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γ-CYCLES IN ARC-COLORED DIGRAPHS

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Abstract

We call a digraph D an m-colored digraph if the arcs of D are colored with m colors. A directed path (or a directed cycle) is called monochromatic if all of its arcs are colored alike. A subdigraph H in D is called rainbow if all of its arcs have different colors. A set $N \subseteq V(D)$ is said to be a kernel by monochromatic paths of D if it satisfies the two following conditions:

(i) for every pair of different vertices $u, v \in N$ there is no monochromatic path in D between them, and

(ii) for every vertex $x \in V(D) - N$ there is a vertex $y \in N$ such that there is an xy-monochromatic path in D.

A γ-cycle in D is a sequence of different vertices $\gamma = (u_0, u_1, \ldots, u_n, u_0)$ such that for every $i \in \{0, 1, \ldots, n\}$:

(i) there is a $u_i u_{i+1}$-monochromatic path, and

(ii) there is no $u_{i+1} u_i$-monochromatic path.

The addition over the indices of the vertices of γ is taken modulo $(n+1)$. If D is an m-colored digraph, then the closure of D, denoted by $\mathfrak{C}(D)$, is the m-colored multidigraph defined as follows: $V(\mathfrak{C}(D)) = V(D)$, $A(\mathfrak{C}(D)) =$

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$A(D) \cup \{(u,v)$ with color $i \mid$ there exists a uv-monochromatic path colored i contained in $D\}$.

In this work, we prove the following result. Let D be a finite m-colored digraph which satisfies that there is a partition $C = C_1 \cup C_2$ of the set of colors of D such that:

(1) $D[\widehat{C}_i]$ (the subdigraph spanned by the arcs with colors in C_i) contains no γ-cycles for $i \in \{1,2\}$;

(2) If $\mathfrak{C}(D)$ contains a rainbow $C_3 = (x_0, z, w, x_0)$ involving colors of C_1 and C_2, then $(x_0, w) \in A(\mathfrak{C}(D))$ or $(z, x_0) \in A(\mathfrak{C}(D))$;

(3) If $\mathfrak{C}(D)$ contains a rainbow $P_3 = (u, z, w, x_0)$ involving colors of C_1 and C_2, then at least one of the following pairs of vertices is an arc in $\mathfrak{C}(D)$: (u,w), (w,u), (x_0,u), (u,x_0), (x_0,w), (z,u), (z,x_0).

Then D has a kernel by monochromatic paths.

This theorem can be applied to all those digraphs that contain no γ-cycles. Generalizations of many previous results are obtained as a direct consequence of this theorem.

Keywords: digraph, kernel, kernel by monochromatic paths, γ-cycle.

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1. Introduction

For general concepts we refer the reader to [1]. Let D be a digraph, and let $V(D)$ and $A(D)$ denote the sets of vertices and arcs of D, respectively. We recall that a subdigraph D_1 of D is a *spanning subdigraph* if $V(D_1) = V(D)$. If S is a nonempty subset of $V(D)$, then the *subdigraph induced* by S, denoted by $D[S]$, is the digraph having vertex set S, and whose arcs are all those arcs of D joining vertices of S. An arc u_1u_2 of D will be called an S_1S_2-*arc* of D whenever $u_1 \in S_1$ and $u_2 \in S_2$.

A set $I \subseteq V(D)$ is *independent* if $A(D[I]) = \emptyset$. A *kernel* N of D is an independent set of vertices such that for each $z \in V(D) - N$ there exists a zN-arc in D, that is, an arc from z toward some vertex in N. A digraph D is a *kernel-prefect digraph* when every induced subdigraph of D has a kernel. Sufficient conditions for the existence of kernels in digraphs have been investigated by several authors, von Neumann and Morgenstern [17]; Richardson [18, 19]; Duchet and Meyniel [4]; Duchet [2, 3]; Galeana-Sánchez and Neumann-Lara [9, 10]. The concept of kernel is very useful in applications.

We call the digraph D an *m-colored digraph* if the arcs of D are colored with m colors. Along this paper, all the paths and cycles will be directed paths and directed cycles. A path is *monochromatic* if all of its arcs are colored alike. A cycle is called a *quasi-monochromatic cycle* if with at most one exception all of its arcs are colored alike. A subdigraph H of D is *rainbow* if all its arcs have distinct colors. A set N of vertices of D is a *kernel by monochromatic paths* if for every pair of vertices of N there is no monochromatic path between them and for every vertex v not in N there is a monochromatic path from v to some vertex in N. If D is an m-colored digraph, then the *closure* of D, denoted by $\mathfrak{C}(D)$, is the m-colored multidigraph defined as follows: $V(\mathfrak{C}(D)) = V(D)$, $A(\mathfrak{C}(D)) = A(D) \cup \{(u,v)$ with color $i \mid$ there exists a uv-monochromatic path colored i contained in $D\}$. Notice that for any digraph D, $\mathfrak{C}(\mathfrak{C}(D)) \cong \mathfrak{C}(D)$, and D has a kernel by monochromatic paths if and only if $\mathfrak{C}(D)$ has a kernel.

