ by Proposition 1.2. ∎

In order to characterize all connected graphs G of size m having $\mathrm{pc}(G) = m-2$ or $\mathrm{pc}(G) = m-3$, we first present two lemmas. For a nontrivial graph G for which $G+uv = G+xy$ for every two pairs $\{u,v\}$, $\{x,y\}$ of nonadjacent vertices of G, the graph $G+e$ is obtained from G by adding the edge e joining two nonadjacent vertices of G.

Lemma 2.2. *If $G = K_{1,m-1}+e$, where $m \ge 4$, then*

$$\mathrm{pc}(G) = \begin{cases} m-2 & \textit{if } m = 4,5, \\ m-3 & \textit{if } m \ge 6. \end{cases}$$

Proof. By Corollary 1.4, $\mathrm{pc}(K_{1,3}+e) = 2$ and so $\mathrm{pc}(K_{1,3}+e) = m-2$ when $m = 4$. First, we show that $\mathrm{pc}(K_{1,4}+e) = 3$. By assigning the colors 1 and 2 to the two bridges of $K_{1,4}+e$ and the color 3 to the remaining edges of $K_{1,4}+e$, we obtain a proper-path 3-coloring of $K_{1,4}+e$ and so $\mathrm{pc}(K_{1,4}+e) \le 3$. Assume, to the contrary, that $K_{1,4}+e$ has a proper-path 2-coloring using the colors 1 and 2. Necessarily, the two bridges uv and uw must be colored differently, say 1 and 2, respectively. Then some vertex x of degree 2 is incident with edges of the same color, say 1. Let $e = xy$. In order for $K_{1,4}+e$ to have a properly colored $x-v$ path, uy must be colored 2. However then, $K_{1,4}+e$ contains no properly colored $y-w$ path. Therefore, $\mathrm{pc}(K_{1,4}+e) = 3 = m-2$. Now suppose that $m \ge 6$ and $G = K_{1,m-1}+e$, where $V(G) = \{v, v_1, v_2, \dots, v_{m-1}\}$, v is the central vertex of $K_{1,m-1}$ and $e = v_{m-2}v_{m-1}$. By Proposition 1.1, $\mathrm{pc}(G) \ge m-3$. Define the coloring $c : E(G) \to \{1,2,\dots,m-3\}$ by $c(vv_i) = i$ for $1 \le i \le m-3$ and $\{c(vv_{m-1}), c(vv_{m-2}), c(v_{m-2}v_{m-1})\} = \{1,2,3\}$. Since c is a proper-path $(m-3)$-coloring of G, it follows that $\mathrm{pc}(G) \le m-3$ and so $\mathrm{pc}(G) = m-3$. ∎

A graph is *unicyclic* if it is connected and contains exactly one cycle. Thus the graph $K_{1,m-1}+e$ considered in Lemma 2.2 is unicyclic, whose lone cycle is a triangle. For an integer $m \ge 5$, let $S_{3,m-3}$ be the double star whose central vertices have degrees 3 and $m-3$. The unicyclic graph U_m of size m is the graph obtained from $S_{3,m-3}$ by adding an edge joining the two neighboring end-vertices of the central vertex of degree 3 in $S_{3,m-3}$ (see Figure 2).

Lemma 2.3. *For each integer $m \ge 5$, $\mathrm{pc}(U_m) = m-3$.*

Proof. Let $H = S_{3,m-3}$ be the double star in U_m. Suppose that u and v are the two central vertices of H with $\deg_H u = 3$ and $\deg_H v = m-3$, where u_1 and u_2 are the two end-vertices adjacent to u and $v_1, v_2, \dots, v_{m-4}$ are the $m-4$

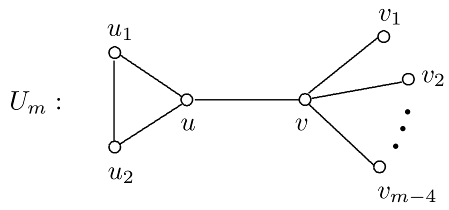


Figure 2. The unicyclic graph U_m of size $m \geq 5$.

