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BOUNDS ON THE DISJUNCTIVE TOTAL DOMINATION NUMBER OF A TREE

Michael A. Henning[1]

Department of Pure and Applied Mathematics
University of Johannesburg
Auckland Park, 2006, South Africa
e-mail: mahenning@uj.ac.za

AND

Viroshan Naicker[2]

Department of Pure and Applied Mathematics
University of Johannesburg
Auckland Park, 2006, South Africa
and
Department of Mathematics
Rhodes University
Grahamstown, 6140 South Africa
e-mail: v.naicker@ru.ac.za

Abstract

Let G be a graph with no isolated vertex. In this paper, we study a parameter that is a relaxation of arguably the most important domination parameter, namely the total domination number, $\gamma_t(G)$. A set S of vertices in G is a disjunctive total dominating set of G if every vertex is adjacent to a vertex of S or has at least two vertices in S at distance 2 from it. The disjunctive total domination number, $\gamma_t^d(G)$, is the minimum cardinality of such a set. We observe that $\gamma_t^d(G) \leq \gamma_t(G)$. A leaf of G is a vertex of degree 1, while a support vertex of G is a vertex adjacent to a leaf. We show that if T is a tree of order n with ℓ leaves and s support vertices, then $2(n-\ell+3)/5 \leq \gamma_t^d(T) \leq (n+s-1)/2$ and we characterize the families of trees which attain these bounds. For every tree T, we show have $\gamma_t(T)/\gamma_t^d(T) < 2$ and this bound is asymptotically tight.

Keywords: total domination, disjunctive total domination, trees.

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1. Introduction

A *total dominating set*, abbreviated a TD-set, of a graph G with no isolated vertex is a set S of vertices of G such that every vertex in $V(G)$ is adjacent to at least one vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. Total domination is now well studied in graph theory. The literature on the subject of total domination in graphs has been surveyed and detailed in the recent book [13]. A survey of total domination in graphs can also be found in [9].

In this paper we continue the study of disjunctive total domination in graphs, a parameter introduced and motivated by the authors in [11, 12] as a relaxation of total domination in graphs. As remarked in [11, 12], given the sheer scale of modern networks (see [6]), many existing domination type structures are expensive to implement. Variations on the theme of dominating and total dominating sets studied to date tend to focus on adding restrictions which in turn raises their implementation costs. As an alternative route a relaxation of the domination number, called *disjunctive domination*, was proposed and studied by Goddard *et al.* [8]. This concept was recently extended in [11] to a relaxation of total domination, called *disjunctive total domination*, which allows for greater flexibility in modeling networks where one trades off redundancy and backup capability with resource optimization.

A set S of vertices in G is a *disjunctive total dominating set*, abbreviated DTD-set, of G if every vertex is adjacent to a vertex of S or has at least two vertices in S at distance 2 from it. For example, the set of seven darkened vertices in the graph G shown in Figure 1 is a DTD-set of G.

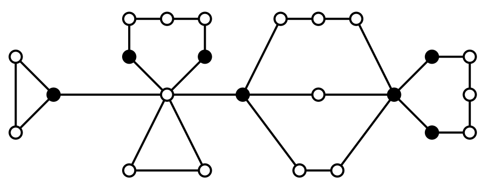


Figure 1. A graph G with $\gamma_t^d(G) = 7$.

We say that a vertex $v \in V$ is *disjunctively totally dominated*, abbreviated DT-*dominated*, by a set S, if v has a neighbor in S or if v is at distance 2 from at least two vertices of S. Further, if v has a neighbor in S, we say S *totally dominates* the vertex v, while if v is at distance 2 from at least two vertices of S, we say S *disjunctively dominates* the vertex v. The *disjunctive total domination number*, $\gamma_t^d(G)$, is the minimum cardinality of a DTD-set in G. A DTD-set of cardinality $\gamma_t^d(G)$ is called a $\gamma_t^d(G)$-set.

1.1. Notation

Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ of order $n(G) = |V|$ and edge set $E = E(G)$ of size $m(G) = |E|$, and let v be a vertex in V. We denote the *degree* of v in G by $d_G(v)$. A *path* on n vertices is denoted by P_n. For two vertices u and v in a connected graph G, the distance $d_G(u, v)$ between u and v is the length of the shortest (u, v)-path in G. The maximum distance among all pairs of vertices of G is the *diameter* of G, denoted by $\text{diam}(G)$. A graph is *nontrivial* if $n(G) \geq 2$. The subgraph of G induced by a set S of vertices in G is denoted by $G[S]$.

A *rooted tree* T distinguishes one vertex r called the root. For each vertex $v \neq r$ of T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. A *descendant* of v is a vertex $u \neq v$ such that the unique (r, u)-path contains v. Thus, every child of v is a descendant of v. Let $D(v)$ denote the set of descendants of v, and define $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted T_v. A *leaf* of T is a vertex of degree 1, while a *support* vertex of T is adjacent to a leaf. A *double star* is a tree with exactly two vertices that are not leaves.

By a *weak partition* of a set we mean a partition of the set in which some of the subsets may be empty. For our purposes we define a *labeling* of a tree T as a weak partition $S = (S_A, S_B, S_C)$ of $V(T)$. We will refer to the pair (T, S) as a *labeled tree*. The *label* or *status* of a vertex v, denoted $\text{sta}(v)$, is the letter $x \in \{A, B, C\}$ such that $v \in S_x$.

The *open neighborhood* of a vertex v is the set $N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is $N_G[v] = \{v\} \cup N_G(v)$. If the graph is clear from the context, we write n, m, $d(v)$, $d(u, v)$, $N(v)$ and $N[v]$ rather than $n(G)$, $m(G)$, $d_G(v)$, $d_G(u, v)$, $N_G(v)$, and $N_G[v]$, respectively. We use the standard notation $[k] = \{1, 2, \ldots, k\}$.

1.2. Known results

Every TD-set is a DTD-set, implying the following observation.

Observation 1 ([11])**.** *For every graph G with no isolated vertex, $\gamma_t^d(G) \leq \gamma_t(G)$.*

The following upper bounds on the total domination number of a graph G in terms of its order n and small minimum degree $\delta(G)$ are given by Theorem 2.

Theorem 2. *Let G be a connected graph of order n. Then the following holds.*

(a) ([2]) *If $n \geq 3$, then $\gamma_t(G) \leq 2n/3$.*
(b) ([10]) *If $n \geq 11$, then $\gamma_t(G) \leq 4n/7$.*
(c) ([1, 3, 14]) *$\gamma_t(G) \leq n/2$.*

If we restrict our attention to the class of trees, then the following lower and upper bounds on the total domination number in terms of the number of leaves and support vertices are shown in [4, 5].

Theorem 3 [4, 5]**.** *If T is a tree with $n \geq 3$ vertices, ℓ leaves and s support vertices, then*

$$\frac{n - \ell + 2}{2} \leq \gamma_t(T) \leq \frac{n + s}{2},$$

and these bounds are tight.

The authors show in [11] that for a connected graph G the upper bound on $\gamma_t^d(G)$ implied by Observation 1 and Theorem 2(a) of two-thirds the order of the graph may be improved ever-so-slightly. However, if the graph G is restricted to minimum degree at least 2, then it is shown in [12] that the upper bound on $\gamma_t^d(G)$ implied by Observation 1 and Theorem 2(b) may be improved significantly to one-half the order of the graph.

Theorem 4. *Let G be a connected graph of order n. Then the following holds.*

(a) ([11]) *If $n \geq 8$, then $\gamma_t(G) \leq 2(n-1)/3$.*
(b) ([12]) *If $n \geq 13$, then $\gamma_t(G) \leq (n-1)/2$.*

The disjunctive total domination number of a path P_n on n vertices is established in [12].