In [22] Sands, Sauer and Woodrow proved that any 2-colored digraph D has a set S of vertices such that: (i) for any $x, y \in S$, there is no monochromatic path between them, and (ii) for every vertex $x \notin S$, there is a monochromatic path from x to a vertex of S (i.e., D has a kernel by monochromatic paths, a concept that was introduced later by Galeana-Sánchez [5]). In particular, they proved that any 2-colored tournament T has a kernel by monochromatic paths. They also raised the following problem: Let T be a 3-colored tournament such that every cycle of length 3 is a quasi-monochromatic cycle; must T have a kernel by monochromatic paths? This question still remains open. In [21] Shen Minggang proved that if T is an m-colored tournament such that every cycle of length 3 is a quasi-monochromatic cycle, and every transitive tournament of order 3 is quasi-monochromatic, then T has a kernel by monochromatic paths. He also proved that this result is the best possible for m-colored tournaments with $m \geq 5$. In fact, he proved that for each $m \geq 5$ there exists an m-colored tournament T such that every cycle of length 3 is quasi-monochromatic and T has no kernel by monochromatic paths. Also for every $m \geq 3$ there exists an m-colored tournament T' such that every transitive tournament of order 3 is quasi-monochromatic and T' has no kernel by monochromatic paths. In 2004 [11] Galeana-Sánchez and Rojas-Monroy presented a 4-colored tournament T such that every cycle of order 3 is quasi-monochromatic, but T has no kernel by monochromatic paths. The known sufficient conditions for the existence of kernel by monochromatic paths in m-colored ($m \geq 3$) tournaments (or nearly tournaments), ask for the monochromaticity or quasi-monochromaticity of certain subdigraphs. More information on m-colored digraphs can be found in [5, 6, 7, 23, 24].

If $\mathcal{C} = (z_0, z_1, \ldots, z_n, z_0)$ is a cycle, we will denote by $\ell(\mathcal{C})$ its length, and if $z_i, z_j \in V(\mathcal{C})$ with $i \leq j$, then we denote by $(z_i, \mathcal{C}, z_j)$ the $z_i z_j$-path contained in $\mathcal{C}$. A sequence of different vertices $\gamma = (u_0, \ldots, u_n, u_0)$ is a *γ-cycle* if for every $i \in \{0, 1, \ldots, n\}$ there is a $u_i u_{i+1}$-monochromatic path, and there is no $u_{i+1} u_i$-monochromatic path. The addition over the indices of the vertices of γ is taken modulo $(n+1)$.

In this paper we prove that if D is a finite m-colored digraph, and if there exists a partition $C = C_1 \cup C_2$ of the set of colors of D such that:

(1) $D[\widehat{C}_i]$ contains no γ-cycles for $i \in \{1, 2\}$, ($\widehat{C}_i$ denotes the set of arcs of D with colors in C_i);

(2) If $\mathfrak{C}(D)$ contains a rainbow $C_3 = (x_0, z, w, x_0)$ involving colors of C_1 and C_2, then $(x_0, w) \in A(\mathfrak{C}(D))$ or $(z, x_0) \in A(\mathfrak{C}(D))$;

(3) If $\mathfrak{C}(D)$ contains a rainbow $P_3 = (u, z, w, x_0)$ involving colors of C_1 and C_2, then at least one of the following pairs of vertices is an arc in $\mathfrak{C}(D)$: (u, w), (w, u), (x_0, u), (u, x_0), (x_0, w), (z, u), (z, x_0).

Then D has a kernel by monochromatic paths.

We will need the following results.

Assertion 1.1. *Let D be a finite or infinite digraph and $u, v \in V(D)$. Every uv-walk in D contains a uv-path.*

Assertion 1.2. *Let D be a finite or infinite digraph. Every closed walk in D contains a cycle.*

Assertion 1.3. *Let D be a finite digraph. If every vertex $v \in V(D)$ fulfills that $\delta^-_D(v) \geq 1$ ($\delta^+_D(v) \geq 1$), then D contains a cycle.*

Theorem 1.4 (Duchet [2]). *If D is a finite digraph such that every cycle of D has at least one symmetrical arc, then D has a kernel.*

Theorem 1.5 (Rojas-Monroy, Villarreal-Valdés [20]). *Let D be a finite or infinite digraph. If every cycle and every infinite outward path has a symmetrical arc, then there exists $x \in V(D)$ which satisfies $(x, u) \in A(D)$ implies $(u, x) \in A(D)$.*

The following lemma has been important to obtain many results on the existence of kernels by monochromatic paths in finite m-colored digraphs [5, 6, 8, 12, 13, 14, 15, 16].

