Discussiones Mathematicae
Graph Theory 36 (2016) 363–382
doi:10.7151/dmgt.1861

LARGE DEGREE VERTICES IN LONGEST CYCLES OF GRAPHS, I

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Abstract

In this paper, we consider the least integer d such that every longest cycle of a k-connected graph of order n (and of independent number α) contains all vertices of degree at least d.

Keywords: longest cycle, large degree vertices, order, connectivity, independent number.

2010 Mathematics Subject Classification: 05C38.

1. Introduction

1.1. Basic notation and terminology

All graphs considered here are simple and finite. For standard graph-theoretic terminology not explained in this paper, we refer the reader to [2]. Let G be a graph. For a vertex $v \in V(G)$ and a subgraph H of G, we use $N_H(v)$ and $d_H(v)$ to denote the set and the number of neighbors of v in H, respectively. We call $N_H(v)$ the *neighborhood* of v in H and $d_H(v)$ the *degree* of v in H. We use $d_H(u,v)$ to denote the distance between two vertices $u, v \in V(H)$ in H. For two subgraphs H and L of a graph G, we set $N_L(H) = \bigcup_{v \in V(H)} N_L(v)$. When no confusion occurs, we will denote $N_G(v)$ and $d_G(v)$ by $N(v)$ and $d(v)$, respectively. We set $N[x] = N(x) \cup \{x\}$.

Throughout this paper, we denote the order, the connectivity and the independent number of a graph G, by $n(G)$, $\kappa(G)$ and $\alpha(G)$, respectively.

1.2. Motivation and main results of this paper

By the definition every Hamilton cycle of a graph passes through every vertex of the graph. Thus, in non-Hamiltonian graphs, a (longest) cycle through some special vertices should be also interesting for the same topic. There are many results on the problem whether a graph has a (longest) cycle through some special vertices, for example, any given vertex set [8]; large degree vertices, see [1, 7, 9]. Unlike most research of the existence of some (longest) cycle passing through special vertices in the literature, we put our attention to the problem to determine the least integer d such that every longest cycle of a graph passes all vertices of degree at least d, using some additional conditions of order, of connectivity or of independence number.

The following known result gave a partly answer for the above problem.

Theorem 1 (Li and Zhang [6])**.** *Let G be a 2-connected graph of order $n \geq 8$. Then every longest cycle of G contains all vertices of degree at least $n-4$.*

We firstly extend Theorem 1 to k-connected graphs for any $k \geq 2$ and shall give a complete answer for the above problem by using the order of a graph and its connectivity.

Theorem 2. *Let G be a graph of connectivity $\kappa(G) \geq k \geq 2$ and of order $n \geq 6k-4$. Then every longest cycle of G contains all vertices of degree at least $n-3k+2$.*

The bound on the degree in Theorem 2 is sharp. We construct a graph as follows. Let $R = 2K_2 \cup (k-2)P_3$, $S = kK_1$ and $T = (n-4k+1)K_1$ are vertex-disjoint. Let R' be the subset of $V(R)$ each vertex of which is either a vertex of a K_2 or a center of a P_3 in R, and let s' be a fixed vertex of S and x a vertex not in $R \cup S \cup T$. Let $L(k,n)$ be the graph with $V(L(k,n)) = \{x\} \cup V(R) \cup V(S) \cup V(T)$, and $E(L(k,n)) = E(R) \cup \{r's', rs, s'x, sx, st, xt : r' \in R', r \in V(R), s \in V(S) \backslash \{s'\}, t \in V(T)\}$. One can check that $L(k,n)$ is k-connected and the degree of x is $n-3k+1$, but there is a longest cycle (in the subgraph induced by $V(R) \cup V(S)$) excluding x.

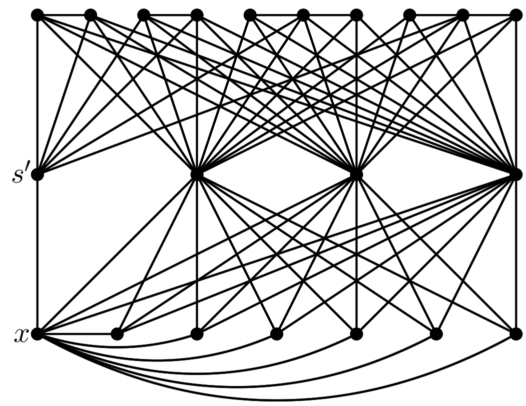

Figure 1. Graph $L(4,21)$.

The bound $n \geq 6k-4$ is also sharp. This can be seen from the complete bipartite graph $K_{3k-3,3k-2}$ of order $6k-5$. However, the longest cycles of $K_{3k-3,3k-2}$ exclude some vertices of degree $3k-3 = n-3k+2$.

Now we define $\varphi(k,n)$ to be the least integer such that every longest cycle of a k-connected graph G of order n contains all vertices of degree at least $\varphi(k,n)$ in G.

To avoid the discussions of the petty cases, we put our considerations on 2-connected graphs, i.e., we always assume that $k \geq 2$. Note that if $n \leq k$, then there are no k-connected graphs of order n. Hence $\varphi(k,n)$ will be meaningless. Is $\varphi(k,n)$ well-defined for all pairs (k,n) with $n \geq k+1$? No. Under the condition that it holds "every k-connected graph on n vertices is Hamiltonian" (e.g., $n = k+1$), $\varphi(k,n)$ does not exist (or we may say $\varphi(k,n) = -\infty$). So we should take the pair (k,n) such that there exist some k-connected graphs of order n which are not Hamiltonian. This implies that $n \geq 2k+1$ from the well-known Dirac's theorem [4]. On the other hand, there indeed exist such graphs when $n \geq 2k+1$ (for example, complete bipartite graphs $K_{k,n-k}$). So $\varphi(k,n)$ is well-defined if and only if $n \geq 2k+1$.

From Theorem 2 and the construction of $L(k,n)$, we have

$$\varphi(k,n) = n-3k+2, \text{ for } n \geq 6k-4.$$

How about the cases when $2k+1 \leq n \leq 6k-5$? First we construct a graph as follows: if n is odd, then let $L(k,n) = K_{(n-1)/2,(n+1)/2}$; if n is even, then let $L(k,n) = K_{n/2-1,n/2+1}$. Note that every longest cycle of $L(k,n)$ excludes some vertices of degree $\lceil n/2 \rceil - 1$. This shows that $\varphi(k,n) \geq \lceil n/2 \rceil$. On the other hand, we have the following result (one may compare it with the results in [1] and [9] where they replaced "every cycle" with "there exists some cycle" under the condition that "G is 2-connected").

Theorem 3. *Let G be a k-connected graph on $n \leq 6k-5$ vertices. Then every longest cycle of G contains all vertices of degree at least $\lceil n/2 \rceil$.*

Instead of Theorems 2 and 3, we shall prove the following theorem in Section 3.

