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# UNIQUE-MAXIMUM COLORING OF PLANE GRAPHS

Igor Fabrici [1]

*Institute of Mathematics*
*P.J. Šafárik University in Košice*
*Slovakia*

and

Frank Göring

*Fakultät für Mathematik*
*Technische Universität Chemnitz*
*Germany*

**Abstract**

A unique-maximum $k$-coloring with respect to faces of a plane graph $G$ is a coloring with colors $1, \ldots, k$ so that, for each face $\alpha$ of $G$, the maximum color occurs exactly once on the vertices of $\alpha$. We prove that any plane graph is unique-maximum 3-colorable and has a proper unique-maximum coloring with 6 colors.

**Keywords:** plane graph, weak-parity coloring, unique-maximum coloring.

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## 1. Introduction

Graphs considered in this paper are simple, finite, and undirected. We use the notation and terminology of Bondy and Murty [1]. A $k$-(*vertex-*)*coloring* of a graph $G$ is a mapping $\varphi : V(G) \to \{1, \ldots, k\}$. A coloring $\varphi$ of $G$ is *proper* if, for any two adjacent vertices $x$ and $y$, $\varphi(x) \neq \varphi(y)$ holds. A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is a generalization of a graph, its (hyper-)edges are subsets of $V$ of arbitrary (positive) size. A (vertex) coloring of hypergraphs can be defined in many ways, so that restricting the definition to simple graphs coincides with proper graph coloring.

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### 1.1. Conflict-free coloring

A coloring of a hypergraph $\mathcal{H}$ is *conflict-free* (*CF*) if, for every edge $e \in \mathcal{E}(\mathcal{H})$, there is a color that occurs exactly once on the vertices of $e$. The *CF chromatic number* of $\mathcal{H}$ is the minimum $k$ for which $\mathcal{H}$ has a CF $k$-coloring. The CF coloring of hypergraphs was introduced (in a geometric setting) by Even *et al.* [8] in connection with frequency assignment problems for cellular networks. For simple graphs, Cheilaris [3] studied the *CF coloring with respect to neighborhoods*, i.e., the coloring in which, for every vertex $x$, there is a color that occurs exactly once in the neighborhood $N(x)$ or in the closed neighborhood $N[x] = N(x) \cup \{x\}$, respectively, and proved the upper bound $2\sqrt{n}$ for the CF chromatic number with respect to neighborhoods of a graph of order $n$. For closed neighbourhood, this bound was improved by Pach and Tardos [14] to $O(\log^{2+\varepsilon} n)$, for any $\varepsilon > 0$. Cheilaris and Tóth [6] and Cheilaris, Pecker, and Zachos [5] studied the *CF coloring* of a graph $G$ *with respect to paths*, i.e., the coloring in which, for every path $P$ of $G$, there is a color that occurs exactly once on the vertices of $P$. Note that the mentioned CF colorings of graphs are special cases of CF coloring of hypergraphs. For more results on CF coloring see, e.g., [6, 10, 12] and for another applications of CF coloring see [16].

### 1.2. Unique-maximum coloring

A coloring of a hypergraph $\mathcal{H}$ is *unique-maximum* (*UM*) if, for every edge $e \in \mathcal{E}(\mathcal{H})$, the maximum color on the vertices of $e$ is unique. The *UM chromatic number* of $\mathcal{H}$ is the minimum $k$ for which $\mathcal{H}$ has a UM $k$-coloring. UM coloring of hypergraphs and its relation to CF coloring was investigated by Cheilaris, Keszegh, and Pálvölgyi [4]. For simple graphs, the *UM coloring with respect to paths* requires that the maximum color on the vertices of any path is unique and it is alternatively known as *ordered coloring* or *vertex ranking*, in which for every path with equally colored end-vertices there is an internal vertex with higher color. Katchalski, McCuaig, and Seager [11] found the upper bound $3(\sqrt{6}+2)\sqrt{n}$ for UM chromatic number with respect to paths of plane graph of order $n$. For more results on UM coloring see, e.g., [5, 6].

### 1.3. Weak-parity coloring

A coloring of a hypergraph $\mathcal{H}$ is *weak-parity* (*WP*) if, for every edge $e \in \mathcal{E}(\mathcal{H})$, there is a color $c$ with the odd number of vertices of $e$ colored $c$. The *WP chromatic number* of $\mathcal{H}$ is the minimum $k$ for which $\mathcal{H}$ has a WP $k$-coloring. The WP coloring of hypergraphs was introduced (under the notion *odd coloring*) by Cheilaris, Keszegh, and Pálvölgyi [4] as a generalization of the *WP coloring* of graphs *with respect to paths* defined originally by Bunde *et al.* [2].