Proposition 5 [12]**.** *For $n \geq 2$, $\gamma_t^d(P_n) = \lceil 2(n+1)/5 \rceil + 1$ if $n \equiv 1 (\bmod 5)$, and $\gamma_t^d(P_n) = \lceil 2(n+1)/5 \rceil$ otherwise.*

1.3. Special families of trees

We introduce here three special families of trees, and one special tree on seven vertices.

The Family $\mathcal{T}$. Let $\mathcal{T}$ be the minimum family of labeled trees that: (i) contains (P_4, S_0^*) where S_0^* is the labeling that assigns to both support vertices of P_4 status A and both leaves status B; and (ii) is closed under the two operations $\mathcal{O}_1$ and $\mathcal{O}_2$ that are defined below which extend the tree T' to a tree T by attaching a tree to the vertex $v \in V(T')$, called the *attacher* of T'. We called the edge that joins v to the vertex of the attached tree, the *attached edge.* A tree in the family $\mathcal{T}$ is shown in Figure 2, where the darkened vertices have status A, the leaves have status B, and the six vertices of degree 2 have status C.

- **Operation $\mathcal{O}_1$.** Let v be a vertex in $T' \in \mathcal{T}$ with $\text{sta}(v) = A$. The resulting tree obtained from T' by adding a new vertex z_1 and the edge vz_1 and letting $\text{sta}(z_1) = B$ is in $\mathcal{T}$.

- **Operation $\mathcal{O}_2$.** Let v be a vertex in $T' \in \mathcal{T}$ with $\text{sta}(v) = B$. The resulting tree obtained from T' by adding a path $z_1z_2z_3z_4z_5$ and the edge vz_1 and letting $\text{sta}(v) = \text{sta}(z_1) = \text{sta}(z_2) = C$, $\text{sta}(z_5) = B$ and $\text{sta}(z_3) = \text{sta}(z_4) = A$ is in $\mathcal{T}$.

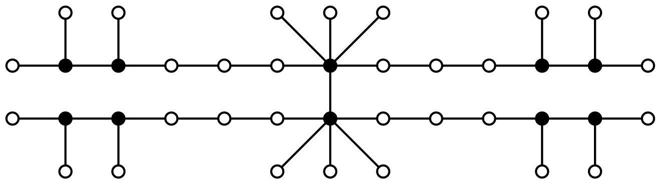
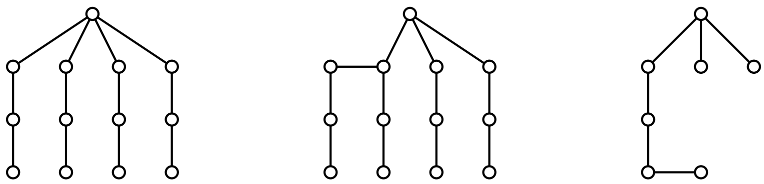


Figure 2. A tree in the family $\mathcal{T}$.

The Families $\mathcal{F}$ and $\mathcal{H}$. For $k \geq 2$, let H_k be the tree obtained from a star $K_{1,k}$ by subdividing every edge exactly twice and let $\mathcal{H}$ be the family of all such trees H_k. We note that $P_7 = H_2 \in \mathcal{H}$. For $k \geq 3$, let F_k be the tree obtained from H_k by deleting an edge uv incident with the central vertex v of H_k and adding the edge uw for some neighbor w of v different from u, and let $\mathcal{F}$ be the family of all such trees F_k. Let T^* be the tree obtained from a star $K_{1,3}$ by subdividing one edge three times, and so T^* has order 7. The trees $H_4 \in \mathcal{H}$, $F_4 \in \mathcal{F}$ and T^* are illustrated in Figure 3(a), 3(b) and 3(c), respectively.



(a) H_4 (b) F_4 (c) T^*

Figure 3. The trees H_4, F_4 and T^*.

2. Main Results

Our aims in this paper are threefold. Firstly, we prove a tight lower bound for the disjunctive total domination number of a tree in terms of its order and the number of leaves in the tree. Further, we provide a constructive characterization of trees that achieve equality in this bound. The key to the constructive characterization is to find a labeling of the vertices that gives a minimum disjunctive total dominating set. In particular, we have the following result, a proof of which is given in Section 4.

Theorem 6. *If T is a nontrivial tree of order n with ℓ leaves, then $\gamma_t^d(T) \geq 2(n-\ell+3)/5$, with equality if and only if $T \in \mathcal{T}$.*

Secondly, we prove a tight upper bound for the disjunctive total domination number of a tree in terms of its order and the number of support vertices in the tree. A proof of Theorem 7 is given in Section 5.

Theorem 7. *If T is a tree of order $n \geq 4$ with s support vertices and $T \neq P_6$, then $\gamma_t^d(T) \leq (n+s-1)/2$, with equality if and only if $T \in \mathcal{F} \cup \mathcal{H} \cup \{P_5, P_{11}, K_{1,3}, T^*\}$.*

As an immediate consequence of Theorems 6 and 7, we have the following result, which shows that if we restrict ourselves to the class of trees, then an analogue to Theorem 3 may be proved for the disjunctive total domination number.

Corollary 8. *If T is a tree of order $n \geq 7$ with ℓ leaves and s support vertices, then*

$$\frac{2(n-\ell+3)}{5} \leq \gamma_t^d(T) \leq \frac{n+s-1}{2},$$

and these bounds are tight.

For $k \geq 3$, if G is obtained from a complete bipartite graph $K_{2,k}$ by attaching a pendant edge to each vertex of degree 2, then $\gamma_t(G) = k+1$ and $\gamma_t^d(G) = 3$. Moreover, if G is obtained from a complete graph K_k by attaching a pendant edge to each vertex in the given complete graph, then $\gamma_t(G) = k$ and $\gamma_t^d(G) = 2$. These examples imply that the ratio $\gamma_t(G)/\gamma_t^d(G)$ can be arbitrarily large, even when restricted to the class of bipartite graphs or claw-free graphs or chordal graphs. Unlike this somewhat negative result, we show that if G belongs to the class of trees, then the ratio $\gamma_t(G)/\gamma_t^d(G)$ is bounded. More precisely, our third aim is to prove the following result, a proof of which is presented in Section 6.

Theorem 9. *For every nontrivial tree T, the ratio $\frac{\gamma_t(T)}{\gamma_t^d(T)} < 2$, and this bound is asymptotically tight.*

3. Preliminary Observations

In this section, we present three preliminary observations.

Observation 10. *Let T be a tree. Then the following holds.*

(a) *Every TD-set in T contains all the support vertices in T.*
(b) *If $\mathrm{diam}(T) \geq 3$, then there is a $\gamma_t(T)$-set that contains no leaf of T.*

Observation 11. *If T is a tree and D is a DTD-set, then the following holds.*

(a) *Every support vertex of T is either contained in D or has at least two neighbors in D.*
(b) *If $\mathrm{diam}(T) \geq 3$, then there is a $\gamma_t^d(T)$-set which contains no leaf of T.*

The following result was first observed in [11].

Observation 12 [11]**.** *If v is a support vertex in a graph G with exactly one neighbor w that is not a leaf, then there is a $\gamma_t^d(G)$-set which contains v. Further, if $d_G(w) = 2$, then there is a $\gamma_t^d(G)$-set which contains both v and w.*

4. Proof of Theorem 6

The following observation establishes properties of trees in the family $\mathcal{T}$.