end-vertices adjacent to v (see Figure 2). Since each of the $m-3$ edges incident with v must be assigned different colors in a proper-path coloring of U_m, it follows that $\text{pc}(U_m) \geq m-3$. On the other hand, the coloring that assigns the color 1 to each edge in $\{uu_1, uu_2, u_1u_2\}$, the color i to the edge vv_i for $1 \leq i \leq m-4$ and the color $m-3$ to the edge uv is a proper-path $(m-3)$-coloring of U_m and so $\text{pc}(U_m) \leq m-3$. Therefore, $\text{pc}(U_m) = m-3$. ■

Theorem 2.4. *Let G be a connected graph of size $m \geq 4$. Then* $\text{pc}(G) = m-2$ *if and only if G is a tree with $\Delta(G) = m-2$ or $G \in \{C_4, K_{1,3}+e, K_{1,4}+e\}$.*

Proof. If G is a tree with $\Delta(G) = m-2$, then $\text{pc}(G) = \Delta(G)$ by Proposition 1.2. By Corollary 1.4 and Lemma 2.2, $\text{pc}(G) = m-2$ if $G \in \{C_4, K_{1,3}+e, K_{1,4}+e\}$. Thus, it remains to verify the converse. Let G be a connected graph of size $m \geq 4$ such that $\text{pc}(G) = m-2$. Assume, to the contrary, that G is not a tree and $G \notin \{C_4, K_{1,3}+e, K_{1,4}+e\}$. Since G is not a tree, G contains a cycle. By Corollary 1.4, $G \neq C_n$ for $n \geq 5$. Let $C = (v_1, v_2, \ldots, v_\ell, v_1)$ be a longest cycle of G. First, suppose that $\ell \geq 5$. Since $\text{pc}(C) = 2$ by Corollary 1.4, it follows by Lemma 1.5 that $\text{pc}(G) \leq m-\ell+2 \leq m-3$, which is impossible. Thus, $\ell = 4$ or $\ell = 3$. If $\ell = 4$, then either G contains $C_4 + e$ as a subgraph or there is a vertex x of G such that xv_i is an edge in G where $1 \leq i \leq 4$. In either case, G contains a subgraph H of size 5 with $\text{pc}(H) = 2$ by Corollary 1.4. It then follows by Lemma 1.5 that $\text{pc}(G) \leq m-3$, which is a contradiction. Finally, if $\ell = 3$, then G contains a subgraph that is isomorphic to one of the three graphs H_1, H_2, H_3 in Figure 3. (Note that G may also contain the graphs G_5 and G_6 shown in Figure 4 as subgraphs. Because H_2 is a subgraph of G_5 and H_1 is a subgraph of G_6, we only need to consider the three graphs H_1, H_2, H_3.) Since $\text{pc}(H_i) = m(H_i) - 3$ for $i = 1, 2, 3$, where $m(H_i)$ is the size of H_i, it follows that $\text{pc}(G) \leq m-3$, a contradiction. ■

Theorem 2.5. *Let G be a connected graph of size $m \geq 5$. Then* $\text{pc}(G) = m-3$ *if and only if* (i) *G is a tree with $\Delta(G) = m-3$,* (ii) *$G = K_{1,m-1} + e$, where $m \geq 6$,* (iii) *$G = U_m$ or* (iv) *G is one of the graphs in Figure* 4.

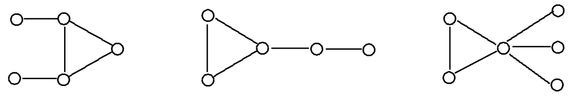


Figure 3. Subgraphs H_1, H_2, H_3 in the proof of Theorem 2.4.

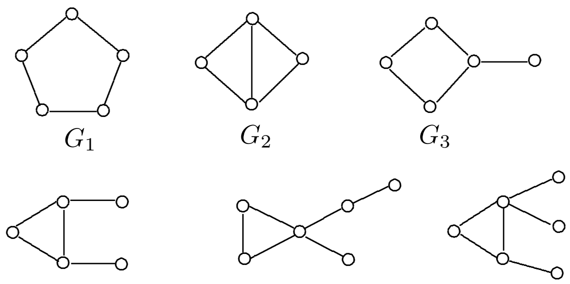


Figure 4. The seven graphs in Theorem 2.5.