Lemma 1.6. *Let D be a finite or infinite m-colored digraph and $\mathfrak{C}(D)$ its closure. Then D contains no γ-cycles if and only if every cycle in $\mathfrak{C}(D)$ has at least one symmetrical arc.*

It follows from Lemma 1.6 and Theorem 1.5 that if D is a finite m-colored digraph which contains no γ-cycles, then D has a kernel by monochromatic paths.

2. γ-Cycles and Monochromatic Paths in Arc-Colored Digraphs

The following three lemmas are about m-colored digraphs containing no γ-cycles, and they are useful to prove our main result.

Lemma 2.1. *Let D be a finite m-colored digraph, and suppose that D contains no γ-cycles. There exists $x_0 \in V(D)$ such that for every $z \in V(D) - \{x_0\}$ if there exists an x_0z-monochromatic path contained in D, then there exists a zx_0-monochromatic path contained in D.*

Proof. Assume, for a contradiction, that D is a digraph as in the hypothesis of the Lemma 2.1, and that there is no vertex x_0 satisfying the assertion of Lemma 2.1.

Let $x_0 \in V(D)$, it follows from our assumptions that there is $x_1 \in V(D) - \{x_0\}$ such that there is an x_0x_1-monochromatic path contained in D and there is no x_1x_0-monochromatic path contained in D. Again from our assumptions there is $x_2 \in V(D) - \{x_1\}$ such that there is an x_1x_2-monochromatic path contained in D and there is no x_2x_1-monochromatic path contained in D. Once chosen $x_0, x_1, \ldots, x_n$; given our supposition we can choose $x_{n+1} \in V(D) - \{x_n\}$ in such a way that there is an x_nx_{n+1}-monochromatic path in D and there is no $x_{n+1}x_n$-monochromatic path in D. Thus, we obtain a sequence of vertices $(x_0, x_1, x_2, x_3, \ldots)$ such that for every $i \in \{0, 1, 2, \ldots\}$ there is an x_ix_{i+1}-monochromatic path contained in D and there is no $x_{i+1}x_i$-monochromatic path contained in D. Since D is a finite digraph, there is $\{i, j\} \subseteq \mathbb{N} \cup \{0\}$ with $i < j$ such that $x_j = x_i$. Let $j_0 = \min\{j \mid x_j = x_i \text{ for some } i < j\}$, and let $i_0 \in \{0, 1, \ldots, j_0 - 1\}$ such that $x_{i_0} = x_{j_0}$ (notice that i_0 is unique because of the definition of j_0). Without loss of generality suppose that $i_0 = 0$ and $j_0 = n$. Thus, $C = (x_0, x_1, \ldots, x_{n-1}, x_n = x_0)$ is a sequence of n different vertices such that for every $i \in \{0, \ldots, n-1\}$ there is an x_ix_{i+1}-monochromatic path contained in D and there is no $x_{i+1}x_i$-monochromatic path contained in D (the indices of the vertices will be taken modulo n). Therefore, $C = (x_0, x_1, \ldots, x_{n-1}, x_n = x_0)$ is a γ-cycle, which contradicts the hypothesis. ■

Let D be an m-colored digraph and let H be a subdigraph of D. We will say that $S \subseteq V(D)$ is a *semikernel by monochromatic paths* modulo H of D if S is independent by monochromatic paths in D and for every $z \in V(D) - S$, if there is a Sz-monochromatic path contained in $D - H$, then there is a zS-monochromatic path contained in D.

Lemma 2.2. *Let D be a finite m-colored digraph. Suppose that there is a partition $C = C_1 \cup C_2$ of the set of colors of D such that $D[\widehat{C}_1]$ contains no γ-cycles. Then there exists $x_0 \in V(D)$ such that $\{x_0\}$ is a semikernel by monochromatic paths* (mod $D[\widehat{C}_2]$) *of D.*

Proof. It follows by applying Lemma 2.1 to $D - \widehat{C}_2$. ■

Let D be a finite m-colored digraph. Suppose that there is a partition $C = C_1 \cup C_2$ of the set of colors of D and $D[\widehat{C}_1]$ contains no γ-cycles.

Denote by

$\mathcal{S} = \{S \mid S \neq \emptyset$ and S is a semikernel by monochromatic paths (mod $D[\widehat{C}_2]$) of $D\}$.

Notice that by Lemma 2.2, there exists a semikernel by monochromatic paths (mod $D[\widehat{C}_2]$) of D, and thus $\mathcal{S} \neq \emptyset$.

Whenever $\mathcal{S} \neq \emptyset$, we will denote by $D_{\mathcal{S}}$ the loopless digraph defined as follows:

(1) $V(D_{\mathcal{S}}) = \mathcal{S}$ (i.e, for every element of $\mathcal{S}$ we put a vertex in $D_{\mathcal{S}}$), and

(2) $(S_1, S_2) \in A(D_{\mathcal{S}})$ if and only if for every $s_1 \in S_1$ there exists $s_2 \in S_2$ such that $s_1 = s_2$ or there exists an s_1s_2-monochromatic path contained in $D[\widehat{C}_2]$ and there is no s_2S_1-monochromatic path contained in D.