Theorem 4. *Let G be a graph of connectivity $\kappa(G) \geq k \geq 2$ and of order $n \geq 2k+1$. Then every longest cycle of G contains all vertices of degree at least*

$$d = \max\left\{ \left\lceil \frac{n}{2} \right\rceil, n-3k+2 \right\}.$$

Now we have a complete formula

$$\varphi(k,n) = \max\left\{ \left\lceil \frac{n}{2} \right\rceil, n-3k+2 \right\}, \text{ for all } n \geq 2k+1.$$

In the following we consider the same problem by using an additional condition of independent number. We use $\varphi(k,\alpha,n)$ to denote the least integer such that for every k-connected graph G of order n and of independent number α, every longest cycle of G contains all vertices of degree at least $\varphi(k,\alpha,n)$. As the analysis above, we should take the triple (k,α,n) such that there exists a k-connected graph of order n and independent number α that is not Hamiltonian. This requires $\alpha \geq k+1$ from Chvátal-Erdös's theorem [3]; and $\alpha \leq n-k$, since every k connected graph of order n has independent number at most $n-k$ (note that an independent set excludes the k neighbors of some vertex). On the other hand, for triple (k,α,n) with $k+1 \leq \alpha \leq n-k$, the graph $kK_1 \vee ((\alpha-1)K_1 \cup K_{n-k-\alpha+1})$ is a k-connected graph of order n and independent number α that is not Hamiltonian. Thus $\varphi(k,\alpha,n)$ is well-defined if and only if $k+1 \leq \alpha \leq n-k$.

By the definition of $\varphi(k,n)$, we can see that

$$\varphi(k,n) = \max\{\varphi(k,\alpha,n) : k+1 \leq \alpha \leq n-k\}, \text{ for all } n \geq 2k+1.$$

Using a result in [10], we can prove the following result.

Theorem 5. *Let G be a k-connected graph of order n and of independent number α. Then every longest cycle of G contains all vertices of degree more than*

$$d_0 = \frac{(\alpha-k)n - k\alpha + k^2 + \alpha^2 - 2\alpha}{\alpha}.$$

Taking $\alpha = k+1$ in the above theorem, we can obtain the following correspondence.

Theorem 6. *Let G be a graph of connectivity $\kappa(G) \geq k \geq 2$, of order $n \geq 2k+1$ and of independent number $k+1$. Then every longest cycle of G contains all vertices of degree at least*

$$d = \left\lfloor \frac{n+1}{k+1} \right\rfloor + k - 1.$$

The bound on d in Theorem 6 is sharp. We construct a graph $L(k, k+1, n)$ by joining each vertex of $R = kK_1$ to all vertices of $S = rK_{q+1} \cup (k+1-r)K_q$, where

$$n - k = q(k+1) + r, \ 0 \leq r \leq k.$$

Note that $L(k, k+1, n)$ has a longest cycle excluding some vertices of degree

$$q + k - 1 = \left\lfloor \frac{n-k}{k+1} \right\rfloor + k - 1 = \left\lfloor \frac{n+1}{k+1} \right\rfloor + k - 2.$$

By Theorem 6, the above equality implies that

$$\varphi(k, k+1, n) = \left\lfloor \frac{n+1}{k+1} \right\rfloor + k - 1, \text{ for all } n \geq 2k+1.$$

Thus, in the following we will assume that $\alpha \geq k+2$. For the case $k = 2$, we have the following result.

Theorem 7. *Let G be a 2-connected graph of order $n \geq 8$ and independent number $\alpha \geq 4$. Then every longest cycle of G contains all vertices of degree at least*

$$d = \left\lfloor \frac{n-5}{\alpha} \right\rfloor (\alpha - 2) + \max\left\{3, n - 4 - \left\lfloor \frac{n-5}{\alpha} \right\rfloor \alpha\right\}$$

i.e.,

$$d = \begin{cases} q(\alpha-2)+3, & 0 \leq r \leq 2, \\ q(\alpha-2)+r+1, & 3 \leq r < \alpha, \end{cases}$$

where

$$n - 5 = q\alpha + r, \ 0 \leq r < \alpha.$$

The bound on d in Theorem 7 is sharp when $q \geq 1$ (i.e., when $n \geq \alpha + 5$). We construct extremal graphs as follows. If $0 \leq r \leq 2$, then let $R = rK_{q+2} \cup (2-r)K_{q+1}$ and $T = (\alpha-2)K_q$; if $3 \leq r < \alpha$, then let $R = 2K_{q+2}$ and $T = (r-2)K_{q+1} \cup (\alpha - r)K_q$. Let s', s, x be three vertices not in $R \cup T$. Let $L(2, \alpha, n)$

be a graph with the vertex set $V(L(2,\alpha,n)) = \{s', s, x\} \cup V(R) \cup V(T)$ and the edge set

$$E(L(2,\alpha,n)) = E(R) \cup E(T) \cup \{s'r, sr, s'x, sx, st, xt : r \in V(R), t \in V(T)\}.$$

One can check that $L(2,\alpha,n)$ is a 2-connected graph of order n and of independent number α, and x has degree $d-1$. But there is a longest cycle of G excluding x. By Theorem 7, this implies that for $n \geq \alpha + 5$,

$$\varphi(2,\alpha,n) = \begin{cases} q(\alpha-2)+3, & 0 \leq r \leq 2, \\ q(\alpha-2)+r+1, & 3 \leq r < \alpha, \end{cases}$$

where

$$n - 5 = q\alpha + r, 0 \leq r < \alpha.$$

For the case $q = 0$, the above construction does not give the exact value of $\varphi(2,\alpha,n)$, since the independent number of the constructed graph is less than α. What is its exact values for this case?

Note that $n \leq \alpha+4$ in this case. Also note that in our assumption $n \geq \alpha+2$. We have three cases: $n = \alpha+2$, $n = \alpha+3$ and $n = \alpha+4$.

Theorem 8. *Let G be a 2-connected graph of independent number $\alpha \geq 4$ and of order n such that $\alpha + 2 \leq n \leq \alpha + 4$. Then every longest cycle of G contains all vertices of degree at least*

$$d = \begin{cases} n-\alpha+1, & n-\alpha = 2, 3, \\ \alpha, & n-\alpha = 4. \end{cases}$$

Now we will show the sharpness of the bound in Theorem 8. For the case $n = \alpha + 2$, consider the graph $L(2,\alpha,\alpha+2) = K_{2,\alpha}$. Note that every longest cycle of $L(2,\alpha,\alpha+2)$ excludes some vertices of degree 2.

For the case $n = \alpha + 3$, consider the graph $L(2,\alpha,\alpha+3) = K_{3,\alpha}$. Note that every longest cycle of $L(2,\alpha,\alpha+3)$ excludes some vertices of degree 3.