## 2. Colorings of Plane Graphs with Respect to Faces

Let $G$ be a plane graph with the face set $F(G)$. For a face $\alpha \in F(G)$, $V(\alpha)$ denotes the set of vertices incident with $\alpha$. The *face-hypergraph* $\mathcal{H}(G)$ of $G$ is the hypergraph with the vertex set $V(G)$ and the edge set $\{V(\alpha) : \alpha \in F(G)\}$, i.e., every face of $G$ corresponds to an edge of $\mathcal{H}(G)$ consisting of the vertices incident with this face. Kündgen and Ramamurthi [13] and Ramamurthi and West [15] considered a coloring of face-hypergraphs as the corresponding face-constrained coloring of plane graphs. Motivated by these papers, we can define the following *colorings* of plane graphs *with respect to faces* as special cases of mentioned colorings for hypergraphs.

- A *WP k-coloring* of $G$ is a colouring such that, for every face $\alpha \in F(G)$, there is a color $c$ with the odd number of vertices of $\alpha$ colored $c$. The minimum $k$ for which $G$ has a (proper) WP $k$-coloring is denoted by $\chi_{\mathrm{wp}}(G)$ ($\chi_{\mathrm{pwp}}(G)$).
- A *CF k-coloring* of $G$ is a coloring such that, for each face $\alpha \in F(G)$, there is a color that occurs exactly once on the vertices of $\alpha$. The minimum $k$ for which $G$ has a (proper) CF $k$-coloring is denoted by $\chi_{\mathrm{cf}}(G)$ ($\chi_{\mathrm{pcf}}(G)$).
- A *UM k-coloring* of $G$ is a coloring such that, for each face $\alpha$ of $G$, the maximum color (of $\alpha$) occurs exactly once on the vertices of $\alpha$. The minimum $k$ for which $G$ has a (proper) UM $k$-coloring is denoted by $\chi_{\mathrm{um}}(G)$ ($\chi_{\mathrm{pum}}(G)$).

A graph $G$ is *WP* (*CF*, *UM*) *k-colorable* if there is a WP (CF, UM) $k$-coloring of $G$.

A simple relation between the chromatic numbers we have defined is the following

**Proposition 1.** *If $G$ is a plane graph, then*
(1) $\chi_{\mathrm{wp}}(G) \le \chi_{\mathrm{cf}}(G) \le \chi_{\mathrm{um}}(G)$,
(2) $\chi(G) \le \chi_{\mathrm{pwp}}(G) \le \chi_{\mathrm{pcf}}(G) \le \chi_{\mathrm{pum}}(G)$.

Czap and Jendroľ proved the following upper bound on WP chromatic number

**Theorem 2** [7]. *If $G$ is a connected plane graph, then $\chi_{\mathrm{wp}}(G) \le 4$.*

and conjectured that this upper bound can be improved.

**Conjecture 3** [7]. *If $G$ is a connected plane graph, then $\chi_{\mathrm{wp}}(G) \le 3$.*

Moreover, they proved this conjecture for 2-connected cubic plane graphs. For the proper WP (CF) coloring, the tight upper bounds are known.

**Theorem 4** [7]**.** *If $G$ is a 2-connected plane graph, then $\chi_{\text{pcf}}(G) \leq 4$; moreover, the bound 4 is tight.*

**Corollary 5** [7]**.** *If $G$ is a 2-connected plane graph, then $\chi_{\text{pwp}}(G) \leq 4$; moreover, the bound 4 is tight.*

For results on corresponding (WP, CF, UM) chromatic indices we refer to [9]. For a better overview, in the next theorem and in the following lemma we use the colors black = 1, blue = 2, and red = 3. We prove Conjecture 3 in the following stronger form:

**Theorem 6.** *Every plane graph has a 3-coloring with colors black, blue and red such that*

(1) *each face is incident with at most one red vertex, and*

(2) *each face that is not incident with a red vertex is incident with exactly one blue vertex.*

Note that the roles of colors red and blue in this theorem are slightly assymetric. While the number of red vertices incident with a face is bounded by one, the theorem gives no bound for the number of blue vertices incident with a face (incident with a red vertex). Indeed, for $n \geq 5$, the $n$-prism (i.e., the cartesian product $C_n \Box K_2$) shows that the number of blue vertices incident with a face has to be unbounded. There are at most two red vertices in total because each vertex is incident with one of the two $n$-gons. Since each vertex is incident with only two quadrangles and each quadrangle is incident with at least one red or blue vertex, there are at least $\frac{n}{2}$ blue or red vertices in the considered coloring. Consequently, at least one of the $n$-gons is incident with at least $\frac{n}{4} - 1$ blue vertices.

