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KERNELS BY MONOCHROMATIC PATHS AND COLOR-PERFECT DIGRAPHS

Hortensia Galeana-Sánchez[1] and Rocío Sánchez-López

Instituto de Matemáticas UNAM, Ciudad Universitaria
Circuito Exterior 04510, México, D.F., México

e-mail: hgaleana@matem.unam.mx
usagitsukinomx@yahoo.com.mx

Abstract

For a digraph D, $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D respectively. In an arc-colored digraph, a subset K of $\mathrm{V}(D)$ is said to be *kernel by monochromatic paths* (mp-kernel) if (1) for any two different vertices x, y in N there is no monochromatic directed path between them (N is mp-independent) and (2) for each vertex u in $V(D) \setminus N$ there exists $v \in N$ such that there is a monochromatic directed path from u to v in D (N is mp-absorbent). If every arc in D has a different color, then a kernel by monochromatic paths is said to be a *kernel*. Two associated digraphs to an arc-colored digraph are the closure and the color-class digraph $\mathscr{C}_C(D)$. In this paper we will approach an mp-kernel via the closure of induced subdigraphs of D which have the property of having few colors in their arcs with respect to D. We will introduce the concept of color-perfect digraph and we are going to prove that if D is an arc-colored digraph such that D is a quasi color-perfect digraph and $\mathscr{C}_C(D)$ is not strong, then D has an mp-kernel. Previous interesting results are generalized, as for example Richardson's Theorem.

Keywords: kernel, kernel perfect digraph, kernel by monochromatic paths, color-class digraph, quasi color-perfect digraph, color-perfect digraph.

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1. Introduction

For general concepts we refer the reader to [2] and [3]. Let D be a digraph with the set of vertices $V(D)$ and the set of arcs $A(D)$. An arc of D of the form (x, x)

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is a *loop*. We say that two digraphs D_1 and D_2 are equal, denoted by $D_1 = D_2$, if $\mathrm{A}(D_1) = \mathrm{A}(D_2)$ and $V(D_1) = V(D_2)$. A *directed walk* in a digraph D is a sequence $(v_1, v_2, \ldots, v_n)$ of vertices of D such that $(v_i, v_{i+1}) \in A(D)$ for each $i \in \{1, \ldots, n-1\}$. If $v_i \neq v_j$ for all i and j such that $\{i, j\} \subseteq \{1, \ldots, n\}$ and $i \neq j$, it is called a *directed path*. A *directed cycle* is a directed walk $(v_1, v_2, \ldots, v_n, v_1)$ such that $v_i \neq v_j$ for all i and j such that $\{i, j\} \subseteq \{1, \ldots, n\}$ and $i \neq j$. If D is an infinite digraph, an *infinite outward path* is an infinite sequence $(v_1, v_2, \ldots)$ of distinct vertices of D such that $(v_i, v_{i+1}) \in A(D)$ for each $i \in \mathbb{N}$. In this paper we are going to write walk, path, cycle instead of directed walk, directed path, directed cycle, respectively. For $S \subseteq V(D)$ we define the *in-neighborhood* of S as $\Gamma^-_D(S) = \{y \in V(D) \mid (y, v) \in A(D) \text{ for some } v \in S\}$. We define the *out-neighborhood* of S as $\Gamma^+_D(S) = \{y \in V(D) \mid (v, y) \in A(D) \text{ for some } v \in S\}$. A digraph D is *strong* if for every pair of vertices x and y there is an xy-path in D. A subdigraph G of D is called a *strong component* if it is strong and it is maximal with respect to this property. A strong component G of D is called *initial* (respectively *terminal*) if $\Gamma^-_D(V(G)) \subseteq V(G)$ (respectively $\Gamma^+_D(V(G)) \subseteq V(G)$). For $S \subseteq V(D)$ the subdigraph of D induced by S, denoted by $D[S]$, has $V(D[S]) = S$ and $A(D[S]) = \{(u, v) \in A(D) \mid \{u, v\} \subseteq S\}$. For a nonempty subset B of $A(D)$, the *edge-induced by B subdigraph*, denoted by $D[B]$, is the subdigraph of D such that $V(D[B]) = \{v \in V(D) \mid (v, w) \in B$ or $(w, v) \in B$ for some $w \in V(D)\}$ and $A(D[B]) = B$. We shall say that a subset $S \subseteq V(D)$ is *independent* if the only arcs in $D[S]$ are loops. A digraph D is a *bipartite digraph* if there is a partition (V_1, V_2) of $V(D)$ such that $D[V_i]$ is an independent set for each $i \in \{1, 2\}$. The *line digraph* of D, denoted by $L(D)$, is the digraph such that $V(L(D)) = A(D)$, and $((u, v), (w, z)) \in A(L(D))$ if and only if $v = w$. A *tournament* is a digraph T such that for any pair of distinct vertices x, y in $V(T)$ exactly one of (x, y) and (y, x) is present. Let D be an m-colored digraph. A digraph D is said to be an *m-colored digraph* if its arcs are colored with the integers $\{1, \ldots, m\}$. A directed path is called *monochromatic* if all of its arcs are colored alike. For an arc (z_1,z_2) of D we will denote by $c(z_1,z_2)$ its color.