Proof. By Proposition 1.2 and Lemmas 2.2 and 2.3, if G is a tree with $\Delta(G) = m-3$ or $G = K_{1,m-1}+e$, where $m \geq 6$, or $G = U_m$ for $m \geq 5$, then $\text{pc}(G) = m-3$. If G is one of the traceable graphs G_i $(1 \leq i \leq 4)$ of size 5 in Figure 4, then $\text{pc}(G) = 2 = m-3$ by Corollary 1.4. If $G = G_i$ for $i = 5, 6$, then G is obtained from $K_{1,4}+e$ by adding a pendant edge $f = xv$ at a vertex v of $K_{1,4}+e$. Observe that for every two vertices u and w of $G-x$, each $u-w$ path of G completely lies in $G-x$. This implies that the restriction of a proper-path coloring of G to the subgraph $G-x$ is also a proper-path coloring of $G-x$. Since $\text{pc}(K_{1,4}+e) = 3$ by Lemma 2.2, it follows that $\text{pc}(G) \geq 3$. Since there is a proper-path 3-coloring of $G = G_i$ for $i = 5, 6$ (as shown in Figure 5) and G has size 6, it follows that $\text{pc}(G) \leq 3$ and so $\text{pc}(G) = 3 = m-3$.

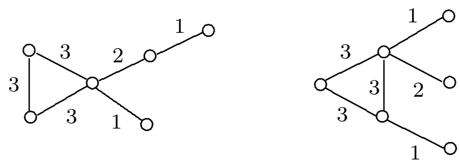


Figure 5. Proper-path 3-colorings of G_5 and G_6 in Figure 4.

For the converse, let G be a connected graph of size $m \geq 5$ such that $\text{pc}(G) = m-3$. Assume, to the contrary, that G is not a tree, $G \neq K_{1,m-1}+e$, where $m \geq 6$, $G \neq U_m$ and G is not any of the graphs shown in Figure 4. Since G is not a tree, it follows that G contains a cycle. By Corollary 1.4, $G \neq C_n$ for $n \geq 6$.

Let $C = (v_1, v_2, \ldots, v_\ell, v_1)$ be a longest cycle of G. First, suppose that $\ell \geq 6$. Since $\text{pc}(C) = 2$ by Corollary 1.4, it follows by Lemma 1.5 that $\text{pc}(G) \leq m - 4$, which is impossible. Thus $\ell \in \{3, 4, 5\}$. If $\ell = 5$, then either G contains $C_5 + e$ as a subgraph or G contains the subgraph obtained from C_5 by adding exactly one pendant edge at a vertex of C_5. In either case, G contains a traceable subgraph H of size 6 with $\text{pc}(H) = 2$ by Corollary 1.4. It then follows by Lemma 1.5 that $\text{pc}(G) \leq m - 4$, which is a contradiction. If $\ell = 4$, then G contains as a subgraph K_4 or one of the graphs of size 6 in Figure 6. Since $\text{pc}(K_4) = 1$ and each of these graphs has proper connection number 2 (where a proper-path 2-coloring of each graph is also shown in Figure 6), it follows by Lemma 1.5 that $\text{pc}(G) \leq m - 4$, which is a contradiction.

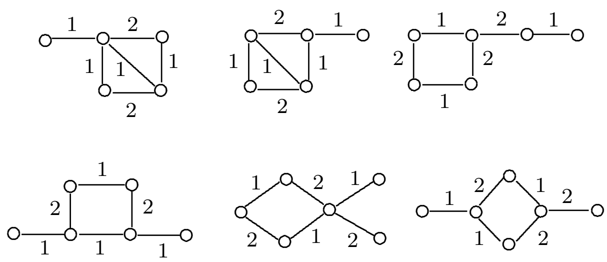


Figure 6. Subgraphs having proper connection number 2 in the proof of Theorem 2.5.

Finally, if $\ell = 3$, then G contains a subgraph that is isomorphic to one of the seven graphs $H_1, H_2, \ldots, H_7$ in Figure 7. For $i = 1, 2, 3, 4$, the graph H_i has size 6 and $\text{pc}(H_i) = 2$; while for $i = 5, 6, 7$, the graph H_i has size 7 and $\text{pc}(H_i) = 3$. A minimum proper-path coloring of each graph H_i $(1 \leq i \leq 7)$ is also shown in Figure 7. Hence $\text{pc}(H_i) = m(H_i) - 4$ for $1 \leq i \leq 7$ and so $\text{pc}(G) \leq m - 4$ by Lemma 1.5, which is a contradiction. ■