Now we consider the case $n = \alpha + 4$. We construct the graph $L(2,\alpha,\alpha+4)$ by combining a cycle C_6 and a star $K_{1,\alpha-3}$ in such a way: choosing two vertices u, v in C_6 with distance 3, joining the center x of the star to u and v, and joining all the end-vertices of the star to v. Note that one longest cycle of $L(2,\alpha,\alpha+4)$ excludes x of degree $\alpha - 1$.

Therefore, we give formulas for $\alpha \geq 4$,

$$\varphi(2,\alpha,\alpha+2) = 3,$$
$$\varphi(2,\alpha,\alpha+3) = 4,$$
$$\varphi(2,\alpha,\alpha+4) = \alpha.$$

The bound of d_0 in Theorem 5 seems not sharp for $\alpha \geq k+2$ (at least it is not sharp when $k = 2$). We propose the following conjecture.

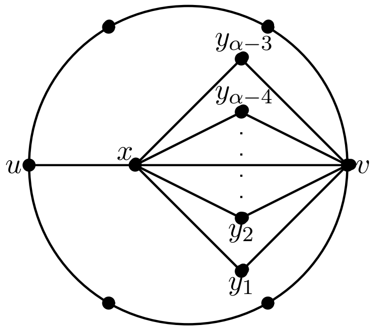

Figure 2. Graph $L(2,\alpha,\alpha+4)$.

Conjecture 9. *Let G be a k-connected graph, $k \geq 3$, of independent number $\alpha \geq k+2$ and of order $n \geq \max\{2\alpha+1, \alpha+3k+1\}$. Then every longest cycle of G contains all vertices of degree at least*

$$d = \begin{cases} q(\alpha-k)+k+1, & 0 \leq r \leq k, \\ q(\alpha-k)+k+2, & k+1 \leq r \leq 2k+1, \\ q(\alpha-k)+r-k+1, & 2k+2 \leq r < \alpha+k, \end{cases}$$

where

$$n-2k-1 = q(\alpha+k)+r,\ 0 \leq r < \alpha+k.$$

We remark that if the conjecture is true, then the bound on d is sharp. We construct a graph as follows. If $0 \leq r \leq k$, then let $R = rK_{2q+2} \cup (k-r)K_{2q+1}$ and $T = (\alpha-k)K_q$; if $k+1 \leq r \leq 2k+1$, then let $R = (r-k-1)K_{2q+3} \cup (2k+1-r)K_{2q+2}$ and $T = K_{q+1} \cup (\alpha-k-1)K_q$; if $2k+2 \leq r < \alpha+k$, then let $R = kK_{2q+3}$ and $T = (r-2k)K_{q+1} \cup (\alpha+k-r)K_q$, and let $S = kK_1$. Let x be a vertex not in $R \cup S \cup T$. Let $L(k,\alpha,n)$ be a graph with $V(L(k,\alpha,n)) = \{x\} \cup V(R) \cup V(S) \cup V(T)$ and

$$E(L(k,\alpha,n)) = E(R) \cup E(T) \cup \{sr, sx, st, xt : r \in V(R), s \in V(S), t \in V(T)\}.$$

One can check that $L(k,\alpha,n)$ is a 2-connected graph of order n and of independent number α, and x has degree $d-1$. But there is a longest cycle of G excluding x.

2. Preliminaries

Let G be a graph and $x, y \in V(G)$. An x-path is a path with x as one of its end vertices; an (x,y)-path is one connecting x and y. If Y is a subset of $V(G)$, then an (x,Y)-path is one connecting x and a vertex in Y with all internal vertices in

$V(G)\backslash Y$; a Y-path is one connecting two vertices in Y with all internal vertices in $V(G)\backslash Y$. For a subgraph H of G, we use the notations (x, H)-path and H-path instead of $(x, V(H))$-path and $V(H)$-path, respectively. It is convenient to denote a path P with end-vertices x, y by $P(x, y)$.

For a cycle C with a given orientation and a vertex x on C, we use x^+ to denote the successor, and x^- the predecessor of x on C. In the following, we always assume that C has an orientation, $\overrightarrow{C}$. For two vertices x, y on C, $\overrightarrow{C}[x, y]$ or $\overleftarrow{C}[y, x]$ denotes the path from x to y along $\overrightarrow{C}$. Similarly, if x, y are two vertices in a path P, $P[x, y]$ denotes the subpath of P between x and y. For an arbitrary path P or cycle C, we use $l(P)$ or $l(C)$ to denote the length (the number of edges) of it.

We first give some lemmas on longest cycles of graphs.

Lemma 10. *Let C be a longest cycle of a graph G, and $P = P(u, v)$ be a C-path. Then $l(\overrightarrow{C}[u, v]) \geq l(P)$.*

Proof. Otherwise, $\overrightarrow{C}[v, u]uPv$ is a cycle longer than C, a contradiction. ■

Lemma 10 can be extended to the following.

Lemma 11. *Let C be a longest cycle of a graph G, H be a component of $G - C$ and $P = P(u, v)$ be a C-path of length at least 2 with all internal vertices in H. Then*

$$l\left(\overrightarrow{C}[u, v]\right) \geq l(P) + 2\left|N_C(H) \cap V\left(\overrightarrow{C}[u^+, v^-]\right)\right|.$$

Proof. We use induction on $|N_C(H) \cap V(\overrightarrow{C}[u^+, v^-])|$. If $N_C(H) \cap V(\overrightarrow{C}[u^+, v^-]) = \emptyset$, then we are done by Lemma 10. Now we suppose that $N_C(H) \cap V(\overrightarrow{C}[u^+, v^-]) \neq \emptyset$.

Let x be a vertex in $N_C(H) \cap V(\overrightarrow{C}[u^+, v^-])$. Let $P' = P'(x, x')$ be an $(x, P - \{u, v\})$-path with all internal vertices in $H - P$. Then $P_1 = P[u, x']x'P'$ and $P_2 = P'x'P[x', v]$ are two C-paths with end-vertices u, x and x, v, respectively, and with all internal vertices in H. Clearly, the length of P_1 and P_2 are at least 2. By induction hypothesis,

$$l\left(\overrightarrow{C}[u, x]\right) \geq l(P_1) + 2\left|N_C(H) \cap V\left(\overrightarrow{C}[u^+, x^-]\right)\right|,$$

$$l\left(\overrightarrow{C}[x, v]\right) \geq l(P_2) + 2\left|N_C(H) \cap V\left(\overrightarrow{C}[x^+, v^-]\right)\right|.$$