In this work every concept defined for uncolored digraphs is also valid for m-colored digraphs when we think of their uncolored version.

A set $K \subseteq V(D)$ is said to be a *kernel* if it is both independent (a vertex in K has no successor in K) and absorbing (a vertex not in K has a successor in K). If every induced subdigraph of D has a kernel, D is said to be a *kernel perfect digraph.* The concept of kernel was first introduced in [17] by von Neumann and Morgenstern in the context of Game Theory as a solution for cooperative n-player games. In [7] Chvátal showed that deciding if a graph possesses a kernel is an NP-complete problem, and in [9] Fraenkel showed that it remains NP-complete for planar directed graphs with indegrees less than or equal to 2, outdegrees less than or equal to 2 and degrees less than or equal to 3. The concept of kernel is important to the theory of digraphs because it arises naturally in applications such as Nim-type games, logic, and facility location, to name a few. Several authors have been investigating sufficient conditions for the existence of kernels in digraphs. For a comprehensive survey see for example [6] and [10].

Among the classical results on kernel theory we have the following theorem, which was proved by Richardson in [19].

Theorem 1 [19]**.** *A finite digraph without cycles of odd length has a kernel.*

Several extensions of Richardson's Theorem have been found in the last years. In the present work we will deduce Richardson's Theorem as a direct consequence of our main result.

Since not every digraph has a kernel it is frequently of interest to examine digraphs without kernels. An interesting class of digraphs without kernel are the critical kernel imperfect digraphs which are defined as follows: A digraph D is said to be *critical kernel imperfect* if D has no kernel but every proper induced subdigraph of D has a kernel.

A structural property of critical kernel imperfect digraph, found by Berge and Duchet, is represented in Theorem 2.

Theorem 2 (Berge and Duchet [5])**.** *Let D be a finite digraph. If D is a critical kernel imperfect digraph, then D is strong.*

The following theorem is obvious, just take a minimal induced subdigraph with no kernel.

Theorem 3. *Let D be a finite digraph. If D has no kernel, then D contains an induced subdigraph which is critical kernel imperfect.*

Theorems 2 and 3 will be useful in the proof of Lemma 11 which shows when a digraph is color-perfect; and Theorem 3 will be useful in the proof of Lemma 14 which characterizes the digraphs D such that every subdigraph of D has a kernel.

In this paper our main study will be on m-colored digraphs and for this the kernel theory will be a powerful tool.

A generalization of the concept of kernel is the concept of mp-kernel since a digraph D has a kernel if and only if the m-colored digraph D, in which every two different arcs have different colors, has an mp-kernel.

Notice that when we say "N is a kernel of an m-colored digraph D" we are thinking of a kernel of the uncolored version of D. That is to say, although the concept of kernel was defined for uncolored digraphs the same concept is valid and can be consider in m-colored digraphs.

The existence of mp-kernels in m-colored digraphs was studied primarily by Sands, Sauer and Woodrow in [21], where their main result on infinite digraphs was the following.

Theorem 4 (Sands, Sauer and Woodrow [21])**.** *Let D be a 2-colored multidigraph without monochromatic infinite outward paths. Then D has an mp-kernel.*

In [13] Galeana-Sánchez established a relation between the concept of kernel and the concept of mp-kernel as follows: the *closure* of D, denoted by $\mathfrak{C}(D)$, is the m-colored multidigraph such that $V(\mathfrak{C}(D)) = V(D)$ and (u,v) is an arc of $\mathfrak{C}(D)$ with color i if and only if there exists a monochromatic path colored i from u to v contained in D. From the definition of closure we have that:

1. $\mathfrak{C}(\mathfrak{C}(D)) = \mathfrak{C}(D)$,
2. N is an mp-kernel of D if and only if N is a kernel of $\mathfrak{C}(D)$.