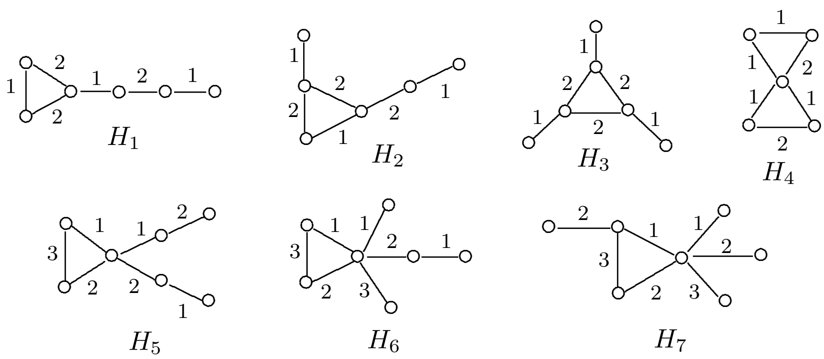


Figure 7. Subgraphs $H_1, H_2, \ldots, H_7$ in the proof of Theorem 2.5.

3. Characterizing Graphs of Size m Having Strong Connection Number $m-1, m-2$ or $m-3$

By Proposition 2.1, the double star $S_{2,m-1}$ of size $m \geq 3$ is the only connected graph of size m with proper connection number $m-1$. Employing an argument similar to the one used in the proof of Proposition 2.1, we now show that this is also true for the strong connection numbers of graphs.

Proposition 3.1. *Let G be a connected graph of size $m \geq 3$. Then* $\text{spc}(G) = m-1$ *if and only if* $G = S_{2,m-1}$.

Proof. If $G = S_{2,m-1}$, then $\text{spc}(G) = \Delta(S_{2,m-1}) = m-1$ by Proposition 1.2. For the converse, let G be a connected graph of size $m \geq 3$ such that $\text{spc}(G) = m-1$. We claim that G is a tree. If this is not the case, then G contains a cycle. Let $C = (v_1, v_2, \ldots, v_\ell, v_1)$, where $\ell \geq 3$, be a cycle of G. If $\ell = 3$, then the coloring that assigns the color 1 to each edge of C and distinct colors from the set $\{2, 3, \ldots, m-2\}$ to the remaining $m-3$ edges is a strong proper-path coloring of G and so $\text{spc}(G) \leq m-2$, which is a contradiction. If $\ell \geq 4$, then the coloring that assigns (i) the color 1 to $v_\ell v_1$ and $v_2 v_3$, (ii) the color 2 to $v_1 v_2$ and $v_3 v_4$ and (iii) distinct colors from the set $\{3, 4, \ldots, m-2\}$ to the remaining $m-4$ edges is a strong proper-path coloring of G and so $\text{spc}(G) \leq m-2$, which is again impossible. Thus, as claimed, G is a tree and so $G = S_{2,m-1}$. ■

In order to characterize all connected graphs G of size m having $\text{spc}(G) = m-2$, we first present two useful lemmas, the first of which was observed in [1].

Lemma 3.2 [1]**.** *For an integer $n \geq 4$,*

$$\text{spc}(C_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

Lemma 3.3. *For an integer $m \geq 4$,* $\text{spc}(K_{1,m-1} + e) = m-2$.

Proof. Let $G = K_{1,m-1} + e$ with $V(G) = \{v, v_1, v_2, \ldots, v_{m-1}\}$, where v is the central vertex of $K_{1,m-1}$ and $e = v_{m-2}v_{m-1}$. Define the coloring $c : E(G) \to \{1, 2, \ldots, m-2\}$ by $c(vv_i) = i$ for $1 \leq i \leq m-3$ and $c(vv_{m-1}) = c(vv_{m-2}) = c(v_{m-2}v_{m-1}) = m-2$. Since c is a strong proper $(m-2)$-coloring of G, it follows that $\text{spc}(G) \leq m-2$. By Proposition 1.1, $\text{spc}(G) \geq m-3$. If G has a strong proper $(m-3)$-coloring c', then c' must assign the same color to an edge $\{vv_i : 1 \leq i \leq m-3\}$ and an edge in $\{vv_{m-1}, vv_{m-2}\}$, say $c'(vv_1) = c'(vv_{m-1})$. However then, there is no proper $v_1 - v_{m-1}$ geodesic in G, which is impossible. Hence $\text{spc}(K_{1,m-1} + e) = m-2$. ■