Note that $l(P_1) + l(P_2) = l(P) + 2l(P') \geq l(P) + 2$, and

$$\left|N_C(H) \cap V\left(\overrightarrow{C}[u^+, x^-]\right)\right| + \left|N_C(H) \cap V\left(\overrightarrow{C}[x^+, v^-]\right)\right| = \left|N_C(H) \cap V\left(\overrightarrow{C}[u^+, v^-]\right)\right| - 1.$$

We have

$$l\left(\overrightarrow{C}[u,v]\right) = l\left(\overrightarrow{C}[u,x]\right) + l\left(\overrightarrow{C}[x,v]\right) \geq l(P) + 2\left|N_C(H) \cap V\left(\overrightarrow{C}[u^+,v^-]\right)\right|.$$

The assertion is proved. ∎

Lemma 12. *Let G be a graph, C be a longest cycle of G and H be a component of $G - C$.*

(1) *If $u \in N_C(H)$, then $u^+, u^- \notin N_C(H)$.*

(2) *If $u, v \in N_C(H)$, then $u^+v^+, u^-v^- \notin E(G)$.*

Proof. The assertion (1) can be deduced from Lemma 10. Now we prove the assertion (2).

Suppose that $u, v \in N_C(H)$. Then let P be a (u,v)-path of length at least 2 with all internal vertices in H. If $u^+v^+ \in E(G)$, then

$$C' = \overrightarrow{C}[u^+,v]vPu\overleftarrow{C}[u,v^+]v^+u^+$$

is a cycle longer than C, a contradiction. Thus we conclude that $u^+v^+ \notin E(G)$, and similarly, $u^-v^- \notin E(G)$. ∎

Let G be a graph and $yz \in E(G)$, we define the *contraction of G at yz*, denoted by $G \cdot yz$, as the graph with $V(G \cdot yz) = V(G)\backslash\{y\}$, and $E(G \cdot yz) = E(G - y) \cup \{xz : xy \in E(G) \text{ and } x \neq z\}$.

Lemma 13. *Let G be a graph and $yz \in E(G)$. If there is a cycle C in $G \cdot yz$, then there is a cycle C' in G with length at least $l(C)$ such that $V(C') \subseteq V(C) \cup \{y\}$.*

Proof. If C does not contain z, then C is also a cycle of G and we are done. So we assume that $z \in V(C)$. By the definition of contraction, $zz^+ \in E(G)$ or $yz^+ \in E(G)$, and $zz^- \in E(G)$ or $yz^- \in E(G)$. Let

$$C' = \begin{cases} C, & \text{if } zz^+ \in E(G) \text{ and } zz^- \in E(G), \\ \overrightarrow{C}[z,z^-]z^-yz, & \text{if } zz^+ \in E(G) \text{ and } zz^- \notin E(G), \\ zyz^+\overrightarrow{C}[z^+,z], & \text{if } zz^+ \notin E(G) \text{ and } zz^- \in E(G), \\ yz^+\overrightarrow{C}[z^+,z^-]z^-y, & \text{if } zz^+ \notin E(G) \text{ and } zz^- \notin E(G). \end{cases}$$

Then C' is a required cycle. ∎

We will use the following theorems from [3, 10].

Theorem 14 (Chvátal and Erdös [3])**.** *If G is a graph of order $n \geq 3$ with $\alpha(G) \leq \kappa(G)$, then G is Hamiltonian.*

Theorem 15 (O, West and Wu [10])**.** *If G is a graph of order n with $\alpha(G) \geq \kappa(G)$, then G has a cycle of length at least*

$$\frac{\kappa(G)(n+\alpha(G)-\kappa(G))}{\alpha(G)}.$$

Theorem 16 (O, West and Wu [10])**.** *If G is separable, then $V(G)$ admits a partition (V_1, V_2) such that $\alpha(G) = \alpha(G[V_1]) + \alpha(G[V_2])$.*

Now we prove some more lemmas.

Lemma 17. *Let G be a nonseparable graph. Then for any two distinct vertices u, v of G, G contains a (u,v)-path of order at least $\lceil n(G)/\alpha(G) \rceil$.*

Proof. If G is complete, then the result is trivially true. Now we assume that G is not complete, i.e., $\alpha(G) \geq 2$. So G is 2-connected. If $\alpha(G) \leq \kappa(G)$, then by Theorem 14, G has a Hamilton cycle C, and either $\overrightarrow{C}[u,v]$ or $\overleftarrow{C}[u,v]$ is a required path. Now we assume that $\alpha(G) > \kappa(G)$.

Let C be a longest cycle of G. By Theorem 15, $l(C) \geq 2n(G)/\alpha(G)$. Since G is 2-connected, we may choose a (u,C)-path P_1 and a (v,C)-path P_2 such that they are vertex-disjoint. Let u' and v' be the end-vertices of P_1 and P_2, respectively, on C (possibly $u = u'$ or $v = v'$, or both). Then $P_1 u' \overrightarrow{C}[u', v'] v' P_2$ or $P_1 u' \overleftarrow{C}[u', v'] v' P_2$ is a required path. ■

Lemma 18. *Let G be a 2-connected graph. Let C be a subgraph of G with at least two vertices, and H be an induced subgraph of $G - C$. Then G contains a C-path P such that*

$$|V(P) \cap V(H)| \geq \left\lceil \frac{n(H)}{\alpha(H)} \right\rceil.$$

Proof. We use induction on $n(H)$. If H has only one vertex, say x, then $n(H) = \alpha(H) = 1$. Since G is 2-connected, there is a C-path passing through x, which is a required path. Now we assume that H has at least two vertices.

Suppose first that H is nonseparable. Let $P_1(u,u')$ and $P_2(v,v')$ be two vertex-disjoint paths between H and C with all internal vertices in $G - (H \cup C)$, where $u, v \in V(H)$ and $u', v' \in V(C)$. By Lemma 17, H contains a (u,v)-path P' of order at least $\lceil n(H)/\alpha(H) \rceil$. Thus $P = P_1 u P' v P_2$ is a required path.