To prepare the proof of this theorem, we provide the following lemma:

**Lemma 7.** *Let $G$ be a plane graph, let $xy \in E(G)$ be a selected edge of $G$ incident with the outer face, and let $c \in \{black, blue\}$. There is a 3-coloring of $G$ with colors black, blue, and red such that*

(1) *vertex $x$ has color $c$,*

(2) *vertex $y$ is black,*

(3) *each edge is incident with at most one blue vertex,*

(4) *no vertex incident with the outer face is red,*

(5) *each inner face is incident with at most one red vertex, and*

(6) *each inner face that is not incident with a red vertex is incident with exactly one blue vertex.*

The proof of Theorem 6 using Lemma 7 is as follows.

***Proof of Theorem* 6.** Let $G$ be a plane graph. Choose a vertex $z \in V(G)$ incident with the outer face and color it red. If $G - z$ is edgeless, then $G$ is a forest (i.e., it has only one face) and we can color all other vertices black. Otherwise, choose an edge $xy$ of the outer face of $G - z$, color $x$ and $y$ black, and apply Lemma 7 on $G - z$ (with the selected edge $xy$ and the color $c = \text{black}$) to obtain colors of the remaining vertices. Any face $\alpha$ of $G$ is either an inner face of $G - z$ and thus colored correctly by Lemma 7, or is incident with the red vertex $z$. Since the vertices of the outer face of $G - z$ are colored black or blue, there is no other red vertex on $\alpha$. ■

***Proof of Lemma* 7.** The proof is by induction on the number of vertices. Let $G$ be a plane graph, let $xy \in E(G)$ be a selected edge of $G$ incident with the outer face $\omega$, and let $c \in \{\text{blue}, \text{black}\}$.

*Case* 1. If $\omega$ is the only face of $G$ (i.e., if $G$ is a forest), the precoloring of $x$ and $y$ can be extended to the required coloring of $G$ by coloring all other vertices black.

*Case* 2. If $G$ is disconnected, denote $G_1$ the component of $G$ containing $xy$ and let $G_2 = G - G_1$. We apply the induction hypothesis to color $G_1$ (with the selected edge $xy$ and the selected color $c$). For an arbitrary edge $x_2y_2 \in E(G_2)$ incident with the outer face of $G_2$ (and thus incident with $\omega$ as well) we color the graph $G_2$ (with the selected edge $x_2y_2$ and the color $c_2 = \text{black}$) by induction hypothesis, or we simply color all vertices of $G_2$ black, if $G_2$ is edgeless.

Hence, we may assume that $G$ is connected and has at least two faces (i.e., $G$ has a cycle and therefore it has at least three vertices and at least three edges).

*Case* 3. Let $U \neq \emptyset$ be the set of vertices incident with no inner face of $G$ (note that, for $u \in U$, every edge incident with $u$ is a bridge of $G$).

*Case* 3.1. If there exists $u \in U \setminus \{x, y\}$, we apply induction hypothesis to color $G - u$ and finally we color $u$ black.

*Case* 3.2. If $x \in U$ and $x$ is a pendant vertex of $G$ (i.e a vertex of degree one) then $y$ has degree at least 2. Let $x'$ be a neighbor of $y$ on $\omega$ which is different from $x$. Now we color $G - x$ (with the selected edge $x'y$ and the color $c' = \text{black}$) by the induction hypothesis. Together with the vertex $x$ colored by $c$ we have a required coloring. (We proceed analogously if $y \in U$ is pendant.)

In the next two cases, let both $x$ and $y$ have degree at least 2.

*Case* 3.3. If $y \in U$ then we apply the induction hypothesis to color $G - y$ (with a selected edge $xy'$ incident with the outer face of $G - y$ and the color $c$) and finally we color $y$ black.

*Case* 3.4. For $U = \{x\}$, let $y_1, \dots, y_k$ be the neighbors of $x$ in $G$ (note that $y$ is one of them). Clearly, all these neighbors have degree at least 2. For $i \in \{1, \dots, k\}$, let $G_i$ be the component of $G - x$ containing $y_i$, let $y_i x_i$ be an edge of $G_i$ incident with the outer face of $G_i$ (and thus incident with $\omega$ as well), and let $c_i =$ black. We apply the induction hypothesis to every graph $G_i$ (with the selected edge $x_i y_i$ and the color $c_i$) and, together with the vertex $x$ colored by $c$, we obtain a required coloring.

Hence, we may assume that each vertex of $G$ is incident with an inner face of $G$.

*Case* 4. Let $B = G[V(\omega)]$ be the graph induced by the vertices incident with the outer face $\omega$ in $G$.