Observation 13. *If $(T, S) \in \mathcal{T}$, then (T, S) has the following properties.*

- (a) *Every support vertex in T has status A.*
- (b) *A vertex in T is a leaf if and only if it has status B in T.*
- (c) *Every vertex of status A has a neighbor of status A.*
- (d) *Every vertex of status C has degree 2 in T and is either adjacent to a vertex of status A or is at distance 2 from two vertices of status A.*
- (e) *The set S_A is a DTD-set in T.*

We are now in a position to determine the disjunctive total domination number of trees in the family $\mathcal{T}$.

Lemma 14. *If $(T, S) \in \mathcal{T}$ has order $n \geq 4$ and ℓ leaves, then $\gamma_t^d(T) = |S_A| = 2(n - \ell + 3)/5$.*

Proof. The proof is by induction on the number of operations k used to construct $(T, S) \in \mathcal{T}$. Let $(T, S) \in \mathcal{T}$, where T is a tree of order $n \geq 4$ with ℓ leaves, and (T, S) is constructed from (P_4, S_0^*) by $k \geq 0$ applications of the operations $\mathcal{O}_1$ and $\mathcal{O}_2$. If $k = 0$, then $T = P_4$ and $\gamma_t^d(T) = |S_A| = 2 = 2(n - \ell + 3)/5$. This establishes the base case. Suppose $k \geq 1$ and assume that if $(T', S') \in \mathcal{T}$, where T' is a tree of order $n' \geq 4$ with ℓ' leaves, and (T', S') is constructed using $k - 1$ applications of the operations $\mathcal{O}_1$ and $\mathcal{O}_2$, then $\gamma_t^d(T') = |S'_A| = 2(n' - \ell' + 3)/5$, where S'_A is the set of vertices labeled A in S'. Let $(T, S) \in \mathcal{T}$, where T is a tree of order $n \geq 4$ with ℓ leaves, and (T, S) is constructed by k applications of the operations $\mathcal{O}_1$ and $\mathcal{O}_2$. Let (T, S) be obtained from $(T', S') \in \mathcal{T}$ by applying either operation $\mathcal{O}_1$ or $\mathcal{O}_2$. Let T' have order $n' \geq 4$ with ℓ' leaves. Further, let $S = (S_A, S_B, S_C)$ and $S' = (S'_A, S'_B, S'_C)$. Applying the inductive hypothesis to (T', S'), the set S'_A is a $\gamma_t^d(T')$-set and $\gamma_t^d(T') = |S'_A| = 2(n' - \ell' + 3)/5$.

Suppose firstly that (T, S) is obtained from (T', S') by Operation $\mathcal{O}_1$. Let v be the attacher vertex in T' and let vz_1 be the attached edge, and so $V(T) \setminus V(T') = \{z_1\}$. In the tree T, we note that $\text{sta}(v) = A$ and $\text{sta}(z_1) = B$. In particular, we note that $S_A = S'_A$, and that $n = n' + 1$ and $\ell = \ell' + 1$. By Observation 13(e), the set S_A is a DTD-set of T, and so $\gamma_t^d(T) \leq |S_A| = |S'_A| =$

$\gamma_t^d(T')$. By Observation 11(b), there exists a $\gamma_t^d(T)$-set, D say, which contains no leaf of T. In particular, $z_1 \notin D$, implying that D is a DTD-set in T'. Thus, $\gamma_t^d(T') \le |D| = \gamma_t^d(T)$. Consequently, $\gamma_t^d(T) = \gamma_t^d(T')$. As observed earlier, $\gamma_t^d(T') = |S'_A| = 2(n' - \ell' + 3)/5$. Since $|S'_A| = |S_A|$ and $n' - \ell' = n - \ell$, this implies that $\gamma_t^d(T) = |S_A| = 2(n - \ell + 3)/5$.

Suppose secondly (T, S) is obtained from (T', S') by Operation $\mathcal{O}_2$. Let v be the attacher vertex in T' and let $z_1z_2z_3z_4z_5$ be the path added to T', where vz_1 is the attached edge. Since $\text{sta}(v) = B$ in the tree T', by Observation 13(b) the vertex v is a leaf in T' and its neighbor in T', say v', has status A in T' (which remains status A in T). Thus, the set $S'_A \cup \{z_3, z_4\}$ is a DTD-set in T, and so $\gamma_t^d(T) \le |S'_A| + 2 = \gamma_t^d(T') + 2$. By Observation 12, there exists a $\gamma_t^d(T)$-set D such that $D \cap \{z_3, z_4, z_5\} = \{z_3, z_4\}$. If $z_2 \in D$, then we can simply replace the vertex z_2 in D with the vertex v'. Hence we may choose D so that $z_2 \notin D$. Analogously, we may choose D so that $z_1 \notin D$, implying that the set $D \cap V(T')$ is a DTD-set of T' and $|D \cap V(T')| = |D| - 2$. Therefore, $\gamma_t^d(T') \le |D| - 2 = \gamma_t^d(T) - 2$. Consequently, $\gamma_t^d(T) = \gamma_t^d(T') + 2$. As observed earlier, $\gamma_t^d(T') = |S'_A| = 2(n' - \ell' + 3)/5$. Since $|S'_A| = |S_A| - 2$, this implies that $\gamma_t^d(T) = |S'_A| + 2 = |S_A|$. Further, since $n' = n - 5$ and $\ell' = \ell$, we note that $n' - \ell' = n - \ell - 5$, implying that $\gamma_t^d(T) = 2(n' - \ell' + 3)/5 + 2 = 2(n - \ell + 3)/5$. ■

We are now in a position to prove Theorem 6. Recall its statement.

Theorem 6. *If T is a nontrivial tree of order n with ℓ leaves, then $\gamma_t^d(T) \ge 2(n - \ell + 3)/5$, with equality if and only if $(T, S) \in \mathcal{T}$ for some labeling S.*

Proof. Let T be tree of order $n \ge 2$ and with ℓ leaves. If $(T, S) \in \mathcal{T}$ for some labeling S, then, by Lemma 14, $\gamma_t^d(T) = 2(n - \ell + 3)/5$. We prove the necessity by induction on $n \ge 2$. If $n \in \{2, 3\}$, then $\gamma_t^d(T) = 2 > 2(n - \ell + 3)/5$. This establishes the base cases. Suppose that $n \ge 4$ and assume that if T' is a tree of order n', where $2 \le n' < n$, and with ℓ' leaves, then $\gamma_t^d(T') \ge 2(n' - \ell' + 3)/5$ with equality only if $(T', S') \in \mathcal{T}$ for some labeling S'. Let T be a tree of order n with ℓ leaves. If $\text{diam}(T) = 2$, then T is a star and $\gamma_t^d(T) = 2 > 2(n - \ell + 3)/5$. Hence we may assume that $\text{diam}(T) \ge 3$, for otherwise the desired result follows.

Suppose that $\text{diam}(T) = 3$. Then, T is a double star. In this case, $\ell \ge 2$, $n - \ell = 2$ and $\gamma_t^d(T) = 2 = 2(n - \ell + 3)/5$. Further, there exists a labeling S of the vertices of T such that (T, S) can be obtained from (P_4, S_0^*) by $\ell - 2 \ge 0$ successive applications of Operation $\mathcal{O}_1$. Thus, $(T, S) \in \mathcal{T}$ for some labeling S. Hence we may assume that $\text{diam}(T) \ge 4$, for otherwise the desired result follows. In particular, we note that $n \ge 5$.