Lemma 2.3. *Let D be a finite m-colored digraph. Suppose that there is a partition $C = C_1 \cup C_2$ of the set of colors of D and $D[\widehat{C}_i]$ contains no γ-cycles for $i \in \{1, 2\}$. Then $D_{\mathcal{S}}$ is an acyclic digraph.*

Proof. Observe that by Lemma 2.2, there exists a semikernel by monochromatic paths (mod $D[\widehat{C}_2]$) of D and therefore $\mathcal{S} \neq \emptyset$. Thus, we can consider the digraph $D_{\mathcal{S}}$. Suppose, for a contradiction, that the digraph $D_{\mathcal{S}}$ contains some cycle, say $\mathcal{C} = (S_0, S_1, \ldots, S_{n-1}, S_0)$ of length $n \geq 2$. Since $\mathcal{C}$ is a cycle in $D_{\mathcal{S}}$, we have that $S_i \neq S_j$ whenever $i \neq j$.

Claim 1. *There exists $i_0 \in \{0, 1, 2, \ldots, n-1\}$ such that for some $z \in S_{i_0}$, $z \notin S_{i_0+1}$ (mod n).*

Proof. Otherwise, for every $i \in \{0, 1, \ldots, n-1\}$ and every $z \in S_i$ we have that $z \in S_{i+1}$, and then $S_i = S_j$ for all $i, j \in \{0, 1, \ldots, n-1\}$. So, $\mathcal{C} = (S_0)$, which is a contradiction, since the digraph is loopless. □

Claim 2. *If there exists $i_0 \in \{0, 1, \ldots, n-1\}$ such that for some $z \in S_{i_0}$ and some $w \in S_{i_0+1}$ (mod n) there exists a zw-monochromatic path, then there exists $j_0 \neq i_0$, $j_0 \in \{0, 1, \ldots, n-1\}$, such that $w \in S_{j_0}$ and $w \notin S_{j_0+1}$(mod n).*

Proof. Suppose without loss of generality that $i_0 = 0$. First, observe that $w \notin S_n = S_0$, since otherwise we have a zw-monochromatic path with $\{z, w\} \subseteq S_0$, contradicting that S_0 is independent by monochromatic paths. Since $w \in S_1$, let $j_0 = \max\{i \in \{0, 1, \ldots, n-1\} \mid w \in S_i\}$ (notice that for both previous observations j_0 is well defined). So, $w \in S_{j_0}$ and $w \notin S_{j_0+1}$. □

It follows from Claim 1 that there exist $i_0 \in \{0, \ldots, n-1\}$ and $t_0 \in S_{i_0}$ such that $t_0 \notin S_{i_0+1}$. It follows from the fact that $(S_{i_0}, S_{i_0+1}) \in A(D_{\mathcal{S}})$ that there exists $t_1 \in S_{i_0+1}$ such that there exists a t_0t_1-monochromatic path contained in $D[\widehat{C}_2]$ and there is no $t_1S_{i_0}$-monochromatic path contained in D. From Claim 2, it follows that there exists an index $i_1 \in \{0, \ldots, n-1\}$ such that $t_1 \in S_{i_1}$ and $t_1 \notin S_{i_1+1}$. Since $(S_{i_1}, S_{i_1+1}) \in A(D_{\mathcal{S}})$ it follows that there exists $t_2 \in S_{i_1+1}$ such that there is a t_1t_2-monochromatic path contained in $D[\widehat{C}_2]$ and there is no $t_2S_{i_1}$-monochromatic path contained in D. Since D is finite, we obtain a sequence of vertices $(t_0, t_1, t_2, \ldots, t_{m-1}, t_0)$ such that there exists a t_it_{i+1}-monochromatic path contained in $D[\widehat{C}_2]$ and there is no $t_{i+1}t_i$-monochromatic path contained in D for each $i \in \{0, 1, 2, \ldots, m-1\}$ (mod m). But this contradicts that $D[\widehat{C}_2]$ contains no γ-cycles. ■

3. The Main Result

The idea of the proof of our main theorem is to select $S \in V(D_{\mathcal{S}})$ such that $\delta^+_{D_{\mathcal{S}}}(S) = 0$ (such S exists since $D_{\mathcal{S}}$ is acyclic) and prove that S is a kernel by monochromatic paths of D.