Now we suppose that H is separable. By Theorem 16, there is a partition (V_1, V_2) of $V(H)$ such that $\alpha(H) = \alpha(G[V_1]) + \alpha(G[V_2])$. Let $H_1 = G[V_1]$ and $H_2 = G[V_2]$. Note that $n(H) = n(H_1) + n(H_2)$. It is not hard to see that

$$\frac{n(H)}{\alpha(H)} \leq \max\left\{\frac{n(H_1)}{\alpha(H_1)}, \frac{n(H_2)}{\alpha(H_2)}\right\} \overset{\text{say}}{=\!=} \frac{n(H_1)}{\alpha(H_1)}.$$

By induction hypothesis, there is a C-path P such that

$$|V(P) \cap V(H)| \geq |V(P) \cap V(H_1)| \geq \left\lceil \frac{n(H_1)}{\alpha(H_1)} \right\rceil \geq \left\lceil \frac{n(H)}{\alpha(H)} \right\rceil,$$

and P is a required path. ∎

Let L_1 be the graph obtained from C_6 by adding a new vertex x, and by adding three edges from x to three pairwise nonadjacent vertices of the C_6, and let $L_2 = K_3 \vee 4K_1$. Set

$$\mathcal{L} = \{G : L_1 \subseteq G \subseteq L_2\}.$$

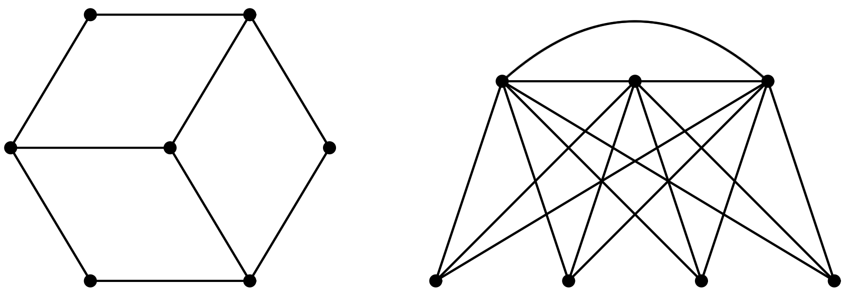

Figure 3. Graphs L_1 and L_2.

We prove the following lemma to complete this section.

Lemma 19. *Let G be a 2-connected graph with $n(G) \leq 7$ and x be a vertex of G with $d(x) \geq 3$. If there is a longest cycle of G excluding x, then $G \in \mathcal{L}$.*

Proof. Let C be a longest cycle of G excluding x and let H be the component of $G - C$ containing x.

We first suppose that H has at least two vertices. By the 2-connectedness of G, there are two independent edges from H to C. Let yy' and zz' be such two edges, where $y, z \in V(H)$ and $y', z' \in V(C)$. Let P be a (y, z)-path of H. Then $P' = y'yPzz'$ is a path with length at least 3. By Lemma 10, $l(\overrightarrow{C}[y', z']) \geq 3$ and $l(\overleftarrow{C}[y', z']) \geq 3$. Thus $l(C) \geq 6$. Note that $n(H) \geq 2$, we have $n(G) \geq 8$, a contradiction.

Thus we conclude that H has the only one vertex x. By Lemma 12, x cannot be adjacent to two successive vertices on C. Since $d(x) \geq 3$, there will be at least three vertices on C adjacent to x, and at least three vertices on C nonadjacent to x. Thus $l(C) \geq 6$. Since $n(G) \leq 7$ and $x \notin V(C)$, we have $l(C) \leq 6$. Thus C

has exactly 6 vertices and x is adjacent to three pairwise nonadjacent vertices of C. This implies that $L_1 \subseteq G$.

Let $C = y_1z_1y_2z_2y_3z_3y_1$ such that $N(x) = \{y_1, y_2, y_3\}$. By Lemma 12, $\{x, z_1, z_2, z_3\}$ is an independent set. This implies that $G \subseteq L_2$. ■

3. Proofs of main results

In this section, we shall present the proof of main results.

***Proof of Theorem* 4.** Let C, with an orientation $\overrightarrow{C}$, be a longest cycle of G. We assume on the contrary that there is a vertex x in $V(G-C)$ with $d(x) \geq d = \max\{\lceil n/2 \rceil, n-3k+2\}$.

An (x, C)-fan is a collection of (x, C)-paths such that they have the only vertex x in common. Since G is k-connected, there is an (x, C)-fan with $s \geq k$ paths $P_i = P_i(x, z_i)$, $1 \leq i \leq s$, where $z_i \in V(C)$. We choose the (x, C)-fan such that s is as large as possible. We suppose that $z_1, z_2, \ldots, z_s$ appear in this order along $\overrightarrow{C}$. Thus

$$l(C) = \sum_{i=1}^{s} l(\overrightarrow{C}[z_i, z_{i+1}]), \tag{1}$$

where the subscripts are taken modulo s.

By Menger's theorem, there is a vertex $y_i \in V(P_i - x)$ such that $S = \{y_i : 1 \leq i \leq s\}$ is a vertex-cut of G separating x and $C - S$. We choose y_i in such a way that $d_{P_i}(x, y_i)$ is as small as possible (note that y_i is possibly equal to z_i). Clearly

$$N_C(x) \subseteq S. \tag{2}$$

Let H be the component of $G - S$ containing x. Then

Claim 20. *For every vertex $y_i \in S$, either $N_H(y_i) = \{x\}$ or $|N_H(y_i)| \geq 2$.*

Proof. Suppose on the contrary that $|N_H(y_i)| = 1$ and $y_i' \neq x$ is the vertex in $N_H(y_i)$. Then y_i' is the neighbor of y_i on $P_i[x, y_i]$. Let $S' = (S \backslash \{y_i\}) \cup \{y_i'\}$. Then S' is a vertex-cut of G separating x and $C - S'$ such that $d_{P_i}(x, y_i') < d_{P_i}(x, y_i)$, contradicting the choice of S. □

If H has only one vertex x, then $d(x) = |S| = s$. By Lemma 12, $l(\overrightarrow{C}[z_i, z_{i+1}]) \geq 2$ for all $i \in \{1, 2, \ldots, s\}$. By (1), $l(C) \geq 2s = 2d(x)$ and

$$n \geq l(C) + 1 \geq 2d(x) + 1 \geq n + 1,$$

a contradiction.

If H has exactly two vertices, then let x' be the vertex in $V(H)\backslash\{x\}$. By Claim 20, every vertex y_i in S is adjacent to x. Hence $N(x) = S \cup \{x'\}$ and $s = d(x) - 1$. Note that $d(x') = d_S(x') + 1$ and $d(x') \geq k$, since G is k-connected. We have $d_S(x') \geq k - 1$. By Lemma 12, $l(\overrightarrow{C}[z_i, z_{i+1}]) \geq 2$ for all i. Moreover, if $x'y_i \in E(G)$, then $P = P_i[z_i, y_i]y_i x' x y_{i+1} P_{i+1}[y_{i+1}, z_{i+1}]$ is a C-path of length at least 3, by Lemma 10, $l(\overrightarrow{C}[z_i, z_{i+1}]) \geq 3$. This implies that $l(C) = \sum_{i=1}^{s} l(\overrightarrow{C}[z_i, z_{i+1}]) \geq 3d_S(x') + 2(s - d_S(x')) \geq 2s + d_S(x') \geq 2d(x) + k - 3 \geq n + k - 3$, and $n \geq l(C) + 2 \geq n + k - 1 \geq n + 1$, a contradiction.

Now it remains to consider the case when H has at least three vertices.

