

## DENSE ARBITRARILY PARTITIONABLE GRAPHS

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### Abstract

A graph  $G$  of order  $n$  is called *arbitrarily partitionable* (AP for short) if, for every sequence  $(n_1, \dots, n_k)$  of positive integers with  $n_1 + \dots + n_k = n$ , there exists a partition  $(V_1, \dots, V_k)$  of the vertex set  $V(G)$  such that  $V_i$  induces a connected subgraph of order  $n_i$  for  $i = 1, \dots, k$ . In this paper we show that every connected graph  $G$  of order  $n \geq 22$  and with  $\|G\| > \binom{n-4}{2} + 12$  edges is AP or belongs to few classes of exceptional graphs.

**Keywords:** arbitrarily partitionable graph, Erdős-Gallai condition, traceable graph, perfect matching.

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### 1. INTRODUCTION AND MAIN RESULT

We use standard notation of graph theory (cf. [8]). In particular,  $|G|$  and  $\|G\|$  will stand for the order and the size of a graph  $G$ , respectively. The minimum degree of a vertex in a graph  $G$  will be denoted by  $\delta(G)$ . By  $c(G)$  we denote the *circumference* of a graph  $G$ , i.e., the length of a longest cycle. If  $G$  and  $H$  are two graphs with disjoint vertex sets, then the *join* of  $G$  and  $H$  is the graph, denoted by  $G \vee H$ , with the vertex set  $V(G \vee H) = V(G) \cup V(H)$  and the edge set

$$E(G \vee H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}.$$

A sequence  $(n_1, \dots, n_k)$  of positive integers is called *admissible* for a graph  $G = (V, E)$  of order  $n$  if  $n_1 + \dots + n_k = n$ . An admissible sequence is said to be *realizable in  $G$*  if there exists a partition of  $V$  into  $k$  parts  $(V_1, \dots, V_k)$  such that  $|V_i| = n_i$  and the subgraph  $G[V_i]$  induced by  $V_i$  is connected, for every  $i = 1, \dots, k$ . Such a partition is called a *realization* of the sequence  $(n_1, \dots, n_k)$  in  $G$ . Note that in fact the ordering of  $(n_1, \dots, n_k)$  is irrelevant, i.e., if this sequence is realizable in  $G$ , then it is also realizable after any permutation of its elements. We say that  $G$  is *arbitrarily partitionable* (AP for short) if every admissible sequence is realizable in  $G$ .

A simple example of an arbitrarily partitionable graph is a path  $P_n$ . Two obvious and well-known facts play a key role in this paper.

**Proposition 1.** *If  $G$  has a spanning subgraph which is AP, then  $G$  is AP itself.*

**Proposition 2.** *Every traceable graph is AP.*

The following easy observation sometimes makes proofs shorter and allows us to assume throughout the paper that every admissible sequence has all elements greater than 1.

**Proposition 3** [15]. *A graph  $G$  is AP if and only if every admissible sequence  $(n_1, \dots, n_k)$  with  $n_i \geq 2$  for  $i = 1, \dots, k$  is realizable in  $G$ .*

The notion of AP graphs was introduced by Barth, Baudon and Puech [1] (and independently by Horňák and Woźniak [13]) to model a problem in the design of computer networks (see [1] for details). The concept of arbitrarily partitionable graphs, sometimes also called *arbitrarily vertex decomposable* or *fully decomposable* or just *decomposable*, has spawned numerous papers. Some of them investigate AP graphs within some classes of graphs (e.g., [1, 2, 9, 7, 13], KPWZ1). Horňák, Tuza and Woźniak [14] introduced the notion of *on-line arbitrarily partitionable* graphs, and then a few other definitions strengthening the condition for AP graphs appeared (e.g., [5, 6, 3, 16]). Here we present only those previous results on AP graphs we make use of in the paper.

A sequence  $(d, \dots, d)$  of length  $\lambda$  will be denoted by  $(d)^\lambda$ . A caterpillar with three leaves is denoted by  $\text{Cat}(a, b)$  if it is obtained from the star  $K_{1,3}$  by substituting two of its edges by paths of orders  $a$  and  $b$ , respectively (see Figure 1). As  $b = n - a$ , we will later also use a shorter notation  $\text{Cat}(a)$ . The following result was proved by Barth *et al.* [1], and independently by Horňák and Woźniak [13].

**Theorem 4.** *The caterpillar  $\text{Cat}(a, b)$ , with  $2 \leq a \leq b$ , is AP if and only if  $a$  and  $b$  are relatively prime. Moreover, each admissible and nonrealizable sequence is of the form  $(d)^k$ , where  $a \equiv b \equiv 0 \pmod{d}$  and  $d > 1$ .*

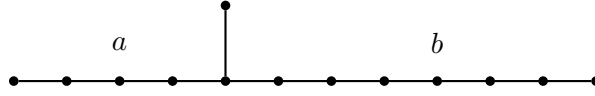


Figure 1.  $\text{Cat}(a, b)$  with  $a = 5, b = 8$ .