On the other hand, it follows from Theorem 4 that if D is a 2-colored digraph then $\mathfrak{C}(D)$ is a kernel-perfect digraph.

Because we can use kernel theory in the closure, this notion is useful in the study of mp-kernels.

Another associated digraph for an m-colored digraph D is the *color-class digraph* $\mathscr{C}_C(D)$ whose vertices are the colors represented in the arcs of D, and $(i, j) \in A(\mathscr{C}_C(D))$ if and only if there exist two arcs (u,v) and (v,w) in D such that (u,v) has color i and (v,w) has color j. From the definition of the color-class digraph it follows that $\mathscr{C}_C(D)$ can have loops.

Clearly, from the definition of closure and color-class digraph we have that $\mathscr{C}_C(D) = \mathscr{C}_C(\mathfrak{C}(D))$.

In [14] Galeana-Sánchez introduced the concept of color-class digraph and with this new associated digraph she showed more conditions which ensure the existence of mp-kernels. In [14] her main result was the following.

Theorem 5 ([14])**.** *Let D be an m-colored finite digraph and $\mathscr{C}_C(D)$ its color-class digraph. If $\mathscr{C}_C(D)$ is a bipartite digraph, then D has an mp-kernel.*

Due to the difficulty of finding mp-kernels in m-colored digraphs, sufficient conditions for the existence of mp-kernels have been obtained mainly by adding a condition on the monochromaticity or quasi-monochromaticity of small subdigraphs like cycles, paths, small sized subtournaments, vertex neighborhoods, and so on. See for example [12, 13, 16, 21].

Theorem 4 ensures that every finite 1-colored or 2-colored digraph has an mp-kernel but for $m \geq 3$ we can find an m-colored digraph without an mp-kernel if we consider the cycle of length 3 whose arcs have different colors. We are going to work with the structure of the digraph $\mathscr{C}_C(D)$ and with the closure of some m'-colored subdigraphs of D (with $m'<m$) in order to guarantee the existence of

at least an mp-kernel. That is to say, we work toward finding mp-kernels using closures of subdigraphs induced by arcs on proper subsets of the colors.

In this work we introduce the following concepts for an m-colored digraph: if $B \subset \{1, \ldots, m\}$ (the symbol $\subset$ denotes a proper subset), then let D_B denote the subdigraph of D induced by the arcs whose colors are in B. D is said to be a *quasi color-perfect digraph* if for every $B \subset V(\mathscr{C}_C(D))$ we have that $\mathfrak{C}(D_B)$ is a kernel-perfect digraph. If D has a kernel by monochromatic paths and it is a quasi color-perfect digraph, then D is said to be a *color-perfect digraph*.

The main result is the following: Let D be a finite m-colored digraph and $\mathscr{C}_C(D)$ its color-class digraph such that $\mathscr{C}_C(D)$ is not strong. If D is a quasi color-perfect digraph, then D has an mp-kernel.

On the other hand, in [21], Sands *et al.* raised the following problem:

Problem 6. Let T be a 3-colored finite tournament such that no cycle of length 3 is colored with three different colors. Must T have an mp-kernel?

In [16] Minggang proved that if T is an m-colored tournament which does not contain C_3 or T_3 (the cycle of order 3, whose arcs are colored with three different colors and the transitive tournament of order 3, whose arcs are colored with three different colors, respectively), then T has an mp-kernel. He also proved that his theorem is the best possible for $m \geq 5$; that is to say, for each $m \geq 5$ he showed an m-colored tournament without C_3 which has no mp-kernel. For $m = 4$, [15] Galeana-Sánchez and Rojas-Monroy proved that if T is a 4-colored tournament without C_3, then T may not have an mp-kernel. The question for $m = 3$ (the problem raised by Sands *et al.*) is still open.

In this paper we will show that if, in Problem 6, the hypothesis on T is on $\mathscr{C}_C(T)$; that is to say, $\mathscr{C}_C(T)$ does not contain cycles of length three, then T has an mp-kernel.

The following results will be useful in this paper.