Theorem 3.4. *Let G be a connected graph of size $m \geq 4$. Then* $\text{spc}(G) = m-2$ *if and only if G is a tree with $\Delta(G) = m-2$ or $G \in \{C_4, C_5, K_{1,m-1} + e\}$.*

Proof. If G is a tree with $\Delta(G) = m - 2$ or $G \in \{C_4, C_5, K_{1,m-1} + e\}$, then $\text{spc}(G) = m - 2$ by Proposition 1.2 and Lemmas 3.2 and 3.3. For the converse, let G be a connected graph of size $m \geq 4$ such that $\text{spc}(G) = m - 2$. Assume, to the contrary, that G is not a tree and $G \notin \{C_4, C_5, K_{1,m-1} + e\}$. Since G is not a tree, it follows that G contains a cycle. Let $C = (v_1, v_2, \ldots, v_\ell, v_1)$ be a longest cycle of G and $\ell \geq 3$. By Lemma 3.2, $G \neq C_n$ for $n \geq 6$. If $\ell \geq 6$, then $\text{spc}(G) \leq m - \ell + 3 \leq m - 3$ by Lemmas 3.2 and 1.5, which is impossible. Therefore, $\ell \in \{5, 4, 3\}$. We consider these three cases. If $\ell = 5$, then either G contains $H_1 = C_5 + e$ as a subgraph or G contains the subgraph H_2 obtained from C_5 by adding a pendant edge. Since H_1 has size 6 with $\text{spc}(H_1) = 2$ and H_2 has size 6 with $\text{spc}(H_2) = 3$, it then follows by Lemma 1.5 that $\text{spc}(G) \leq m - 3$. If $\ell = 4$, then either G contains $F_1 = C_4 + e$ as a subgraph or G contains the subgraph F_2 obtained from C_4 by adding a pendant edge. In each case, the size of F_i is 5 and $\text{spc}(F_i) = 2$ for $i = 1, 2$. It then follows by Lemma 1.5 that $\text{spc}(G) \leq m - 3$, which is impossible.

Finally, assume that $\ell = 3$. Since $G \neq K_{1,m-1} + e$ where $m - 1 \geq 3$, there are two vertices x and y of G that do not lie on C such that either (1) x is adjacent to a vertex v_1, say, of C and $xy \in E(G)$ or (2) x and y are adjacent to different vertices of C, say $xv_1, yv_2 \in E(G)$. First, suppose that (1) occurs. If $yv_1 \notin E(G)$, then the coloring that assigns (i) the color 1 to xy, v_1v_2 and v_1v_3, (ii) the color 2 to xv_1 and v_2v_3, and (iii) distinct colors from the set $\{3, 4, \ldots, m - 3\}$ to the remaining $m - 5$ edges is a strong proper-path coloring of G and so $\text{spc}(G) \leq m - 3$, which is impossible. If $yv_1 \in E(G)$, then the coloring that assigns (i) the color 1 to each edge of C, (ii) the color 2 to xy, xv_1 and yv_1, and (iii) distinct colors from the set $\{3, 4, \ldots, m - 4\}$ to the remaining $m - 6$ edges is a strong proper-path coloring of G and so $\text{spc}(G) \leq m - 4$, which is impossible. Next suppose that (2) occurs. Then the coloring that assigns (i) the color 1 to each edge of C, (ii) the color 2 to xv_1 and yv_2, and (iii) distinct colors from the set $\{3, 4, \ldots, m-3\}$ to the remaining $m - 5$ edges is a strong proper-path coloring of G and so $\text{spc}(G) \leq m - 3$, which is impossible. ■

In order to determine all connected graphs of size $m \geq 5$ with strong connection number $m - 3$, we first describe three classes of such graphs of size m. For an integer $m \geq 5$, let H_m be the graph of size m obtained from a 4-cycle C_4 by adding $m - 4$ pendant edges at a vertex of C_4 and let F_m be the graph of size m obtained from $K_{1,m-2} + e$ by adding a pendant edge at an end-vertex of $K_{1,m-2} + e$ (see Figure 8).