*Case* 4.1. If $B$ contains a cut-vertex $x_2$, then we split the graph $G$ on $x_2$ into two subgraphs $G_1$ and $G_2$ so that $xy \in E(G_1)$. More formally, let $M$ be the component of $G - x_2$ containing $x$ or $y$ (note that either $x$ and $y$ belong to the same component of $G - x_2$ or $x_2 \in \{x, y\}$), let $G_2 = G[V(G) \setminus V(M)]$, and let $G_1 = G[V(M) \cup \{x_2\}]$. Moreover, let $y_2$ be a neighbor of $x_2$ on the outerface of $G_2$. There is a required 3-coloring $\varphi_1$ of $G_1$ (with the selected edge $xy$ and the color $c$) and a required 3-coloring of $G_2$ (with the selected edge $x_2 y_2$ and the color $c_2 = \varphi_1(x_2) \in \{\text{black}, \text{blue}\}$, as $x_2$ is incident with the outer face of $G_1$), both by induction hypothesis.

*Case* 4.2. If $B$ contains an inner edge $x_2 y_2$ (i.e., an edge not incident with $\omega$—in this case, $\{x_2, y_2\}$ is a 2-vertex-cut of $G$), then we split the graph $G$ on $x_2 y_2$ into two subgraphs $G_1$ and $G_2$ so that $xy \in E(G_1)$. More formally, let $M$ be the component of $G - x_2 - y_2$ containing $x$ or $y$, let $G_2 = G[V(G) \setminus V(M)]$, and let $G_1 = G[V(M) \cup \{x_2, y_2\}]$. There is a required 3-coloring $\varphi_1$ of $G_1$ (with the selected edge $xy$ and the color $c$) and thereafter a required 3-coloring of $G_2$ (with the selected edge $x_2 y_2$ and the color $c_2 =$ black, if $\varphi_1(x_2) = \varphi_1(y_2) =$ black, or $c_2 =$ blue, if $\varphi_1(x_2) =$ blue or $\varphi_1(y_2) =$ blue, respectively), both by induction hypothesis.

Hence, we may assume that $B$ is a cycle and $y$ has a neighbor $v$ on $B$ that is different from $x$.

*Case* 4.3. If $G = B$ then we color vertex $x$ by $c$, vertex $v$ black or blue, but different from $x$, and all other vertices (inclusively $y$) black.

*Case* 4.4. If $G \neq B$, let $\alpha$ be the inner face of $G$ incident with $yv$. Because $G[V(\alpha)] \neq B$, $\alpha$ has a vertex $u \notin V(B)$ (i.e., not incident with $\omega$). We apply induction hypothesis (with the selected edge $xy$ and the color $c$) on $G - u \setminus yv$ obtained from $G$ by deleting the vertex $u$ and the edge $yv$ and finally we color $u$ red to obtain a required coloring. The vertices of the outer face of $G - u \setminus yv$ are exactly the vertices incident with $\omega$ (in $G$) together with the vertices incident with the faces containing vertex $u$ (in $G$). Obviously, none of them is colored red and therefore $\omega$ is incident with no red vertex. Any inner face of $G$ is either an inner face of $G - u \setminus yv$ and thus colored correctly by induction hypothesis, or it is incident with the red vertex $u$ (which is its unique red vertex). Moreover, there is no edge in $G$ incident with two blue vertices. Namely, every edge of $G$ is either an edge of $G - u \setminus yv$ and thus colored correctly by induction hypothesis, or it is incident with the red vertex $u$, or it is the edge $yv$, where $y$ is black. ■

With the fact that, for odd $n$, the $n$-prism is not WP 2-colorable (because for any WP 2-coloring it holds: from the pair of opposite edges of every quadrangle, one edge is monochromatic and the other one is bichromatic—a contradiction, see [7]), the following theorem is a direct consequence of Theorem 6.

**Theorem 8.** *If $G$ is a plane graph, then $\chi_{\text{wp}}(G) \le \chi_{\text{cf}}(G) \le \chi_{\text{um}}(G) \le 3$; moreover, the bound 3 is tight for all three chromatic numbers.*

With the help of the Four Color Theorem, we use Theorem 6 to prove the following upper bound on proper UM coloring.

**Theorem 9.** *If $G$ is a plane graph, then $\chi_{\text{pum}}(G) \le 6$.*

***Proof.*** Let $\varphi'$ be a UM 3-coloring of $G$ with colors black = 1, blue = 5, and red = 6, and let $\varphi''$ be a proper 4-coloring of $G$ with colors 1, 2, 3, 4. The coloring $\varphi$ defined by $\varphi(x) = \max\{\varphi'(x), \varphi''(x)\}$, for $x \in V(G)$, is a proper UM 6-coloring of $G$. ■

We believe that the following strengthening of the Four Color Theorem holds.

**Conjecture 10.** *If $G$ is a plane graph, then $\chi_{\text{pum}}(G) \le 4$.*

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