Theorem 7 [2]**.** *Let D be a finite strong digraph. D is a bipartite digraph if and only if each cycle of D has an even length.*

Theorem 8 [18]**.** *Let D be a finite digraph. If D is a bipartite digraph, then D has a kernel.*

Theorem 9 [14]**.** *Let D be an m-colored finite digraph and $\mathscr{C}_C(D)$ its color-class digraph. If D is strong, then $\mathscr{C}_C(D)$ is strong.*

For the rest of the work D is a finite digraph without loops.

2. Main Results

Recall that for $B \subset \{1, \ldots, m\}$, D_B denotes the subdigraph of D induced by the arcs whose colors are in B. Notice that D_B is an arc-colored digraph whose arc-coloring is the m-coloring of D restricted to the arcs of D_B.

In the proofs we will use the following facts for an m-colored digraph.

1. $\mathfrak{C}(\mathfrak{C}(D)) = \mathfrak{C}(D)$,
2. N is an mp-kernel of D if and only if N is a kernel of $\mathfrak{C}(D)$,
3. $\mathscr{C}_C(D) = \mathscr{C}_C(\mathfrak{C}(D))$.

Theorem 10. *Let D be an m-colored digraph and $\mathscr{C}_C(D)$ its color-class digraph such that $\mathscr{C}_C(D)$ is not strong. If D is a quasi color-perfect digraph, then D has an mp-kernel* (*and thus D is a color-perfect digraph*).

Proof. Let H be a terminal strong component of $\mathscr{C}_C(D)$ (notice that $V(H) \subset V(\mathscr{C}_C(D))$ because $\mathscr{C}_C(D)$ is not strong). The digraph $D_{V(H)}$ will be denoted by D'. Since D is a quasi color-perfect digraph and $V(H) \subset V(\mathscr{C}_C(D))$, it follows that $\mathfrak{C}(D')$ has a kernel, say N_1 (which is an mp-kernel of D'). Consider the set $N_2 = (V(D) \setminus V(D'))$.

If $N_1 \cup N_2$ is an mp-independent set in D, then $N_1 \cup N_2$ is an mp-kernel of D. Therefore, suppose that $N_1 \cup N_2$ is not an mp-independent set in D.

We are going to use the following notation.

Let $\{u, v\} \subseteq V(D)$ and $S \subseteq V(D)$. We will write: $u \rightsquigarrow_i v$ if there exists a uv-monochromatic path with color i in D; $u \rightsquigarrow_{mono} v$ if there exists a uv-monochromatic path in D; $u \rightsquigarrow_{mono} S$ if there exists a uS-monochromatic path in D; $u \not\rightsquigarrow_i v$ is the denial of $u \rightsquigarrow_i v$; $u \not\rightsquigarrow_{mono} S$ is the denial of $u \rightsquigarrow_{mono} S$.

In order to prove Theorem 10 consider the following claims.

Claim 1. $V(D') \setminus N_1 \neq \emptyset$. Since $V(H) \neq \emptyset$, we have that $A(D') \neq \emptyset$. So, $V(D')$ is not an independent set in $\mathfrak{C}(D')$, which implies that $V(D') \setminus N_1 \neq \emptyset$.

Claim 2. If u and v are two different vertices in $N_1 \cup N_2$ such that $u \rightsquigarrow_i v$ for some $i \in V(\mathscr{C}_C(D))$, then $i \in V(\mathscr{C}_C(D)) \setminus V(H)$.

Proof. Consider two cases on u and v.

Case 1. $\{u, v\} \subseteq N_1$. Since N_1 is an independent set in $\mathfrak{C}(D')$, from the definition of D' it follows that there exist no monochromatic paths with color k from u to v in D' for each $k \in V(H)$. So, $i \in V(\mathscr{C}_C(D)) \setminus V(H)$.

Case 2. $\{u, v\} \cap N_2 \neq \emptyset$. If $u \in N_2$, then it follows that there exists no $w \in V(D)$ such that $(u, w) \in A(D)$ and $c(u, w) \in V(H)$ (because $u \notin V(D')$). So, $i \in V(\mathscr{C}_C(D)) \setminus V(H)$. Analogously if $v \in N_2$ we can prove that $i \in V(\mathscr{C}_C(D)) \setminus V(H)$. □

Claim 3. If u and v are two different vertices in $N_1 \cup N_2$ such that $u \rightsquigarrow_i v$ for some $i \in V(\mathscr{C}_C(D)) \setminus V(H)$, then $w \not\rightsquigarrow_j u$ for every $j \in V(H)$ and for every $w \in V(D) \setminus (N_1 \cup N_2)$.