By $b(x)$ we denote the number of vertices in $V(G)\backslash N[x]$. Then $b(x) = n - 1 - d(x) \leq 3k - 3$. Hence, by (2),

$$l(C) \leq s + b(x) \leq s + 3k - 3. \tag{3}$$

Claim 21. *Every vertex in $V(H)\backslash\{x\}$ is not a cut-vertex of H.*

Proof. Suppose, otherwise, that $x' \neq x$ is a cut-vertex of H. Let H_1 and H_2 be two components of $H - x'$ such that $x \in V(H_1)$.

We claim that for every vertex $y_i \in S$, $N_{H_1}(y_i) \neq \emptyset$. Otherwise, every (x, y_i)-path with all internal vertices in H will pass through x', so is $P_i[x, y_i]$. Let $S' = (S\backslash\{y_i\}) \cup \{x'\}$. Then S' is a vertex-cut of G separating x and $C - S'$ such that $d_{P_i}(x, x') < d_{P_i}(x, y_i)$, a contradiction. Thus as we claimed, $N_{H_1}(y_i) \neq \emptyset$.

For every $y_i \in S$, let w_i be a vertex in $N_{H_1}(y_i)$. Now we claim that $l(\overrightarrow{C}[z_i, z_{i+1}]) \geq 4$ for those i such that $N_{H_2}(y_i) \neq \emptyset$. Suppose $N_{H_2}(y_i) \neq \emptyset$. Let w'_i be a neighbor of y_i in H_2. Then H has a (w'_i, w_{i+1})-path P of length at least 2. Thus $P' = P_i[z_i, y_i]y_i w'_i P w_{i+1} y_{i+1} P_{i+1}[y_{i+1}, z_{i+1}]$ is a path of length at least 4 with all internal vertices in $G - C$. By Lemma 10, $l(\overrightarrow{C}[z_i, z_{i+1}]) \geq 4$.

Note that $|N_S(H_2)| \geq k - 1$, since G is k-connected and $N_S(H_2) \cup \{x'\}$ is a vertex-cut. Therefore,

$$l(C) \geq 4(k - 1) + 2(s - k + 1) = 2s + 2k - 2 \geq s + 3k - 2,$$

contradicting (3). □

Claim 22. *H is a star with center x.*

Proof. Suppose, otherwise, H has an x-path $xx'x''$ (say) of length 2. Then there is an (x'', S)-fan with k internally disjoint paths $Q_i = Q_i(x'', y_{j_i})$, $1 \leq j_1 < j_2 < \cdots < j_k \leq s$, such that they have the only vertex x'' in common. We set $S' = \{y_{j_i} : 1 \leq i \leq k\}$.

Note that at most one path of Q_i passes through x. We will prove that $l(\overrightarrow{C}[z_{j_i}, z_{j_i+1}]) \geq 4$ for those j_i such that $y_{j_i} \in S'$ and Q_i does not pass through x.

Suppose that $y_{j_i} \in S'$ and Q_i does not pass through x. Let w_{j_i} be the neighbor of y_{j_i} on Q_i. Then $w_{j_i} \neq x$. If $l(Q_i) \geq 2$, then let v_{j_i} be a neighbor of w_{j_i} on the path $Q_i[x'', w_{j_i}]$; if $l(Q_i) = 1$, then ($w_{j_i} = x''$ and) we let $v_{j_i} = x'$. Then $v_{j_i} \neq x$. By Claim 20, y_{j_i+1} has a neighbor w'_{j_i+1} in H other that w_{j_i}. We claim that H has a (w_{j_i}, w'_{j_i+1})-path of length at least 2. Otherwise $w_{j_i}w'_{j_i+1} \in E(G)$ and $w_{j_i}w'_{j_i+1}$ is a cut-edge of H. By Claim 21, every vertex of $V(H)\backslash\{x\}$ is not a cut-vertex of H. This implies that $w'_{j_i+1} = x$ and w_{j_i} has only one neighbor x in H, contradicting the fact that $v_{j_i} \in N_H(w_{j_i})$ and $v_{j_i} \neq x$. Thus as we claimed, H has a (w_{j_i}, w'_{j_i+1})-path P of length at least 2. Thus $P' = P_{j_i}[z_{j_i}, y_{j_i}]y_{j_i}w_{j_i}Pw'_{j_i+1}y_{j_i+1}P_{j_i+1}[y_{j_i+1}, z_{j_i+1}]$ is a path of length at least 4 with all internal vertices in $G - C$. By Lemma 10, $l(\overrightarrow{C}[z_{j_i}, z_{j_i+1}]) \geq 4$.

Thus we conclude that there are at least $k-1$ segments $\overrightarrow{C}[z_i, z_{i+1}]$ of length at least 4. Hence

$$l(C) \geq 4(k-1) + 2(s-k+1) = 2s + 2k - 2 \geq s + 3k - 2,$$

a contradiction. □

By Claim 22, $H = K_{1,n(H)-1}$. Let

$$S_0 = \{y_i \in S : N_H(y_i) = \{x\}\}, \qquad S_2 = S\backslash(S_0 \cup S_1),$$
$$S_1 = \{y_i \in S : |N_H(y_i)\backslash\{x\}| = 1\}, \qquad s_i = |S_i|, \ i \in \{0,1,2\}.$$

Thus $s = s_0 + s_1 + s_2$.

Let y_{j_i}, $1 \leq j_1 < j_2 < \cdots < j_{s_1+s_2} \leq s$, be the vertices in $S_1 \cup S_2$. Since G is k-connected,

$$s_1 + s_2 \geq |N_S(x')| \geq k - 1 \tag{4}$$

for any $x' \in V(H)\backslash\{x\}$, and

$$s_1 + (n(H)-1)s_2 \geq |E(H-x, S)| \geq (k-1)(n(H)-1). \tag{5}$$

If $s_1 + s_2 = 1$, then without loss of generality we assume that $x'y_1 \in E(G)$, where $x' \in V(H)\backslash\{x\}$ and $y_1 \in S_1 \cup S_2$. Note that $\{x, y_1\}$ is a vertex-cut of G, implying that $k = 2$. Since $z_1P_1[z_1, y_1]y_1x'xy_2P_2[y_2, z_2]$ is a path of length at least 3, by Lemma 10, $l(\overrightarrow{C}[z_1, z_2]) \geq 3$ and by symmetry, $l(\overrightarrow{C}[z_1, z_s]) \geq 3$. Thus

$$l(C) \geq 3 + 3 + 2(s-2) = 2s + 2 > s + 3k - 3,$$

a contradiction. Now we conclude that $s_1 + s_2 \geq 2$.