In what follows, we shall adopt the following notation. Our aim is to prune the tree T by deleting certain vertices of T to produce a nontrivial tree T' to which we apply the inductive hypothesis. We denote the order of such a pruned tree T' by n' and the number of leaves in T' by ℓ'. We let D' denote a $\gamma_t^d(T')$-set. Further, if $\mathrm{diam}(T') \geq 3$, then by Observation 11(b) the set D' is chosen to contain no leaf of T'. We now proceed with a series of claims that we may assume are satisfied by the tree T, for otherwise the desired result hold.

Claim A. *Every support vertex in T has exactly one leaf neighbor.*

Proof. Let v be a support vertex of T that is adjacent to $k \geq 2$ leaves. Let u be a redleaf neighbor of v and let T' be obtained from T by removing every leaf neighbor of v except for the leaf u. Then, $n' = n - k + 1$ and $\ell' = \ell - k + 1$. Since $\mathrm{diam}(T') = \mathrm{diam}(T) \geq 4$, we note that $n' \geq 5$. By Observation 11(a), either $v \in D'$ or v has at least two neighbors in D'. In both cases, the set D' is DTD-set in T, implying that $\gamma_t^d(T) \leq |D'| = \gamma_t^d(T')$. Conversely, by Observation 11(b), we can choose a $\gamma_t^d(T)$-set which contains no leaf. Such a set is a DTD-set of T', implying that $\gamma_t^d(T') \leq \gamma_t^d(T)$. Consequently, $\gamma_t^d(T) = \gamma_t^d(T') \geq 2(n' - \ell' + 3)/5 = 2(n - \ell + 3)/5$. Further, suppose that $\gamma_t^d(T) = 2(n - \ell + 3)/5$. Then, $\gamma_t^d(T') = 2(n' - \ell' + 3)/5$, implying that $(T', S') \in \mathcal{T}$ for some labeling S'. By Observation 13(a), the labeling S' assigns to the support vertex v the label A. Let S be the labeling obtained from S' by labeling each deleted leaf with the label B. Then, (T, S) can be obtained from (T', S') by repeated applications of Operation $\mathcal{O}_1$, implying that $(T, S) \in \mathcal{T}$. Hence, we may assume that every support vertex is adjacent to exactly one leaf, for otherwise the desired result follows. □

We now root the tree T at a vertex r on a longest path in T. Necessarily, r is a leaf. Let u be a vertex at maximum distance from r. Necessarily, u is a leaf. Let v be the parent of u, let w be the parent of v, let x be the parent of w, and let y be the parent of x. Since u is a vertex at maximum distance from the root r, every child of v is a leaf. By Claim A, every support vertex in T has exactly one leaf neighbor. In particular, this implies that $d_T(v) = 2$. We state this formally as follows.

Claim B. $d_T(v) = 2$.

By Observation 11(b), there exists a $\gamma_t^d(T)$-set that contains no leaf of T. Let D be such a $\gamma_t^d(T)$-set.

Claim C. $d_T(w) = 2$.

Proof. Suppose that $d_T(w) \geq 3$. Suppose that w has a child, v', different from v that is not a leaf. Then, v' is a support vertex. Since every support vertex in T has exactly one leaf neighbor, $d_T(v') = 2$. Let u' be the leaf neighbor of v'. By our choice of the $\gamma_t^d(T)$-set D, we note that $D \cap \{u, u', v, v'\} = \{v, v'\}$.

Suppose that $w \in D$. We now consider the tree $T' = T - \{u, v\}$. Then, $n' = n - 2$ and $\ell' = \ell - 1$. The set $D \setminus \{v\}$ is a DTD-set of T', implying that $\gamma_t^d(T') \le |D| - 1 = \gamma_t^d(T) - 1$. Thus, $\gamma_t^d(T) \ge \gamma_t^d(T') + 1 \ge 2(n' - \ell' + 3)/5 + 1 > 2(n - \ell + 3)/5$. Hence we may assume that $w \notin D$, for otherwise the desired result follows.

Since $w \notin D$, at least one neighbor of w, different from both v and v', belongs to D. We now consider the tree $T' = T - \{u, u', v\}$. Then, $n' = n - 3$ and $\ell' = \ell - 1$. The set $(D \setminus \{v, v'\}) \cup \{w\}$ is a DTD-set of T', implying that $\gamma_t^d(T') \le |D| - 1 = \gamma_t^d(T) - 1$. Thus, $\gamma_t^d(T) \ge \gamma_t^d(T') + 1 = 2(n' - \ell' + 3)/5 + 1 = 2(n - \ell + 1)/5 + 1 > 2(n - \ell + 3)/5$. Hence, we may assume that every child of w different from v is a leaf, for otherwise the desired result holds. Therefore, by Claim A, $d_T(w) = 3$ and the child of w different from v is a leaf. Let v' be the child of w different from v.

We now consider the tree $T' = T - v'$. In this case, $n' = n - 1$ and $\ell' = \ell - 1$. By our choice of the $\gamma_t^d(T)$-set D, we note that $D \cap \{u, v, v', w\} = \{v, w\}$. Thus the set D is a DTD-set of T', implying that $\gamma_t^d(T) = |D| \ge \gamma_t^d(T') \ge 2(n' - \ell' + 3)/5 + 1 = 2(n - \ell + 3)/5$. Further, suppose that $\gamma_t^d(T) = 2(n - \ell + 3)/5$. Then, $\gamma_t^d(T') = 2(n' - \ell' + 3)/5$, implying that $(T', S') \in \mathcal{T}$ for some labeling S'. By Observation 13(a), the labeling S' assigns to the support vertex v the label A. Further, by Observation 13(b) and 13(c), the labeling S' assigns to the vertices u and w the labels B and A, respectively. Let S be the labeling obtained from S' by labeling the deleted leaf v' with the label B. Then, (T, S) can be obtained from (T', S') by applying Operation $\mathcal{O}_1$, implying that $(T, S) \in \mathcal{T}$. Hence, we may assume that $d_T(w) = 2$, for otherwise the desired result follows. □

By Claim C, $d_T(w) = 2$. By our choice of the $\gamma_t^d(T)$-set D, we note that $D \cap \{u, v, w\} = \{v, w\}$.

Claim D. $d_T(x) = 2$.

Proof. Suppose that $d_T(x) \ge 3$. If the vertex x has a descendant $u' \ne u$ at distance 3 from it, and if $u'v'w'x$ denotes the (u', x)-path, then u' is a leaf and analogous arguments as in Claims B and C show that $d_T(v') = d_T(w') = 2$. In this case, our choice of the set D implies that D contains both v' and w'. If the vertex x has a descendant v' at distance 2 from it that is a leaf, and if $v'w'x$ denotes the (v', x)-path, then since D contains no leaf, the set D contains the vertex w'. If the vertex x has a child w' that is a leaf, then either $x \in D$ or D contains at least two vertices in $N(x)$.

We now consider the tree $T' = T - \{u, v, w\}$. Then, $n' = n - 3$ and $\ell' = \ell - 1$. Since $d_T(x) \ge 3$, our earlier observations imply that the set D contains at least one vertex in $N[x]$ different from w. If $x \in D$, then let $D^* = (D \setminus \{v, w\}) \cup \{y\}$. If $x \notin D$, then let $D^* = (D \setminus \{v, w\}) \cup \{x\}$. In both cases, the set D^* is a

DTD-set of T', implying that $\gamma_t^d(T') \le |D^*| \le |D| - 1 = \gamma_t^d(T) - 1$. Thus, $\gamma_t^d(T) \ge \gamma_t^d(T') + 1 \ge 2(n' - \ell' + 3)/5 + 1 > 2(n - \ell + 3)/5$. □

By Claim D, $d_T(x) = 2$. If $y = r$, then $T = P_5$ and T is the path $uvwxy$. In this case, $n = 5$, $\ell = 2$ and $\gamma_t^d(T) = 3 > 2(n - \ell + 3)/5$. Hence we may assume that $y \ne r$, for otherwise the desired result holds. Let z be the parent of y.