Lemma 3.5. *For an integer $m \geq 5$, $\text{spc}(H_m) = \text{spc}(F_m) = m - 3$.*

Proof. Let $V(H_m) = \{u, v, w, x, v_1, v_2, \ldots, v_{m-4}\}$ where (u, v, w, x, u) is the 4-cycle in H_m and vv_i is an edge of H_m for $1 \leq i \leq m - 4$. Define the coloring $c : E(H_m) \to \{1, 2, \ldots, m-3\}$ by $c(vv_i) = i$ for $1 \leq i \leq m-4$ and $c(uv) = c(vw) =$

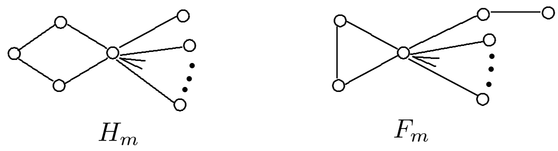


Figure 8. The graphs H_m and F_m of size $m \geq 5$.

$m-3$, $c(ux) = 1$ and $c(xw) = 2$. Since c is a strong proper $(m-3)$-coloring of H_m, it follows that $\text{spc}(H_m) \leq m-3$. By Proposition 1.1, $\text{spc}(H_m) \geq m-4$. If H_m has a strong proper $(m-4)$-coloring c', then c' must assign the same color to an edge $\{vv_i : 1 \leq i \leq m-4\}$ and an edge in $\{vu, vw\}$, say $c'(vu) = c(vv_1)$. However then, there is no proper $u - v_1$ geodesic in H_m, which is impossible. Hence $\text{spc}(H_m) = m-3$.

A similar argument shows that $\text{spc}(F_m) = m-3$ for $m \geq 5$. ■

Recall that U_m denotes the unicyclic graph of size $m \geq 5$ shown in Figure 2. An argument similar to the one used in the proof of Lemma 2.3 yields the following lemma.

Lemma 3.6. *For each integer* $m \geq 5$, $\text{spc}(U_m) = m-3$.

Theorem 3.7. *Let* G *be a connected graph of size* $m \geq 5$. *Then* $\text{spc}(G) = m-3$ *if and only if* (i) G *is a tree with* $\Delta(G) = m-3$, (ii) $G \in \{H_m, F_m, U_m\}$ *or* (iii) G *is one of the four graphs in Figure* 9.

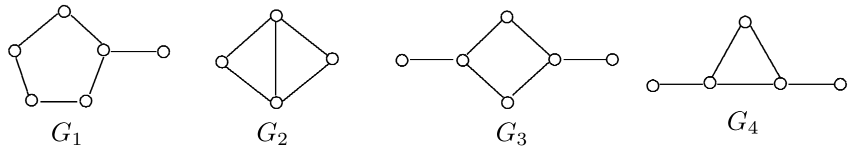


Figure 9. The graphs in Theorem 3.7.

Proof. By Proposition 1.2 and Lemmas 3.5 and 3.6, if G is a tree with $\Delta(G) = m-3$ or $G \in \{H_m, F_m, U_m\}$, then $\text{pc}(G) = m-3$. Also, it is easy to see that $\text{pc}(G_i) = m-3$ for $1 \leq i \leq 4$ for each graph G_i in Figure 9. A minimum strong proper-path coloring of G_i $(1 \leq i \leq 4)$ is shown in Figure 10.

It remains to verify the converse. Let G be a connected graph of size $m \geq 5$ such that $\text{spc}(G) = m-3$. Assume, to the contrary, that G is not a tree, $G \notin \{H_m, F_m, U_m\}$ and G is not any of the graphs shown in Figure 9. Since G is not a tree, it follows that G contains a cycle. Let $C = (v_1, v_2, \ldots, v_\ell, v_1)$ be a longest cycle of G. First, suppose that $\ell \geq 6$. Since $\text{spc}(C) = 2$ if ℓ is

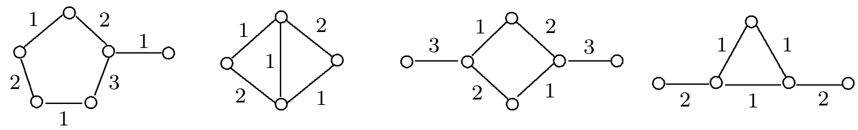


Figure 10. Minimum strong proper-path colorings of the graphs in Figure 9.