Theorem 3.1. *Let D be a finite m-colored digraph. If there exists a partition $C = C_1 \cup C_2$ of the set of colors of D such that:*

(1) *$D[\widehat{C}_i]$ contains no γ-cycles for $i \in \{1, 2\}$;*

(2) *If $\mathfrak{C}(D)$ contains a rainbow $C_3 = (x_0, z, w, x_0)$ involving colors of C_1 and C_2, then $(x_0, w) \in A(\mathfrak{C}(D))$ or $(z, x_0) \in A(\mathfrak{C}(D))$;*

(3) *If $\mathfrak{C}(D)$ contains a rainbow $P_3 = (u, z, w, x_0)$ involving colors of C_1 and C_2, then at least one of the following pairs of vertices is an arc in $\mathfrak{C}(D)$: (u, w), (w, u), (x_0, u), (u, x_0), (x_0, w), (z, u), (z, x_0).*

Then D has a kernel by monochromatic paths.

Proof. Consider the digraph $D_{\mathcal{S}}$ of the digraph D. Since $D_{\mathcal{S}}$ is a finite digraph, and from Lemma 2.3 it contains no cycles, it follows that $D_{\mathcal{S}}$ has at least one vertex of zero outdegree. Let $S \in V(D_{\mathcal{S}})$ be such that $\delta^+_{D_{\mathcal{S}}}(S) = 0$. We will prove that S is a kernel by monochromatic paths of D.

Suppose, for a contradiction, that S is not a kernel by monochromatic paths of D. Since $S \in V(D_{\mathcal{S}})$, we have that S is independent by monochromatic paths.

Let $X = \{z \in V(D) \mid \text{there is no } zS\text{-monochromatic path in } D\}$. It follows from our assumption that $X \neq \emptyset$. Consider $D - \widehat{C}_2$ and its closure $\mathfrak{C}(D - \widehat{C}_2)$. Note that $D[\widehat{C}_1]$ is a subdigraph of $D - \widehat{C}_2$ which satisfies $A(D[\widehat{C}_1]) = A(D - \widehat{C}_2)$. Since $D[\widehat{C}_1]$ contains no γ-cycles, we have that $D - \widehat{C}_2$ contains no γ-cycles either. Lemma 1.6 implies that every cycle in $\mathfrak{C}(D - \widehat{C}_2)$ has at least one symmetrical arc. Let $H = \mathfrak{C}(D - \widehat{C}_2)[X]$ be the subdigraph of $\mathfrak{C}(D - \widehat{C}_2)$ induced by X. We have that H also satisfies that every cycle has at least one symmetrical arc, by Theorem 1.5 there is a vertex x_0 which satisfies that $(x_0, u) \in A(H)$ implies $(u, x_0) \in A(H)$.

Let $T = \{z \in S \mid$ there is no zx_0-monochromatic path in $D[\widehat{C}_2]\}$. From the definition of T, we have that for every $z \in (S - T)$ there exists a zx_0-monochromatic path contained in $D[\widehat{C}_2]$.

Claim 1. *$T \cup \{x_0\}$ is independent by monochromatic paths.*

Proof. Since $T \subseteq S$ with $S \in \mathcal{S}$ and $x_0 \in X$, it remains to prove that there is no wx_0-monochromatic path in $D[\widehat{C}_1]$ for $w \in T$. Suppose that such path there exists. Since S is a semikernel by monochromatic paths (mod $D[\widehat{C}_2]$), there is an x_0S-monochromatic path in D, but this is a contradiction with the definition of X. □

Claim 2. *For each $z \in V(D) - (T \cup \{x_0\})$, if there exists a $(T \cup \{x_0\})z$-monochromatic path contained in $D[\widehat{C}_1]$, then there exists a $z(T \cup \{x_0\})$-monochromatic path contained in D.*

Proof. *Case* 1. There exists a Tz-monochromatic path contained in $D[\widehat{C}_1]$. Since $T \subseteq S$ and $S \in \mathcal{S}$, it follows that there exists a zS-monochromatic path contained in D. We may suppose that there exists a $z(S-T)$-monochromatic path contained in D (otherwise we are done). Let α_1 be a uz-monochromatic path contained in $D[\widehat{C}_1]$ with $u \in T$, and let α_2 be a zw-monochromatic path with $w \in (S - T)$ contained in D. Since $w \in (S - T)$, it follows from the definition of T that there exists a wx_0-monochromatic path α_3 contained in $D[\widehat{C}_2]$.

Moreover, color(α_1) $\neq$ color(α_2) (color(α) denotes the color used in the arcs of α), otherwise there exists a uw-monochromatic path contained in $\alpha_1 \cup \alpha_2$, with $\{u, w\} \subseteq S$, in contradiction with the fact that S is independent by monochromatic paths. In addition, we will suppose that color(α_2) $\neq$ color(α_3), since when color(α_2) = color(α_3) we have $\alpha_2 \cup \alpha_3$ contains a zx_0-monochromatic path and Claim 2 is proved. Also color(α_1) $\neq$ color(α_3) as color(α_1) $\in C_1$ and color(α_3) $\in C_2$.