Claim E. $N[y] \cap D = \emptyset$.

Proof. Suppose that $x \in D$. Thus, $D \cap \{u, v, w, x\} = \{v, w, x\}$. In this case, we consider the tree $T' = T - \{u, v\}$. Then, $n' = n - 2$ and $\ell' = \ell$. Further, the set $D \setminus \{v\}$ is a DTD-set of T', implying that $\gamma_t^d(T') \le |D| - 1 = \gamma_t^d(T) - 1$. Thus, $\gamma_t^d(T) \ge \gamma_t^d(T') + 1 \ge 2(n' - \ell' + 3)/5 + 1 > 2(n - \ell + 3)/5$. Hence we may assume that $x \notin D$.

Suppose next that $y \in D$. Thus, $D \cap \{u, v, w, x, y\} = \{v, w, y\}$. In this case, we again consider the tree $T' = T - \{u, v\}$. As before, $n' = n - 2$ and $\ell' = \ell$. The set $(D \setminus \{v, w\}) \cup \{x\}$ is a DTD-set of T', implying that $\gamma_t^d(T') \le |D| - 1 = \gamma_t^d(T) - 1$. Thus, $\gamma_t^d(T) \ge \gamma_t^d(T') + 1 \ge 2(n' - \ell' + 3)/5 + 1 > 2(n - \ell + 3)/5$. Hence we may assume that $y \notin D$.

Suppose finally that some neighbor x' of y belongs to D. By our earlier assumptions, $D \cap \{x, y\} = \emptyset$. We now consider the tree $T' = T - \{u, v, w, x\}$. Then, $n' = n - 4$ and $\ell' \le \ell$. In this case, the set $D \setminus \{v, w\}$ is a DTD-set of T', implying that $\gamma_t^d(T') \le |D| - 2 = \gamma_t^d(T) - 2$. Thus, $\gamma_t^d(T) \ge \gamma_t^d(T') + 2 \ge 2(n' - \ell' + 3)/5 + 2 > 2(n - \ell + 3)/5$. Hence we may assume that $N[y] \cap D = \emptyset$, for otherwise the desired result follows. □

By Claim E, $N[y] \cap D = \emptyset$. In particular, this implies that $d_T(z) \ge 2$.

Claim F. $d_T(y) = 2$.

Proof. Suppose that $d_T(y) \ge 3$. If the vertex y has a descendant $u' \ne u$ at distance 4 from it, and if $u'v'w'x'y$ denotes the (u', y)-path, then u' is a leaf and analogous arguments as in Claims B, C and D show that $d_T(v') = d_T(w') = d_T(x') = 2$. By Claim E, we note that $N[y] \cap D = \emptyset$. By our choice of the set D which was chosen to contain no leaf of T, this implies that every leaf that is a descendant of y is at distance 4 from y. This in turn, together with our earlier observations, implies that every descendant of y at distance 1, 2 or 3 from y has degree 2 in T.

Let $k = d_T(y) - 1$. By supposition, $k \ge 2$. By our earlier observations, the tree T_y can be obtained from a star $K_{1,k}$ by subdividing every edge exactly three times. We now consider the tree $T' = T - V(T_y)$ of order $n' = n - 4k - 1$ with $\ell' \le \ell - k + 1$ leaves. We note that $n' - \ell' \ge n - \ell - 3k - 2$. Let D^* be the restriction of D to the tree T', and so $D^* = V(T') \cap D$. By our choice of the set D and by Claim E, we note that D^* is a DTD-set of T' and that $|D^*| = |D| - 2k$, implying that

$\gamma_t^d(T') \le |D| - 2k = \gamma_t^d(T) - 2k$. Thus, $\gamma_t^d(T) \ge \gamma_t^d(T') + 2k \ge 2(n' - \ell' + 3)/5 + 2k \ge 2(n - \ell - 3k + 1)/5 + 2k = 2(n - \ell + 2k + 1)/5 > 2(n - \ell + 3)/5$. □

By Claim F, $d_T(y) = 2$. As in the proof of Claim F, we now consider the tree $T' = T - V(T_y) = T - \{u, v, w, x, y\}$. In this case, $n' = n - 5$. If $d_T(z) = 2$, then $\ell' = \ell$, while if $d_T(z) \ge 3$, then $\ell' = \ell - 1$. In both cases, $\ell' \le \ell$, and so $n' - \ell' \ge n - \ell - 5$. As in the proof of Claim F, we let D^* be the restriction of D to the tree T', and so $D^* = D \setminus \{v, w\}$. The set D^* is a DTD-set of T', implying that $\gamma_t^d(T) \ge \gamma_t^d(T') + 2 \ge 2(n' - \ell' + 3)/5 + 2 \ge 2(n - \ell - 2)/5 + 2 = 2(n - \ell + 3)/5$. Further, suppose that $\gamma_t^d(T) = 2(n - \ell + 3)/5$. Then, $\gamma_t^d(T') = 2(n' - \ell' + 3)/5$ and $\ell' = \ell$, implying that $(T', S') \in \mathcal{T}$ for some labeling S' and that z is a leaf in T'. By Observation 13(b), the labeling S' assigns to the leaf z the label B. Let S be the labeling obtained from S' by labeling the deleted vertices y, x, w, v and u with the labels C, C, A, A, and B, respectively, and relabeling the vertex z with the label C. Then, (T, S) can be obtained from (T', S') by applying Operation $\mathcal{O}_2$ with z as the attacher of T', implying that $(T, S) \in \mathcal{T}$. This completes the proof of Theorem 6. ■

5. Proof of Theorem 7

In this section, we prove Theorem 7. Recall its statement.

Theorem 7. *If T is a tree of order $n \ge 4$ with s support vertices and $T \neq P_6$, then $\gamma_t^d(T) \le (n + s - 1)/2$, with equality if and only if $T \in \mathcal{F} \cup \mathcal{H} \cup \{P_5, P_{11}, K_{1,3}, T^*\}$.*

Proof. Let $T \neq P_6$ be a tree of order $n \ge 4$ with s support vertices. We proceed by induction on n. If $n = 4$, then $T = P_4$ or $T = K_{1,3}$. If $T = P_4$, then $s = 2$ and $\gamma_t^d(T) = 2 < (n + s - 1)/2$, while if $T = K_{1,3}$, then $s = 1$ and $\gamma_t^d(T) = 2 = (n + s - 1)/2$. This establishes the base case. Suppose that $n \ge 5$ and assume that if $T' \neq P_6$ is a tree of order n', where $4 \le n' < n$, and with s' support vertices, then $\gamma_t^d(T') \le (n' + s' - 1)/2$ with equality if and only if $T' \in \mathcal{F} \cup \mathcal{H} \cup \{P_5, P_{11}, K_{1,3}, T^*\}$. Let $T \neq P_6$ be a tree of order n with s support vertices.