Claim 23. *For every vertex $y_{j_i} \in S_1 \cup S_2$,*

$$l(\overrightarrow{C}[z_{j_i}, z_{j_{i+1}}]) \geq \begin{cases} 3 + 2|N_C(x) \cap V(\overrightarrow{C}[z_{j_i}^+, z_{j_{i+1}}^-])|; & y_{j_i} \in S_1, \\ 4 + 2|N_C(x) \cap V(\overrightarrow{C}[z_{j_i}^+, z_{j_{i+1}}^-])|; & y_{j_i} \in S_2, \end{cases}$$

where the subsubscripts are taken modulo $s_1 + s_2$.

Proof. For any $y_{j_i} \in S_1 \cup S_2$, we let w_{j_i} be a vertex in $N_H(y_{j_i}) \setminus \{x\}$. If $y_{j_i} \in S_1$, then by Claim 20, $y_{j_i}x \in E(G)$. Thus

$$P = P_{j_i}[z_{j_i}, y_{j_i}]y_{j_i}xw_{j_{i+1}}y_{j_{i+1}}P_{j_{i+1}}[y_{j_{i+1}}, z_{j_{i+1}}]$$

is a C-path of length at least 3. If $y_{j_i} \in S_2$, then let w'_{j_i} be a vertex in $N_H(y_{j_i}) \setminus \{x, w_{j_{i+1}}\}$. Thus $P = P_{j_i}[z_{j_i}, y_{j_i}]y_{j_i}w'_{j_i}xw_{j_{i+1}}y_{j_{i+1}}P_{j_{i+1}}[y_{j_{i+1}}, z_{j_{i+1}}]$ is a C-path of length at least 4. Note that $N_C(H) \cap V(\overrightarrow{C}[z_{j_i}^+, z_{j_{i+1}}^-]) = N_C(x) \cap V(\overrightarrow{C}[z_{j_i}^+, z_{j_{i+1}}^-])$. By Lemma 11, we have the assertion. □

Note that $\sum_{i=1}^{s_1+s_2} |N_C(x) \cap V(\overrightarrow{C}[z_{j_i}^+, z_{j_{i+1}}^-])| = s_0$. By Claim 23,

$$l(C) = \sum_{i=1}^{s_1+s_2} l(\overrightarrow{C}[z_{j_i}, z_{j_{i+1}}]) \geq 2s_0 + 3s_1 + 4s_2 = 2s + s_1 + 2s_2.$$

By (4) and (5), we have

$$\begin{aligned} l(C) &\geq 2s + s_1 + 2s_2 = 2s + \frac{n(H)-3}{n(H)-2}(s_1+s_2) + \frac{1}{n(H)-2}(s_1 + (n(H)-1)s_2) \\ &\geq 2s + \frac{n(H)-3}{n(H)-2}(k-1) + \frac{n(H)-1}{n(H)-2}(k-1) = 2s + 2k - 2 \geq s + 3k - 2, \end{aligned}$$

a contradiction.

The proof is complete. ■

Proof of Theorem 5. If $\alpha \leq \kappa(G)$, then G is Hamiltonian by Theorem 14 and we are done. Now suppose that $\alpha > \kappa(G)$. Let C be a longest cycle of G with an orientation, $\overrightarrow{C}$. Assume for contradiction that there exists a vertex x of degree more than d_0 such that $x \notin V(C)$. Let H be the component of $G - C$ containing x. Then $|N_C(H)| \geq k$, since G is k-connected. Let $N_C(H) = \{z_1, z_2, \ldots, z_s\}$, where $s = |N_C(H)|$. Hence

$$d(x) \leq |V(H - x)| + |N_C(H)| \leq n + s - l(C) - 1. \tag{6}$$

By Lemma 12, $\{x, z_1^+, z_2^+, \ldots, z_s^+\}$ is an independent set of G. Thus, we obtain that $s+1 \le \alpha$. Therefore, by (6) and by the hypothesis of $d(x) > d_0$ and by Theorem 15,

$$\begin{aligned} d_0 < d(x) \le n + s - l(C) - 1 &\le n + \alpha - 1 - \frac{\kappa(G)(n+\alpha-\kappa(G))}{\alpha} - 1 \\ &= n + \alpha - 2 - \frac{\kappa(G)(n+\alpha-\kappa(G))}{\alpha} = d_0, \end{aligned}$$

a contradiction. This completes the proof of Theorem 5. ■

In order to use the induction method, we prove the following stronger theorem instead of Theorem 7.

Theorem 24. *Suppose $\alpha \ge 4$ and $n \ge 3$ are two integers and d is defined as in Theorem 7. Let G be a 2-connected graph with $n(G) \le n$ and $\alpha(G) \le \alpha$. Then every longest cycle of G contains all the vertices of degree at least d, unless $G \in \mathcal{L}$.*

Proof. We use induction on $n(G)$. If G has only three or four vertices, then G is Hamiltonian and the result is trivially true. Now we assume that G has at least five vertices and assume that the assertion holds for all graphs with order less than $n(G)$. This implies that $n \ge 5$ and $q \ge 0$.

Suppose that $q = 0$. Then $r = n - 5$. If $n \le 7$, then $r \le 2$ and $d = 3$. By Lemma 19, $G \in \mathcal{L}$ or every longest cycle contains all vertices of degree at least d. If $n \ge 8$, then $r \ge 3$ and $d = r + 1 = n - 4$. By Theorem 1, every longest cycle contains all vertices of degree at least d. Thus we are done. So in the following, we assume that $q \ge 1$ (i.e., $n \ge \alpha + 5$).

Let C be a longest cycle of G. We suppose on the contrary that there is a vertex x in $V(G-C)$ with $d(x) \ge d$. Let H be the component of $G - C$ containing x.

Let $b = n - 1 - d$. Then

$$b = \begin{cases} 2q + r + 1, & 0 \le r \le 2, \\ 2q + 3, & 3 \le r < \alpha. \end{cases}$$

By $b(x)$ we denote the number of vertices in $V(G) \backslash N[x]$. Then

$$b(x) \le b \le 2q + 3. \tag{7}$$

Suppose first that H has only one vertex x. By Lemma 12, x is nonadjacent to every vertex of $N_C^+(x)$. Thus $b \ge b(x) \ge d(x) \ge d$. By comparing the formulas of b and d, we can see that $r = 2$ and $\alpha = 4$. Since $q \ge 1$, we have $d \ge \alpha + 1 \ge 5$. But in this case $N_C^+(x)$ is an independent set with $d(x) \ge 5$ vertices, a contradiction. This implies that H has at least two vertices.