So, we obtain that (u, z, w, x_0) is a rainbow P_3 in $\mathfrak{C}(D)$ involving colors of both C_1 and C_2, and by the hypothesis there exists at least one of the following monochromatic paths in D: from u to w; from w to u; from x_0 to u; from u to x_0; from x_0 to w; from z to u; from z to x_0. If there exists a zu-monochromatic path or a zx_0-monochromatic path in D, then Claim 2 is proved. So, we will demonstrate that is not possible the existence of the other paths.

(i) There is no uw-monochromatic path in D, since $\{u, w\} \subseteq S$ and S is a semikernel by monochromatic paths (mod $D[\widehat{C}_2]$) of D.

(ii) There is no wu-monochromatic path in D, (the same reason as in (i)).

(iii) There is no x_0u-monochromatic path in D as $T \cup \{x_0\}$ is independent by monochromatic paths in D.

(iv) There is no ux_0-monochromatic path in D (the same reason as in (iii)).

(v) There is no x_0w-monochromatic path in D, since $x_0 \in X$ and $w \in S$.

Case 2. There exists an x_0z-monochromatic path contained in $D[\widehat{C}_1]$. Let α_1 be such a path. Suppose that $z \in X$, then (x_0, z) is an arc in H (recall $H = \mathfrak{C}(D - \widehat{C}_2)[X]$). The choice of x_0 implies that $(z, x_0) \in A(H)$. By the definition of the closure of an m-colored digraph and the fact that H is an induced subdigraph of $\mathfrak{C}(D - \widehat{C}_2)$ we conclude that there is a zx_0-monochromatic path in $D - \widehat{C}_2$, and this path is a zx_0-monochromatic path in D. Now, assume that $z \notin X$. It follows from the definition of X that there exists some zS-monochromatic path contained in D, let α_2 be such a path, say that α_2 ends in w. We will suppose that $w \in (S - T)$. Since $w \in (S - T)$, by the definition of T, we have that there exists a wx_0-monochromatic path contained in $D[\widehat{C}_2]$, let α_3 be such a path.

Again, we have that color$(\alpha_1) \neq$ color(α_2), otherwise there exists an x_0w-monochromatic path contained in D, contradicting that $x_0 \in X$ and $w \in S$. In addition, we may suppose that color$(\alpha_2) \neq$ color(α_3), since if color$(\alpha_2) =$ color(α_3), then D contains a zx_0-monochromatic path and Claim 2 is proved. Also color$(\alpha_1) \neq$ color(α_3), since $\alpha_1 \subseteq D[\widehat{C}_1]$ and $\alpha_3 \subseteq D[\widehat{C}_2]$.

Then (x_0, z, w, x_0) is a rainbow C_3 in $\mathfrak{C}(D)$ which involves colors of both C_1 and C_2, and from hypothesis there exist an x_0w-monochromatic path or a zx_0-monochromatic path in D. Since $x_0 \in X$ and $w \in S$, it follows directly from the definitions of X and S that there is no x_0w-monochromatic path in D. Then there is a zx_0-monochromatic path in D, and Claim 2 is proved. □

We conclude from Claims 1 and 2 that $T \cup \{x_0\} \in \mathcal{S}$ and therefore $T \cup \{x_0\} \in V(D_{\mathcal{S}})$. We have that $(S, T \cup \{x_0\}) \in A(D_{\mathcal{S}})$, since $T \subseteq T \cup \{x_0\}$, and for each $s \in S - T$ there exists an sx_0-monochromatic path contained in $D[\widehat{C}_2]$, and there is no x_0S-monochromatic path contained in D. But this contradicts the fact that $\delta^+_{D_{\mathcal{S}}}(S) = 0$. Therefore S is a kernel by monochromatic paths in D and Theorem 3.1 is proved. ■

Remark 3.2. Theorem 3.1 can be applied to all those digraphs that contain no γ-cycles. Generalizations of many previous results are obtained as a direct consequence of this theorem.

Now, we give some definitions and next we give a list of digraphs that contains no γ-cycles.

Definition. A digraph D is *n-quasitransitive* if for every $\{u, v\} \subseteq V(D)$ such that there is a uv-directed path of length n, we have $(u, v) \in A(D)$ or $(v, u) \in A(D)$.

Definition. We denote by $A^+(u)$ the set of arcs of D that have u as the initial end-point, and $A^+(u)$ is *monochromatic* if all of its elements have the same color.

Definition. We denote by T_4 the digraph such that $V(T_4) = \{u, v, w, x\}$ and $A(T_4) = \{(u, v), (v, x), (x, w), (u, w)\}$, see Figure 1.

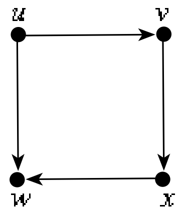

Figure 1. T_4.