If $\mathrm{diam}(T) = 2$, then T is a star, $s = 1$, and $\gamma_t^d(T) = 2 < (n + s - 1)/2$. If $\mathrm{diam}(T) = 3$, then T is a double star, $s = 2$, and $\gamma_t^d(T) = 2 < (n + s - 1)/2$. If T is a path P_n (where recall that $n \ge 5$ and $P \neq P_6$), then as an immediate consequence of Proposition 5, we have that $\gamma_t^d(T) \le (n + 1)/2$ with equality if and only if $T \in \{P_5, P_7, P_{11}\}$. Hence, recalling that $P_7 = H_2 \in \mathcal{H}$, we may assume that $\mathrm{diam}(T) \ge 4$ and that T is not a path, for otherwise the desired result follows. In particular, we note that $n \ge 6$.

In what follows, we shall adopt the following notation. Our aim is to prune the tree T by deleting certain vertices of T to produce a nontrivial tree $T' \neq P_6$ to which we apply the inductive hypothesis. We denote the order of such a pruned tree T' by n' and the number of support vertices in T' by s'. We let D' denote a $\gamma_t^d(T')$-set. Further, if $\text{diam}(T') \geq 3$, then by Observation 11(b) the set D' is chosen to contain no leaf of T'. If $n' \geq 4$ and $T' \neq P_6$, then applying the inductive hypothesis to T' we have $|D'| \leq (n' + s' - 1)/2$. We now proceed with a series of claims that we may assume are satisfied by the tree T, for otherwise the desired result holds.

Claim G. *Every support vertex in T has exactly one leaf neighbor.*

Proof. Let v be a support vertex of T that is adjacent to $k \geq 2$ leaves. Let T' be obtained from T by removing all but one leaf neighbor of v. Then, $n' = n - k + 1$ and $s' = s$. Since $\text{diam}(T') = \text{diam}(T) \geq 4$, we note that $n' \geq 5$. By Observation 11, either $v \in D'$ or v has at least two neighbors in D'. Thus, D' is a DTD-set of T, and so $\gamma_t^d(T) \leq |D'| = \gamma_t^d(T')$. If $T' \neq P_6$, then $|D'| \leq (n' + s' - 1)/2$, implying that $\gamma_t^d(T) \leq (n + s - k)/2 < (n + s - 1)/2$. If $T' = P_6$ and $k \geq 3$, then $n = k + 6$, $s = 2$ and $\gamma_t^d(T) = 4 < (k + 6)/2 = (n + s - 1)/2$. If $T' = P_6$ and $k = 2$, then $T = T^*$ and $\gamma_t^d(T) = 4 = (n + s - 1)/2$. Hence, we may assume that every support vertex is adjacent to exactly one leaf, for otherwise the desired result follows. □

We now root the tree T at a vertex r on a longest path in T. Necessarily, r is a leaf. Let u be a vertex at maximum distance from r. Necessarily, u is a leaf. Let v be the parent of u, let w be the parent of v, let x be the parent of w, and let y be the parent of x. Since u is a vertex at maximum distance from the root r, every child of v is a leaf. By Claim G, every support vertex in T has exactly one leaf neighbor. In particular, this implies that $d_T(v) = 2$. We state this formally as follows.

Claim H. $d_T(v) = 2$.

Claim I. $d_T(w) = 2$.

Proof. Suppose $d_T(w) \geq 3$. Let $T' = T - \{u, v\}$. Then, $n' = n - 2$ and $s' = s - 1$. Since $\text{diam}(T') \geq \text{diam}(T) - 1 \geq 3$, we note that $n' \geq 4$. The vertex w is either a support vertex in T' or, by Claim G, every child of w in T is a support vertex of degree 2. If w has a leaf neighbor in T', then $w \in D'$ or at least two neighbors of w belong to D'. If w has a child, v', of degree 2, then $v' \in D'$ and $w \in D'$ or at least two neighbors of w different from v' belong to D'. In both cases, $w \in D'$ or at least two neighbors of w belong to D'. Thus the set $D' \cup \{v\}$ is a DTD-set of T, implying that $\gamma_t^d(T) \leq |D'| + 1 = \gamma_t^d(T') + 1$. If $T' \neq P_6$, then $\gamma_t^d(T) \leq \gamma_t^d(T') + 1 \leq (n' + s' - 1)/2 + 1 = (n + s - 2)/2 < (n + s - 1)/2$. If $T' = P_6$, then either w is a support vertex in T' or a central vertex in T'. In both cases, $n = 8$, $s = 3$, and $\gamma_t^d(T) = 4 < (n + s - 1)/2$. □

By Claim I, $d_T(w) = 2$. By Observation 11(b), there exists a $\gamma_t^d(T)$-set that contains no leaf of T. Let D be such a $\gamma_t^d(T)$-set. We note that $D \cap \{u, v, w\} = \{v, w\}$.

Claim J. $d_T(x) = 2$.

Proof. Suppose that $d_T(x) \geq 3$. In this case, we consider the tree $T' = T - \{u, v, w\}$. Then, $n' = n - 3$ and $s' = s - 1$. By Claim G, $n' \geq 4$. If $T' = P_6$, then either x is a support vertex in T' or a central vertex in T'. In both cases, $n = 9$, $s = 3$, and $\gamma_t^d(T) = 5 < (n + s - 1)/2$. Hence we may assume that $T' \neq P_6$, and so $|D'| \leq (n' + s' - 1)/2$. The set $D' \cup \{v, w\}$ is a DTD-set of T, implying that $\gamma_t^d(T) \leq |D'| + 2 \leq (n' + s' - 1)/2 + 2 = (n + s - 1)/2$.

Further, suppose that $\gamma_t^d(T) = (n + s - 1)/2$. Then, $\gamma_t^d(T') = (n' + s' - 1)/2$, implying that $T' \in \mathcal{F} \cup \mathcal{H} \cup \{P_5, P_{11}, K_{1,3}, T^*\}$. By Claim G, $T' \neq K_{1,3}$. If $T' = T^*$, then by Claim G, the vertex x is a leaf of T, and so $d_T(x) = 2$, a contradiction. Hence, $T' \neq T^*$. If $T' = P_5$, then either x is a support vertex in T' or a central vertex in T'. In both cases, $n = 8$, $s = 3$, and $\gamma_t^d(T) = 4 < (n + s - 1)/2$. If $T' = P_{11}$, then either x is a support vertex in T' or the distance from x to a leaf of T' is 2 or 3. In all three cases, $n = 14$, $s = 3$, and $\gamma_t^d(T) \leq 7 < (n + s - 1)/2$. Therefore, we may assume that $T' \in \mathcal{H} \cup \mathcal{F}$, for otherwise the desired result follows.

Suppose that $T' \in \mathcal{H}$. Then, $T' = H_k$ for some $k \geq 2$ (possibly, $k = 2$, in which case $T' = P_7$). Then, $n = 3k + 4$ and $s = k + 1$. Let z denote the central vertex of T'. Since $d_T(x) \geq 3$, we note that x has degree at least 2 in T'. Suppose that x is a support vertex of T'. Let z' denote the common neighbor of x and z, and let $L(T)$ denote the set of leaves in T. In this case, the set $V(T) \setminus (L(T) \cup \{z, z'\})$ is a DTD-set of T of cardinality $2k + 1$, implying that $\gamma_t^d(T) \leq 2k + 1 < (n + s - 1)/2$. Hence we may assume that either $x = z$ or x is a neighbor of z. If $x = z$, then $T = H_{k+1} \in \mathcal{H}$. If x is a neighbor of z, then $T = F_{k+1} \in \mathcal{F}$.