Proof. Proceeding by contradiction, suppose that there exist $w \in V(D) \setminus (N_1 \cup N_2)$ and $j \in V(H)$ such that $w \rightsquigarrow_j u$. Then we have that there exists $z_1 \in V(D)$ such that $(z_1, u) \in A(D)$ and $c(z_1, u) = j$. On the other hand, since $u \rightsquigarrow_i v$, it follows that there exists $z_2 \in V(D)$ such that $(u, z_2) \in A(D)$ and $c(u, z_2) = i$. Thus, from the definition of $\mathscr{C}_C(D)$ we have that $(j, i) \in A(\mathscr{C}_C(D))$, contradicting that H is a terminal strong component of $\mathscr{C}_C(D)$. □

Let $T = \{w \in (N_1 \cup N_2) \mid w \rightsquigarrow_{mono} z \text{ for some } z \in (N_1 \cup N_2)\}$ and $N_3 = (N_1 \cup N_2) \setminus T$. Notice that $T \neq \emptyset$, because $N_1 \cup N_2$ is not an mp-independent set in D.

Claim 4. $N_3 \neq \emptyset$.

Proof. Proceeding by contradiction, suppose that $N_3 = \emptyset$. Then in particular, let z be a vertex in $V(D') \setminus N_1$ (such vertex exists by Claim 1). Since N_1 is an mp-kernel of D', it follows that there exists $u \in N_1$ such that $z \rightsquigarrow_i u$ for some $i \in V(H)$ (by the definition of D'). On the other hand, since $T = (N_1 \cup N_2)$ (because we are supposing that $N_3 = \emptyset$), it follows from the definition of T that for $u \in N_1$ there exists $v \in (N_1 \cup N_2)$ such that $u \rightsquigarrow_j v$ for some $j \in V(\mathscr{C}_C(D)) \setminus V(H)$ (by Claim 2), which contradicts Claim 3. □

Claim 5. N_3 is an mp-independent set in D.

Proof. It follows from the definition of N_3. □

Claim 6. For each $w \in V(D') \setminus N_1$ there exists $z \in N_3$ such that $w \rightsquigarrow_{mono} z$.

Proof. Let $w \in V(D') \setminus N_1$. Since N_1 is an mp-kernel of D', it follows that there exists $z \in N_1$ such that $w \rightsquigarrow_i z$ for some $i \in V(H)$. On the other hand, by the definition of T and Claim 3 we have that $z \notin T$. So, $z \in N_3$. □

Let $T' = \{z \in T \mid z \rightsquigarrow_{mono} N_3\}$. If $T \setminus T' = \emptyset$, then it follows from Claims 5, 6 and the definition of T' that N_3 is an mp-kernel of D. Therefore, suppose that $T \setminus T' \neq \emptyset$. The next claim follows from the construction of N_3 and the definition of $T \setminus T'$.

Claim 7. There exist no monochromatic paths between N_3 and $T \setminus T'$ in D. Consider the digraph $\mathfrak{C}(D)[T \setminus T']$ (the subdigraph of $\mathfrak{C}(D)$ induced by the set $T \setminus T'$). Notice that $\mathfrak{C}(D)[T \setminus T']$ is an arc-colored subdigraph of $\mathfrak{C}(D)$ whose arc coloring is the m coloring of $\mathfrak{C}(D)$ restricted to the arcs of $\mathfrak{C}(D)[T \setminus T']$.

Consider the following claims on $\mathfrak{C}(D)[T \setminus T']$.

Claim 8. $c_{\mathfrak{C}(D)}(u, v) \in V(\mathscr{C}_C(D)) \setminus V(H)$ for each $(u, v) \in A(\mathfrak{C}(D)[T \setminus T'])$ ($c_{\mathfrak{C}(D)}(u, v)$ denotes the color of the arc (u, v) in $\mathfrak{C}(D)$).

Proof. Let (u,v) be an arc of $A(\mathfrak{C}(D)[T \setminus T'])$ and suppose that $c_{\mathfrak{C}(D)}(u,v) = j$ for some $j \in V(\mathscr{C}_C(D))$. Then, it follows from the definition of $\mathfrak{C}(D)$ that $u \rightsquigarrow_j v$, which implies that $j \in V(\mathscr{C}_C(D)) \setminus V(H)$ (by Claim 2). □

Let $B = V(\mathscr{C}_C(D)) \setminus V(H)$. The digraph $\mathfrak{C}(D_B)$ will be denoted by D''.