even and $\mathrm{spc}(C) = 3$ if ℓ is odd by Lemma 3.2, it follows that $\mathrm{spc}(C) \le \ell - 4$. Hence $\mathrm{spc}(G) \le m - 4$ by Lemma 1.5, which is impossible. Thus, $\ell \in \{3, 4, 5\}$. First suppose that $\ell = 5$. Since $G \ne C_5$, it follows that G contains a subgraph isomorphic to one of the graphs R_1, R_2, R_3, R_4 in Figure 11, where a minimum strong proper-path coloring is also shown for each graph. Thus $\mathrm{spc}(R_1) = 2$ and $\mathrm{spc}(R_i) = 3$ for $i = 2, 3, 4$. Since $\mathrm{spc}(R_i) \le m(R_i) - 4$ where $m(R_i)$ is the size of R_i for $i = 1, 2, 3, 4$, it follows that $\mathrm{spc}(G) \le m - 4$ by Lemma 1.5, which is a contradiction.

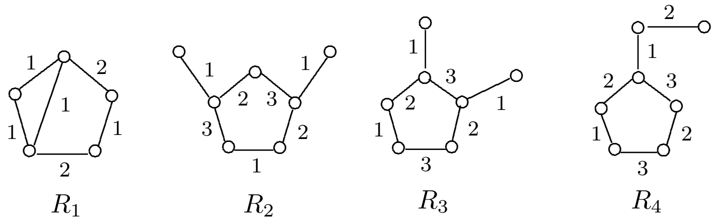
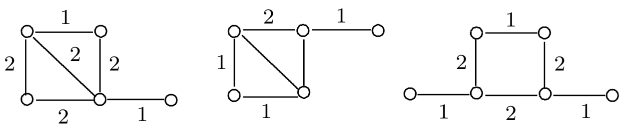
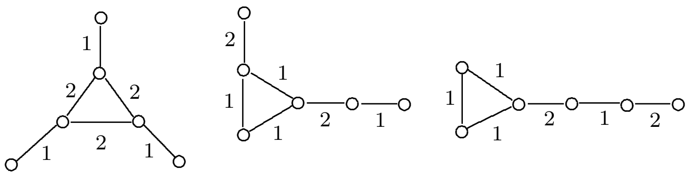


Figure 11. The subgraphs R_1, R_2, R_3, R_4 in the proof of Theorem 3.7.



(a)



(b)

Figure 12. The subgraphs of G in the proof of Theorem 3.7.

Next, suppose that $\ell = 4$. Then G contains a subgraph that is isomorphic to (i) K_4 or (ii) one of the three graphs of size 6 in Figure 12(a). Since $\mathrm{spc}(K_4) = 1$ and the strong connection number of each of the graphs in Figure 12(a) is 2 (a minimum strong proper-path coloring for each graph is shown in the figure as well), it follows by Lemma 1.5 that $\mathrm{pc}(G) \leq m-4$, a contradiction. Finally suppose that $\ell = 3$. Then G contains a subgraph isomorphic to one of the three graphs of size 6 in Figure 12(b), where a minimum strong proper-path coloring is also shown for each graph. Since the strong connection number of each of these graphs is 2, $\mathrm{spc}(G) \leq m-4$ by Lemma 1.5, which is a contradiction. ∎

Acknowledgment

We greatly appreciate the valuable suggestions made by an anonymous referee that resulted in an improved paper.

References

[1] E. Andrews, E. Laforge, C. Lumduanhom and P. Zhang, *On proper-path colorings in graphs*, J. Combin. Math. Combin. Comput., to appear.

[2] G. Chartrand, G.L. Johns, K.A. McKeon and P. Zhang, *Rainbow connection in graphs*, Math. Bohem. **133** (2008) 85–98.

[3] G. Chartrand, L. Lesniak and P. Zhang, Graphs & Digraphs, 5th Edition (Chapman & Hall/CRC, Boca Raton, FL, 2010).

[4] G. Chartrand and P. Zhang, Chromatic Graph Theory (Chapman & Hall/CRC, Boca Raton, FL, 2008).
doi:10.1201/9781584888017

[5] A.B. Ericksen, *A matter of security*, Graduating Engineer & Computer Careers (2007) 24–28.

[6] X.L. Li and Y.F. Sun, Rainbow Connections of Graphs (Springer, Boston, MA, 2012).
doi:10.1007/978-1-4614-3119-0

[7] X.L. Li, Y.F. Sun and Y. Zhao, *Characterization of graphs with rainbow connection number $m-2$ and $m-3$*, Australas. J. Combin. **60** (2014) 306–313.

Received 15 October 2014
Revised 5 August 2015
Accepted 10 August 2015