Suppose finally that $T' \in \mathcal{F}$. Then, $T' = F_k$ for some $k \geq 3$, and $n = 3k + 4$ and $s = k + 1$. Let T' be the tree obtained from the vertex disjoint union of k paths P_3 on three vertices, where the ith path is given by $a_i b_i c_i$ for $i \in [k]$, by adding a new vertex z and joining z to $a_2, a_3, \ldots, a_k$ and adding the edge $a_1 a_2$. Since $d_T(x) \geq 3$, the vertex x has degree at least 2 in T'. Let $A = \{a_1, a_2, \ldots, a_k\}$ and let $B = \{b_1, b_2, \ldots, b_k\}$. Suppose that $x \in B$. Then, $x = b_i$ for some $i \in [k]$. In this case, then set $A \cup (B \setminus \{b_i\}) \cup \{v, w\}$ is a DTD-set of T of cardinality $2k+1$, implying that $\gamma_t^d(T) \leq 2k + 1 < (n + s - 1)/2$. Suppose $x \in A$. If $x = a_i$ where $i \in \{1, 2\}$, let $R = (A \setminus \{a_i\}) \cup B \cup \{v, w\}$. If $x = a_i$ where $i \in \{3, \ldots, k\}$, let $R = (A \setminus \{a_2, a_i\}) \cup B \cup \{v, w, z\}$. In both cases, R is a DTD-set of T of cardinality $2k + 1$, implying that $\gamma_t^d(T) \leq 2k + 1 < (n + s - 1)/2$. If $x = z$, then $T = F_{k+1} \in \mathcal{H}$ and $\gamma_t^d(T) = 2(k + 1) = (n + s - 1)/2$. Hence, we may assume that $d_T(x) = 2$, for otherwise the desired result follows. □

By Claim J, $d_T(x) = 2$. If $y = r$, then $T = P_5$, contrary to our earlier assumption that T is not a path. Hence, $y \neq r$. Let z be the parent of y. Since T is not a path, we note that $n \geq 7$. We now consider the tree $T' = T - V(T_x)$ of order $n' = n - 4 \geq 3$. If $n' = 3$, then $T = T^*$. Hence, we may assume that $n' \geq 4$. Suppose that $T' = P_6$, and so $n = 10$. Since T is not a path, the vertex y is not a leaf in T', and so $s = 3$. If y is a support vertex of T', then $\gamma_t^d(T) \leq 5 < (n + s - 1)/2$. If y is a central vertex of T', then $T' = F_3 \in \mathcal{F}$. Hence we may assume that $T' \neq P_6$.

If y is a leaf in T', then $s' = s$. If y is not a leaf in T', then $s' = s - 1$. In both cases, $s' \leq s$. By our earlier assumptions, $n' \geq 4$ and $T' \neq P_6$. Thus, $|D'| \leq (n' + s' - 1)/2$. The set $D' \cup \{v, w\}$ is a DTD-set of T, implying that $\gamma_t^d(T) \leq |D'| + 2 = (n' + s' - 1)/2 + 2 \leq (n + s - 1)/2$.

Further, suppose that $\gamma_t^d(T) = (n + s - 1)/2$. Then, $\gamma_t^d(T') = (n' + s' - 1)/2$, implying that $T' \in \mathcal{F} \cup \mathcal{H} \cup \{P_5, P_{11}, K_{1,3}, T^*\}$. Further, $s' = s$, and so y is a leaf in T'. Since T is not a path, this implies that $T' \notin \{P_5, P_{11}\}$. By Claim G, $T' \neq K_{1,3}$. If $T' = T^*$, then $n = 11$, $s = 3$ and $\gamma_t^d(T) \leq 6 < (n + s - 1)/2$. Therefore, we may assume that $T' \in \mathcal{H} \cup \mathcal{F}$, for otherwise the desired result follows.

Suppose that $T' \in \mathcal{H}$. Then, $T' = H_k$ for some $k \geq 2$. Thus, $n = 3k + 5$. Since y is a leaf in T' but not in T, we note that $s = k$. Further, we note that since T is not a path, necessarily $k \geq 3$. Let A be the set of neighbors of the central vertex of T', and let B be the set of support vertices of T'. Let b be the neighbor of y in T'. Then the set $A \cup (B \setminus \{b\}) \cup \{v, w\}$ is a DTD-set of T of cardinality $2k + 1$, implying that $\gamma_t^d(T) \leq 2k + 1 < (n + s - 1)/2$.

Suppose finally that $T' \in \mathcal{F}$. Then, $T' = F_k$ for some $k \geq 3$, and $n = 3k + 5$ and $s = k$. We adopt the notation for T' described in the last paragraph of the proof of Claim J. As before, we define $A = \{a_1, a_2, \ldots, a_k\}$ and let $B = \{b_1, b_2, \ldots, b_k\}$. If $y = c_i$ where $i \in \{1, 2\}$, let $R = A \cup (B \setminus \{b_i\}) \cup \{v, w\}$. If $y = c_i$ where $i \in \{3, \ldots, k\}$, let $R = (A \setminus \{a_2\}) \cup (B \setminus \{b_i\}) \cup \{v, w, z\}$. In both cases, R is a DTD-set of T of cardinality $2k+1$, implying that $\gamma_t^d(T) \leq 2k+1 < (n+s-1)/2$. This completes the proof of Theorem 7. ■

6. Proof of Theorem 9

In this section, we present a proof of Theorem 9. The following result shows that the total domination of a tree is strictly less than twice its disjunctive total domination number.

Theorem 15. *If T is a nontrivial tree, then $\gamma_t(T) < 2\gamma_t^d(T)$.*

Proof. We proceed by induction on the order $n \geq 2$ of a tree. This base cases when $n \in \{2, 3\}$ are trivially true. Suppose that $n \geq 4$ and assume that if T'

is a nontrivial tree of order less than n, then $\gamma_t(T') < 2\gamma_t^d(T')$. Let T be a tree of order n. Let D denote a $\gamma_t^d(T)$-set. If $\gamma_t^d(T) = 2$, then $\gamma_t(T) = 2$. Suppose that $\gamma_t^d(T) = 3$ and let $D = \{v_1, v_2, v_3\}$. Suppose that D is not a TD-set in T. Renaming vertices of D, if necessary, we may assume that v_1 is isolated in $T[D]$. Since D is a DTD-set of T, the vertex v_1 is at distance 2 from both v_2 and v_3. If the common neighbor of v_1 and v_2 is different from the common neighbor of v_1 and v_3, then $d(v_2, v_3) = 4$ and neither v_2 nor v_3 is DT-dominated by D, a contradiction. Hence, v_1, v_2 and v_3 are all adjacent to a common vertex, which when added to the set D forms a TD-set of T, implying that $\gamma_t(T) \le 4 < 2\gamma_t^d(T)$. Therefore, we may assume that $\gamma_t^d(T) \ge 4$, for otherwise the desired result follows. If $\text{diam}(T) \le 3$, then $\gamma_t^d(T) = 2$, a contradiction. Hence, $\text{diam}(T) \ge 4$. Thus, by Observation 11(b), the set D is chosen to contain no leaf of T.

Our aim in what follows is to prune the tree T by deleting certain vertices of T to produce a nontrivial tree T' to which we apply the inductive hypothesis to show that $\gamma_t(T') < 2\gamma_t^d(T')$. Let S' be a $\gamma_t(T')$-set. We now proceed with a series of claims that we may assume are satisfied by the tree T, for otherwise the desired result holds.