Claim 9. $\mathfrak{C}(D)[T \setminus T']$ is an induced subdigraph of D''.

Proof. We will first prove that $(T \setminus T') \subseteq V(D'')$. Let $w \in (T \setminus T')$. Then it follows that there exists $x \in N_1 \cup N_2$ such that $w \rightsquigarrow_i x$ for some $i \in V(\mathscr{C}_C(D)) \setminus V(H)$ (by definition of T and Claim 2). So, there exists $y \in V(D)$ such that $(w,y) \in A(D)$ and $c(w,y) = i$, which implies that $(w,y) \in \{(u,v) \in A(D) \mid c(u,v) \in V(\mathscr{C}_C(D)) \setminus V(H)\}$. Therefore $w \in V(D'')$.

We are now going to prove that $(u,v) \in A(\mathfrak{C}(D)[T \setminus T'])$ if and only if $(u,v) \in A(D'')$, with $\{u,v\} \subseteq T \setminus T'$.

(*necessity*) Let (u,v) be an arc of $A(\mathfrak{C}(D)[T \setminus T'])$. Then it follows from Claim 8 that $c_{\mathfrak{C}(D)}(u,v) = j$ for some $j \in V(\mathscr{C}_C(D)) \setminus V(H)$. This implies that there exists a uv-monochromatic path with color j in D, say P. Since P is contained in D_B, it follows that $(u,v) \in A(D'')$.

(*sufficiency*) Let (u,v) be an arc of $A(D'')$. Since D'' is a subdigraph of $\mathfrak{C}(D)$, it follows that $(u,v) \in A(\mathfrak{C}(D))$, which implies that $(u,v) \in A(\mathfrak{C}(D)[T \setminus T'])$. □

Since $\mathfrak{C}(D)[T \setminus T']$ is an induced subdigraph of D'' and D'' is a kernel perfect digraph, it follows that $\mathfrak{C}(D)[T \setminus T']$ has a kernel, say N_4.

Claim 10. N_4 is an mp-independent set in D.

Proof. Proceeding by contradiction, suppose that there exist x and y in N_4, with $x \neq y$, such that $x \rightsquigarrow_j y$ for some $j \in V(\mathscr{C}_C(D)) \setminus V(H)$ (by Claim 2). Then, it follows that $(x,y) \in A(\mathfrak{C}(D))$. Since $\{x,y\} \subseteq N_4 \subseteq T \setminus T'$, it follows that $(x,y) \in A(\mathfrak{C}(D)[T \setminus T'])$, which contradicts that N_4 is an independent set in $\mathfrak{C}(D)[T \setminus T']$. □

Claim 11. $N_3 \cup N_4$ is an mp-kernel of D.

Proof. Since $N_3 \cup N_4$ is an mp-independent set in D (by Claims 5, 7 and 10), it remains to prove that $N_3 \cup N_4$ is an mp-absorbent set in D. Let $x \in V(D) \setminus (N_3 \cup N_4)$.

We have the following cases: $x \in V(D') \setminus N_1, x \in T'$ or $x \in (T \setminus T') \setminus N_4$. Then, it follows from Claim 6, the definition of T' and because N_4 is a kernel of $\mathfrak{C}(D)[T \setminus T']$ that there exists $w \in (N_3 \cup N_4)$ such that $x \rightsquigarrow_{mono} w$. □

Therefore, D has an mp-kernel. So, D is a color-perfect digraph. ■

Lemma 11. *Let D be an m-colored digraph and $\mathscr{C}_C(D)$ its color-class digraph. If every subdigraph of $\mathfrak{C}(D_{V(H)})$ has a kernel for every strong component H of $\mathscr{C}_C(D)$, then D is a color-perfect digraph.*

Proof. Consider two cases on $\mathscr{C}_C(D)$.

Case 1. $\mathscr{C}_C(D)$ is strong. Notice that in this case it follows from the hypothesis that every subdigraph of $\mathfrak{C}(D)$ has a kernel. Therefore, in particular we have that D has an mp-kernel. We will prove that D is a quasi color-perfect digraph.