Definition. A digraph D is called a *bipartite tournament* if its set of vertices can be partitioned into two sets V_1 and V_2 such that: (i) every arc of D has an end-point in V_1, and the other end-point in V_2, and (ii) for every $x_1 \in V_1$ and every $x_2 \in V_2$, we have $|\{(x_1, x_2), (x_2, x_1)\} \cap A(D)| = 1$.

Definition. $\widetilde{T}_6$ is the bipartite tournament defined as follows:

1. $V(\widetilde{T}_6) = \{u, v, w, x, y, z\}$,
2. $A(\widetilde{T}_6) = \{(u, w), (v, w), (w, x), (w, z), (x, y), (y, u), (y, v), (z, y)\}$,

with $\{(u, w), (w, x), (y, u), (z, y)\}$ coloured 1 and $\{(v, w), (w, z), (x, y), (y, v)\}$ colored 2, see Figure 2.

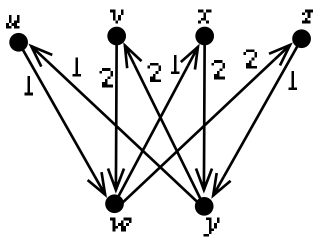

Figure 2. $\widetilde{T}_6$.

Definition. If v is a vertex of an m-coloured tournament T, we denote by $\xi(v)$ the set of colours assigned to the arcs with v as an end-point.

Definition. $\widetilde{T}_8$ is the digraph defined as follows:

1. $V(\widetilde{T}_8) = \{s, t, u, v, w, x, y, z\}$,
2. $A(\widetilde{T}_8) = \{(s, t), (s, x), (t, u), (t, y), (u, v), (u, z), (v, w), (v, s), (w, x), (w, t),$ $(x, y), (x, u), (y, z), (y, v), (z, s), (z, w)\}$,

and each other arc in $\widetilde{T}_8$ colored 2, see Figure 3.

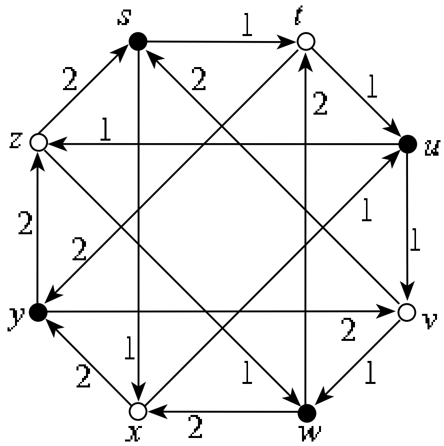

Figure 3. $\widetilde{T_8}$.

A list of theorems proving the existence of digraphs without γ-cycles.

Theorem 3.3 (Galeana-Sánchez, Gaytán-Gómez, Rojas-Monroy [8]). *Let D be a finite m-colored digraph such that every cycle in D is monochromatic. Then D contains no γ-cycle.*

Theorem 3.4 (Galeana-Sánchez, Rojas-Monroy, Zavala [16]). *Let D be a finite m-colored 3-quasitransitive digraph such that for every vertex u of D, $A^+(u)$ is monochromatic. If every C_3, C_4 and T_4 contained in D is quasi-monochromatic, then there is no γ-cycles in D.*

Theorem 3.5 (Galeana-Sánchez [5]). *Let T be a finite m-colored tournament. If each directed cycle contained in T and of length at most 4 is a quasi-monochromatic cycle, then there is no γ-cycles in T.*

Theorem 3.6 (Galeana-Sánchez [6]). *Let D be a finite m-colored digraph resulting from the deletion of a single arc (x, y) of some m-colored tournament T (i.e., $D \cong T - (x, y)$). If every directed cycle contained in D of length at most 4 is quasi-monochromatic, then there is no γ-cycles in D.*

Theorem 3.7 (Galeana-Sánchez and Rojas-Monroy [14]). *Let T be a finite m-colored bipartite tournament. Assume that every directed cycle of length 4 is quasi-monochromatic, every directed cycle of length 6 is monochromatic, and T has no subtournament isomorphic to $\widetilde{T_6}$. Then there is no γ-cycles in T.*

Theorem 3.8 (Galeana-Sánchez and Rojas-Monroy [12]). *Let T be a finite m-colored bipartite tournament. If every directed cycle of length 4 in T is monochromatic, then there is no γ-cycles in T.*

Theorem 3.9 (Galeana-Sánchez and Rojas-Monroy [13]). *Let T be a finite* 3-*colored tournament such that every directed cycle of length* 3 *is quasi-monochromatic, and for each $v \in V(T)$ we have $|\xi(v)| \leq 2$, then there is no γ-cycles in T.*

Theorem 3.10 (Galeana-Sánchez and Rojas-Monroy [15]). *Let T be a finite m-colored bipartite tournament such that, every C_4 is quasi-monochromatic, every T_4 is quasi-monochromatic, and T has no induced subdigraph isomorphic to $\widetilde{T}_8$. Then T has no γ-cycles.*

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References

[1] C. Berge, Graphs (North-Holland, Amsterdam, 1985).