Claim K. *Every support vertex in T has exactly one leaf neighbor.*

Proof. Let v be a support vertex of T that is adjacent to at least two leaves. Let u be a leaf neighbor of v and let $T' = T - u$. Since $\text{diam}(T') = \text{diam}(T) \ge 4$, by Observation 10(b) there exists a $\gamma_t(T')$-set, S', that contains no leaf of T. By Observation 10(a), the support vertex $v \in S'$. Thus, S' is a TD-set of T, and so $\gamma_t(T) \le |S'| = \gamma_t(T')$. By Observation 11(b), $u \notin D$, implying that D is a DTD-set of T', and so $\gamma_t^d(T') \le |D| = \gamma_t^d(T)$. Thus, $\gamma_t(T) \le \gamma_t(T') < 2\gamma_t^d(T') \le 2\gamma_t^d(T)$. $\square$

We now root the tree T at a vertex r on a longest path in T. Necessarily, r is a leaf. Let u be a vertex at maximum distance from r. Necessarily, u is a leaf. Let v be the parent of u, let w be the parent of v, let x be the parent of w, and let y be the parent of x. Since u is a vertex at maximum distance from the root r, every child of v is a leaf. By Claim K, every support vertex in T has exactly one leaf neighbor. In particular, this implies that $d_T(v) = 2$. We state this formally as follows.

Claim L. $d_T(v) = 2$.

Claim M. $d_T(w) = 2$.

Proof. Suppose $d_T(w) \ge 3$. Let v' be a child of w different from v. Suppose firstly that v' is a leaf. In this case, let $T' = T - v'$. By Observation 11(b), $v' \notin D$, implying that D is a DTD-set of T', and so $\gamma_t^d(T') \le |D| = \gamma_t^d(T)$. Since v is a support vertex of degree 2 in T', we note that $\{v, w\} \subseteq S'$. Thus, S' is a TD-set of T, and so $\gamma_t(T) \le |S'| = \gamma_t(T')$. Therefore, $\gamma_t(T) \le \gamma_t(T') < 2\gamma_t^d(T') \le 2\gamma_t^d(T)$.

Suppose secondly that v' is not a leaf. Then, by Claim K, $d_T(v') = 2$. Let u' be the leaf neighbor of v'. In this case, let $T' = T - \{u, u', v\}$. By our choice of D, we note that $D \cap \{u, u', v, v'\} = \{v, v'\}$. If $w \in D$, then $D \setminus \{v\}$ is a DTD-set of T', implying that $\gamma_t^d(T') \le |D| - 1 = \gamma_t^d(T) - 1$. If $w \notin D$, then in order to disjunctively dominate v, the set D contains at least one neighbor of w different from v and v', implying that the set $(D \setminus \{v, v'\}) \cup \{w\}$ is a DTD-set of T', and so once again $\gamma_t^d(T') \le |D| - 1 = \gamma_t^d(T) - 1$. We now consider the $\gamma_t(T')$-set S'. Since w is a support vertex in T' (with leaf neighbor v'), we note that $w \in S'$. Thus, $S' \cup \{v, v'\}$ is a TD-set of T, implying that $\gamma_t(T) \le |S'| + 2 = \gamma_t(T') + 2$. Therefore, $\gamma_t(T) \le \gamma_t(T') + 2 < 2\gamma_t^d(T') + 2 \le 2\gamma_t^d(T)$. □

By Claim M, $d_T(w) = 2$. We note that $D \cap \{u, v, w\} = \{v, w\}$.

Claim N. $N[x] \cap D = \{w\}$.

Proof. Suppose that $|N[x] \cap D| \ge 2$. Thus, D contains x or at least one neighbor of x different from w. Let $T' = T - \{u, v, w\}$. If $x \in D$, let $D' = (D \setminus \{v, w\}) \cup \{y\}$. If $x \notin D$, let $D' = (D \setminus \{v, w\}) \cup \{x\}$. In both cases, D' is a DTD-set of T', implying that $\gamma_t^d(T') \le |D| - 1 = \gamma_t^d(T) - 1$. Every $\gamma_t(T')$-set can be extended to a TD-set of T by adding to it the vertices v and w, implying that $\gamma_t(T) \le \gamma_t(T') + 2$. Therefore, $\gamma_t(T) \le \gamma_t(T') + 2 < 2\gamma_t^d(T') + 2 \le 2\gamma_t^d(T)$. □

By Claim N, $N[x] \cap D = \{w\}$.

Claim O. $d_T(x) = 2$.

Proof. Suppose that $d_T(x) \ge 3$. If the vertex x has a descendant $u' \ne u$ at distance 3 from it, and if $u'v'w'x$ denotes the (u', x)-path, then u' is a leaf and analogous arguments as in Claims L and M show that $d_T(v') = d_T(w') = 2$. By our choice of the set D which was chosen to contain no leaf of T, this implies that $\{v', w'\} \in D$. If x has a descendant v' at distance 2 from it that is a leaf, then the common neighbor of x and v' belongs to D. If x has a child that is a leaf, then D contains x or at least two neighbors of x. In all cases, we contradict Claim N. □

By Claim O, $d_T(x) = 2$. If $y = r$, then $T = P_5$ and $\gamma_t^d(T) = 3$, a contradiction. Hence, $y \ne r$. Let z be the parent of y. We now consider the tree $T' = T - \{u, v, w, x\}$. If $x \in D$, let $D' = (D \setminus \{v, w, x\}) \cup \{y, z\}$. If $x \notin D$ and $y \in D$, let $D' = (D \setminus \{v, w\}) \cup \{z\}$. If neither x nor y belong to D but D contains a neighbor of y, let $D' = (D \setminus \{v, w\}) \cup \{y\}$. If $D \cap N[y] = \emptyset$, let $D' = (D \setminus \{v, w\}) \cup \{z\}$. In all cases, D' is a DTD-set of T' and $|D'| \le |D| - 1$, implying that $\gamma_t^d(T') \le |D| - 1 = \gamma_t^d(T) - 1$. Every $\gamma_t(T')$-set can be extended to a

TD-set of T by adding to it the vertices v and w, implying that $\gamma_t(T) \leq \gamma_t(T')+2$. Therefore, $\gamma_t(T) \leq \gamma_t(T') + 2 < 2\gamma_t^d(T') + 2 \leq 2\gamma_t^d(T)$. This completes the proof of Theorem 15. ■

We show next that the bound of Theorem 15 is asymptotically tight.

Proposition 16. *For every fixed $\epsilon > 0$, there exists a tree T such that $\gamma_t(T) > (2 - \epsilon)\gamma_t^d(T)$.*

Proof. Let $\epsilon > 0$ be given. Let $k > \max\{2, \frac{3}{\epsilon} - 3\}$. Let T_k' be obtained from a path $v_1 v_2 \cdots v_{2k+1}$ by adding a pendant edge to each of v_2 and v_{2k}, and let T_k be obtained from T_k' by adding a pendant edge to every vertex of T_k'. The tree T_4, for example, is illustrated in Figure 4, where the darkened vertices form a minimum DTD-set of T_4. Letting $T = T_k$, it is a simple exercise to show that $\gamma_t(T) = 2k + 3$ and $\gamma_t^d(T) = k + 3$. Since $\epsilon > \frac{3}{k+3}$, we note that $(2 - \epsilon)\gamma_t^d(T) < \left(2 - \frac{3}{k+3}\right)(k + 3) = 2k + 3 = \gamma_t(T)$. ■

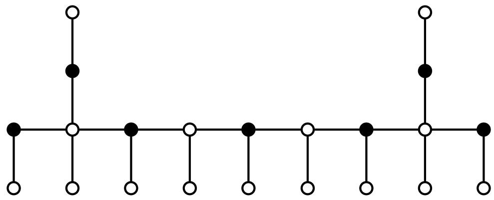


Figure 4. The tree T_4.

Theorem 9 is an immediate consequence of Theorem 15 and Proposition 16.

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