Let $B \subset V(\mathscr{C}_C(D))$. We are going to prove that $\mathfrak{C}(D_B)$ is a kernel-perfect digraph. The digraph $\mathfrak{C}(D_B)$ will be denoted by D'. Let G be an induced subdigraph of D'. Since every subdigraph of $\mathfrak{C}(D)$ has a kernel, it follows that G has a kernel (because D' is a subdigraph of $\mathfrak{C}(D)$). Thus, D' is a kernel-perfect digraph. Therefore, D is a quasi color-perfect digraph, which implies that D is a color-perfect digraph.

Case 2. $\mathscr{C}_C(D)$ is not strong. We will prove that D is a quasi color-perfect digraph. Let $B \subset V(\mathscr{C}_C(D))$. We are going to prove that $\mathfrak{C}(D_B)$ is a kernel-perfect digraph. The digraph $\mathfrak{C}(D_B)$ will be denoted by D'.

Proceeding by contradiction, suppose that D' is not a kernel-perfect digraph. Then D' contains a critical kernel imperfect digraph, say G (by Theorem 3). Since G is strong (by Theorem 2), it follows that $\mathscr{C}_C(G)$ is strong (by Theorem 9). On the other hand, since $\mathscr{C}_C(G)$ is a subdigraph of $\mathscr{C}_C(D')$ and $\mathscr{C}_C(D')$ is a subdigraph of $\mathscr{C}_C(\mathfrak{C}(D))$ (because G is a subdigraph of D' and D' is a subdigraph of $\mathfrak{C}(D)$), it follows that $\mathscr{C}_C(G)$ is a subdigraph of $\mathscr{C}_C(D)$, (recall that $\mathscr{C}_C(D) = \mathscr{C}_C(\mathfrak{C}(D))$). So, we have that there exists a strong component H of $\mathscr{C}_C(D)$ such that $\mathscr{C}_C(G)$ is a subdigraph of H.

Since every subdigraph of $\mathfrak{C}(D_{V(H)})$ has a kernel and G is a subdigraph of $\mathfrak{C}(D_{V(H)})$, it follows that G has a kernel, which is not possible because G is a critical kernel imperfect digraph. Thus, D' is a kernel-perfect digraph. Therefore, D is a quasi color-perfect digraph, which implies that D is a color-perfect digraph (by Theorem 10). ■

Note 12. The assumption "$\mathscr{C}_C(D)$ is not strong" in Theorem 10 is necessary. For example, consider the cycle C_3 of length three whose arcs have different colors. Clearly, this digraph has no mp-kernel, its color-class digraph is strong and C_3 is a quasi color-perfect digraph.

Note 13. The assumption "every subdigraph of $\mathfrak{C}(D_{V(H)})$ has a kernel for every strong component H of $\mathscr{C}_C(D)$" in Lemma 11 is necessary as the example in Figure 1 shows.

Lemma 14. *Let D be a digraph. Every subdigraph of D has a kernel if and only if every strong component of D is either a bipartite digraph or trivial.*

Proof. (*necessity*) Let D' be a strong component of D. Suppose that D' is not a trivial digraph. Since every odd cycle has no kernel, it follows from the

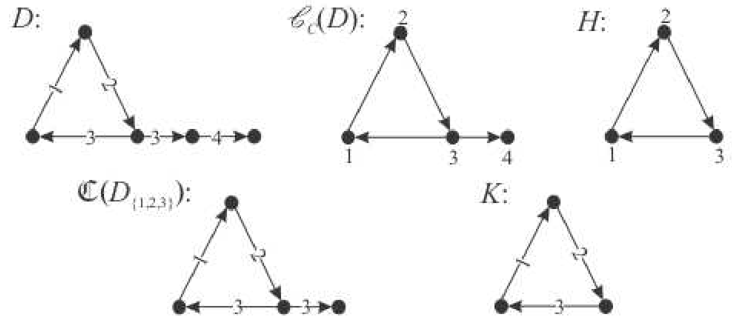

Figure 1. H is a strong component of $\mathscr{C}_C(D)$, K is a subdigraph of $\mathfrak{C}(D_{\{1,2,3\}})$ which has no kernel and D has no mp-kernel.

hypothesis that D', in particular, has no odd cycles, which implies that D' is a bipartite digraph (by Theorem 7).

(*sufficiency*) Let H be a subdigraph of D. We are going to prove that H has a kernel.