Note that

$$d-\alpha=\begin{cases}(q-1)(\alpha-2)+1, & 0\le r\le 2,\\ (q-1)(\alpha-2)+r-1, & 3\le r<\alpha.\end{cases}$$

We have $d-\alpha\ge(q-1)(\alpha-2)+1$, and

$$\left\lceil\frac{d-\alpha}{\alpha-2}\right\rceil\ge\left\lceil\frac{(q-1)(\alpha-2)+1}{\alpha-2}\right\rceil=q. \tag{8}$$

Suppose that there is some component of $G-C$ other than H. Let G' be the graph obtained from G by removing all other components of G, i.e., $G'=G[V(C)\cup V(H)]$. Then G' is 2-connected, $n(G')<n(G)$, $\alpha(G')\le\alpha(G)$, and $d_{G'}(x)=d(x)$. By induction hypothesis, every longest cycle of G' contains x. This implies that there is a cycle in G', and then in G, longer than C, a contradiction. Hence we conclude that there is only one component H of $G-C$, i.e., $G-C=H$.

Claim 25. $N(x)=(V(H)\cup N_C(H))\backslash\{x\}$.

Proof. Suppose that there is a vertex y in H such that $xy\notin E(G)$. We choose a vertex $z\in N(y)$ in such a way that if $G-y$ is 2-connected, then let z be an arbitrary neighbor of y; if $G-y$ is separable, then let z be a neighbor of y which is an inner-vertex of some end-block of $G-y$. In any case, $\{y,z\}$ is not a vertex-cut and thus $G'=G\cdot yz$ is 2-connected. Note that $n(G')<n(G)$, $\alpha(G')\le\alpha(G)$, and $d_{G'}(x)=d(x)$. By induction hypothesis, every longest cycle of G' contains x. This implies that there is a cycle in G' longer than C. But if G' contains such a cycle, then so is G by Lemma 13, a contradiction. This implies that x is adjacent to all the vertices in $V(H)\backslash\{x\}$.

Note that every vertex in $V(H)\backslash\{x\}$ is not a cut-vertex of H. Suppose that there is a vertex z in $N_C(H)$ such that $xz\notin E(G)$. It is not difficult to see that there is a neighbor y of z in H such that $\{y,z\}$ is not a vertex-cut of G. Thus $G'=G\cdot yz$ is 2-connected. Note that $n(G')<n(G)$, $\alpha(G')\le\alpha(G)$, and $d_{G'}(x)=d(x)$. By induction hypothesis, every longest cycle of G' contains x. This implies that there is a cycle in G', and then in G, longer than C, a contradiction. Now we conclude that x is adjacent to all the vertices in $(V(H)\cup N_C(H))\backslash\{x\}$. □

By Claim 25, $\alpha(H)=\alpha(H-x)$ and $d_H(x)=n(H-x)$. By Lemma 18, there is a C-path $P=P(u,v)$ such that

$$|V(P)\cap V(H-x)|\ge\left\lceil\frac{n(H-x)}{\alpha(H-x)}\right\rceil=\left\lceil\frac{d_H(x)}{\alpha(H)}\right\rceil.$$

By Claim 25, we can choose P such that it satisfies the above inequality and $x\in V(P)$. Thus

$$|V(P)|\ge|V(P)\cap V(H-x)|+|\{u,v,x\}|\ge\left\lceil\frac{d_H(x)}{\alpha(H)}\right\rceil+3.$$

By Claim 25, $d_H(x) = d(x) - |N_C(H)| \geq d - |N_C(H)|$. Note that the union of $N_C^+(H)$ and an independent set of H form an independent set of G. This implies that $\alpha(H) \leq \alpha(G) - |N_C(H)| \leq \alpha - |N_C(H)|$. Together with the above inequality, we have

$$|V(P)| \geq \left\lceil \frac{d - |N_C(H)|}{\alpha - |N_C(H)|} \right\rceil + 3 = \left\lceil \frac{d - \alpha}{\alpha - |N_C(H)|} \right\rceil + 4 \geq \left\lceil \frac{d - \alpha}{\alpha - 2} \right\rceil + 4.$$

By (8), $l(P) = |V(P)| - 1 \geq q + 3$. By Lemma 11,

$$l(C) = l(\overrightarrow{C}[u, v]) + l(\overrightarrow{C}[v, u]) \geq 2l(P) + 2(|N_C(H)| - 2) \geq 2q + 2|N_C(H)| + 2.$$

Thus

$$b(x) = |V(C) \backslash N_C(H)| \geq 2q + 2|N_C(H)| + 2 - |N_C(H)| \geq 2q + 4,$$

contradicting (7).

The proof is complete. ■

***Proof of Theorem* 8.** The case $n = \alpha + 2$ is trivial. The only 2-connected graphs with independent number α and order $\alpha + 2$ are $K_{2,\alpha}$ and $K_{1,1,\alpha}$. Note that every longest cycle of them contains all (the two) vertices with degree at least 3. For the case $n = \alpha + 4$, the bound on d in Theorems 7 and 8 are equal. So the result can be deduced by Theorem 7 immediately.

Now we consider the case $n = \alpha + 3$. Let G be a 2-connected graph with independent number α and order $\alpha + 3$, let C be an arbitrary longest cycle of G, and let x be a vertex of G of degree at least 4. If C contains x, then we have nothing to prove. So we assume that $x \in V(G - C)$. If x is an isolated vertex of $G - C$, then $d_C(x) = d(x) \geq 4$. By Lemma 12, $l(C) \geq 8$. Thus

$$\begin{aligned}\alpha(G) &\leq \alpha(G[V(C)]) + \alpha(G - C) \leq \alpha(C) + |V(G - C)| = \left\lfloor \frac{l(C)}{2} \right\rfloor + n - l(C) \\ &= n - \left\lceil \frac{l(C)}{2} \right\rceil \leq n - 4,\end{aligned}$$

a contradiction. Thus we conclude that x has a neighbor x' in $G - C$. Since G is 2-connected, G has a C-path P passing through the edge xx'. Note that $l(P) \geq 3$, and by Lemma 10, $l(C) \geq 6$. Thus

$$\begin{aligned}\alpha(G) &\leq \alpha(G[V(C)]) + \alpha(G - C) \leq \left\lfloor \frac{l(C)}{2} \right\rfloor + n - l(C) - 1 \\ &= n - \left\lceil \frac{l(C)}{2} \right\rceil - 1 \leq n - 4,\end{aligned}$$

a contradiction. ■

Acknowledgments

This work is supported by the Natural Science Funds of China (11471037, 11171129, 11271300). The first author is supported by the Doctorate Foundation of Northwestern Polytechnical University (CX201202) and the project NEXLIZ – CZ.1.07/2.3.00/30.0038, which is co-financed by the European Social Fund and the state budget of the Czech Republic. The results were finished while the second author visited Department of Mathematics, University of West Bohemia, Plzeň, Czech Republic, under the project P202/12/G061 of the Czech Science Foundation and he enjoyed the wonderful stay in Plzeň. The second author is also supported by Specialized Research Fund for the Doctoral Program of Higher Education (No. 20131101110048).

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Received 8 December 2014
Revised 13 May 2015
Accepted 12 July 2015