[2] P. Duchet, *Graphes Noyau-Parfaits*, Ann. Discrete Math. **9** (1980) 93–101.
doi:10.1016/S0167-5060(08)70041-4

[3] P. Duchet, *Classical perfect graphs: An introduction with emphasis on triangulated and interval graphs*, Ann. Discrete Math. **21** (1984) 67–96.

[4] P. Duchet and H. Meyniel, *A note on kernel-critical graphs*, Discrete Math. **33** (1981) 103–105.
doi:10.1016/0012-365X(81)90264-8

[5] H. Galeana-Sánchez, *On monochromatic paths and monochromatics cycles in edge colored tournaments*, Discrete Math. **156** (1996) 103–112.
doi:10.1016/0012-365X(95)00036-V

[6] H. Galeana-Sánchez, *Kernels in edge-coloured digraphs*, Discrete Math. **184** (1998) 87–99.
doi:10.1016/S0012-365X(97)00162-3

[7] H. Galeana-Sánchez and J.J. García-Ruvalcaba, *Kernels in the closure of coloured digraphs*, Discuss. Math. Graph Theory **20** (2000) 243–254.
doi:10.7151/dmgt.1123

[8] H. Galeana-Sánchez, G. Gaytán-Gómez and R. Rojas-Monroy, *Monochromatic cycles and monochromatic paths in arc-colored digraphs*, Discuss. Math. Graph Theory **31** (2011) 283–292.
doi:10.7151/dmgt.1545

[9] H. Galeana-Sánchez and V. Neumann-Lara, *On kernels and semikernels of digraphs*, Discrete Math. **48** (1984) 67–76.
doi:10.1016/0012-365X(84)90131-6

[10] H. Galeana-Sánchez and V. Neumann-Lara, *On kernel-perfect critical digraphs*, Discrete Math. **59** (1986) 257–265.
doi:10.1016/0012-365X(86)90172-X

[11] H. Galeana-Sánchez and R. Rojas-Monroy, *A counterexample to a conjecture on edge-coloured tournaments*, Discrete Math. **282** (2004) 275–276. doi:10.1016/j.disc.2003.11.015

[12] H. Galeana-Sánchez and R. Rojas-Monroy, *On monochromatic paths and monochromatic 4-cycles in edge coloured bipartite tournaments*, Discrete Math. **285** (2004) 313–318. doi:10.1016/j.disc.2004.03.005

[13] H. Galeana-Sánchez and R. Rojas-Monroy, *Monochromatic paths and at most 2-coloured arc sets in edge-coloured tournaments*, Graphs Combin. **21** (2005) 307–317. doi:10.1007/s00373-005-0618-z

[14] H. Galeana-Sánchez and R. Rojas-Monroy, *Monochromatic paths and quasi-monochromatic cycles in edge-coloured bipartite tournaments*, Discuss. Math. Graph Theory **28** (2008) 285–306. doi:10.7151/dmgt.1406

[15] H. Galeana-Sánchez and R. Rojas-Monroy, *Independent domination by monochromatic paths in arc coloured bipartite tournaments*, AKCE Int. J. Graphs Comb. **6** (2009) 267–285.

[16] H. Galeana-Sánchez, R. Rojas-Monroy and B. Zavala, *Monochromatic paths and monochromatic sets of arcs in 3-quasitransitive digraphs*, Discuss. Math. Graph Theory **29** (2009) 337–347. doi:10.7151/dmgt.1450

[17] J. von Neumann and O. Morgenstern, Theory of Games and Economic Behavior (Princeton University Press, Princeton, 1944).

[18] M. Richardson, *Solutions of irreflexive relations*, Ann. of Math. **58** (1953) 573–590. doi:10.2307/1969755

[19] M. Richardson, *Extension theorems for solutions of irreflexive relations*, Proc. Natl. Acad. Sci. USA **39** (1953) 649–655. doi:10.1073/pnas.39.7.649

[20] R. Rojas-Monroy and J.I. Villarreal-Valdés, *Kernels in infinite diraphs*, AKCE Int. J. Graphs Comb. **7** (2010) 103–111.

[21] Shen Minggang, *On monochromatic paths in m-coloured tournaments*, J. Combin. Theory Ser. B **45** (1988) 108–111. doi:10.1016/0095-8956(88)90059-7

[22] B. Sands, N. Sauer and R. Woodrow, *On monochromatic paths edge-coloured digraphs*, J. Combin. Theory Ser. B **33** (1982) 271–275. doi:10.1016/0095-8956(82)90047-8

[23] I. Włoch, *On imp-sets and kernels by monochromatic paths of the duplication*, Ars Combin. **83** (2007) 93–99.

[24] I. Włoch, *On kernels by monochromatic paths in the corona of digraphs*, Cent. Eur. J. Math. **6** (2008) 537–542. doi:10.2478/s11533-008-0044-6

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