Proceeding by contradiction, suppose that H has no kernel. Then H contains a critical kernel imperfect digraph (by Theorem 3), say G. Since G is a strong digraph (by Theorem 2), it follows that there exists a strong component of D, say G', such that G is a subdigraph of G'. Since G' is a bipartite digraph, it follows that G is a bipartite digraph. Thus, G has a kernel (by Theorem 8), contradicting that G is a critical kernel imperfect digraph. Therefore, H has a kernel. ■

Corollary 15. *Let D be an m-colored digraph and $\mathscr{C}_C(D)$ its color-class digraph. If every strong component of $\mathfrak{C}(D_{V(H)})$ is a bipartite digraph for every strong component H of $\mathscr{C}_C(D)$, then D is a color-perfect digraph.*

Proof. It follows from Lemma 14 that every subdigraph of $\mathfrak{C}(D_{V(H)})$ has a kernel, which implies that D is a color-perfect digraph (by Lemma 11). ■

The following corollary will be useful in providing an alternative proof of Richardson's Theorem.

Corollary 16. *Let D be an m-colored digraph and $\mathscr{C}_C(D)$ its color-class digraph. If every strong component of $\mathscr{C}_C(D)$ is either a bipartite digraph or trivial, then D is a color-perfect digraph.*

Proof. Let H be a strong component of $\mathscr{C}_C(D)$. We will prove that every subdigraph of $\mathfrak{C}(D_{V(H)})$ has a kernel.

If H is trivial, then clearly every subdigraph of $\mathfrak{C}(D_{V(H)})$ has a kernel (because $D_{V(H)}$ is a 1-colored digraph). Suppose that $|V(H)| \geq 2$ and H is a bipartite digraph. The digraph $\mathfrak{C}(D_{V(H)})$ will be denoted by D'. Let G be a subdigraph of D'. We are going to prove that G has a kernel.

If G is 1-colored, then $\mathfrak{C}(G)$ has a kernel. Since G is an induced subdigraph of D', it follows that $\mathfrak{C}(G) = G$. Thus, G has a kernel.

Suppose that G is m'-colored, with $m' \geq 2$. Since $\mathscr{C}_C(\mathfrak{C}(D_{V(H)})) = \mathscr{C}_C(D_{\mathrm{V}(H)})$ and $\mathscr{C}_C(D_{V(H)}) = H$, it follows that $\mathscr{C}_C(G)$ is a bipartite digraph (because $\mathscr{C}_C(G)$ is a subdigraph of $\mathscr{C}_C(D') = H$ and H is a bipartite digraph), which implies that G has an mp-kernel (by Theorem 5). Thus, G has a kernel (because $\mathfrak{C}(G) = G$).

Therefore, it follows from Lemma 11 that D is a color-perfect digraph. ■

Theorem 17 (Richardson)**.** *Let D be a digraph. If D has no odd cycles, then D has a kernel.*

Proof. Let D' be the edge-colored digraph obtained from D by assigning a different color to each arc of D, $L(D')$ the line digraph of D' and $\mathfrak{C}(D)$ the closure of D. In this case, since $\mathscr{C}_C(D') = L(D')$ and $L(D')$ has no odd cycles (because D' has no odd cycles), it follows that every strong component of $\mathscr{C}_C(D')$ is either a bipartite digraph or trivial. So, it follows from Corollary 16 that, in particular, D' has an mp-kernel, which is a kernel of D. ■

Corollary 18. *Let D be an m-colored digraph. If $\mathscr{C}_C(D)$ has no odd cycles of length at least* 3, *then D has an mp-kernel.*

Proof. It follows from Corollary 16 that D has an mp-kernel. ■

Corollary 19. *Let D be a* 3*-colored digraph and $\mathscr{C}_C(D)$ its color-class digraph. If $\mathscr{C}_C(D)$ is not strong, then D has an mp-kernel.*

Proof. Clearly, D is a quasi color-perfect digraph, which implies that D has an mp-kernel (by Theorem 10). ■

The following Corollary shows that Sands, Sauer and Woodrow's question is true if $\mathscr{C}_C(T)$ has no cycles of length 3.

Corollary 20. *Let T be a* 3*-colored tournament and $\mathscr{C}_C(T)$ its color-class digraph. If $\mathscr{C}_C(T)$ has no cycles of length* 3, *then T has an mp-kernel.*

Proof. It follows from Corollary 19. ■

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