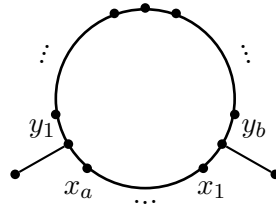


Figure 2.  $\text{Sun}(a, b)$ .

A *sun with  $r$  rays* is a graph of order  $n \geq 2r$  with  $r$  pendant vertices  $u_1, \dots, u_r$  whose deletion yields a cycle  $C_{n-r}$ , and each vertex  $v_i$  on  $C_{n-r}$  adjacent to  $u_i$  is of degree three. If the sequence of vertices  $v_i$  is situated on the cycle  $C_{n-r}$  in such a way that there are exactly  $a_i \geq 0$  vertices, each of degree two, between  $v_i$  and  $v_{i+1}, i = 1, \dots, r$  (the indices taken modulo  $r$ ), then this sun is denoted by  $\text{Sun}(a_1, \dots, a_r)$ . Suns with two and three rays are presented in Figures 2 and 3, respectively. Kalinowski, Piłśniak, Woźniak and Ziolo characterized all AP suns with at most three rays.

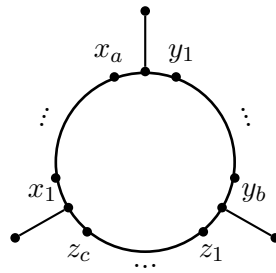


Figure 3.  $\text{Sun}(a, b, c)$ .

**Theorem 5** [15]. *A sun with two rays  $\text{Sun}(a, b)$  is AP if and only if at most one of the numbers  $a$  and  $b$  is odd. Moreover,  $\text{Sun}(a, b)$  of order  $n$  is not AP if and only if  $(2)^{n/2}$  is the unique admissible and nonrealizable sequence.*

**Theorem 6** [15]. *A sun with three rays  $\text{Sun}(a, b, c)$  is AP if and only if none of the following three conditions is fulfilled:*

- (1) *at most one of the numbers  $a, b, c$  is even,*
- (2)  $a \equiv b \equiv c \equiv 0 \pmod{3}$ ,
- (3)  $a \equiv b \equiv c \equiv 2 \pmod{3}$ .

*Moreover, if  $\text{Sun}(a, b, c)$  is not AP, then at least one of the following three sequences  $(2)^{n/2}$ ,  $(3)^{n/3}$ ,  $(3, (2)^{(n-3)/2})$  is admissible and nonrealizable.*

In this paper we consider the following question. How many edges in a connected graph  $G$  guarantee that a graph is AP or belongs to few families of exceptional graphs?

Dense AP graphs were already investigated in another context. This was initiated by Marczyk who proved in [18], [19] some Ore-type sufficient conditions for a graph to be AP. The best result in this direction is due to Horňák, Marczyk, Schiermeyer and Woźniak.

**Theorem 7** [12]. *Every connected graph  $G$  of order  $n \geq 20$  such that the degree sum of each pair of nonadjacent vertices is at least  $n - 5$  is AP if and only if  $G$  admits a perfect matching or a quasi-perfect matching (i.e., a matching omitting exactly one vertex).*

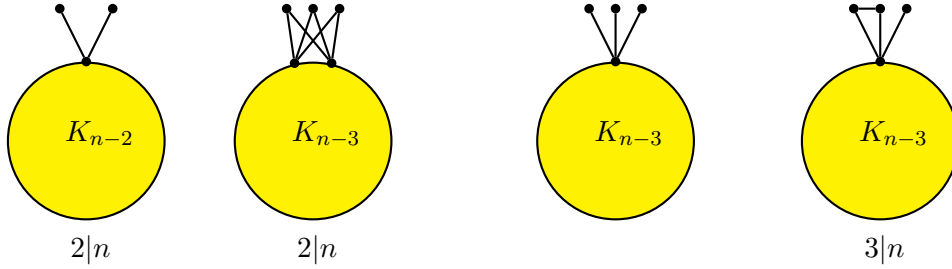


Figure 4. Four graphs such that every non-AP graph  $G$  with  $\|G\| > \binom{n-4}{2} + 12$  is a spanning subgraph of one of them (below each graph, requirements on the order  $n$  are given).

Let us formulate now our main result.

**Theorem 8.** *If  $G$  is a connected graph of order  $n \geq 22$  and size*

$$\|G\| > \binom{n-4}{2} + 12,$$

then  $G$  is AP unless  $G$  is a spanning subgraph of one of the graphs depicted in Figure 4.

It is easily seen that none of four graphs in Figure 4 is AP whenever its order  $n$  meets the divisibility condition given below the graph. By Proposition 1, every spanning subgraph is non-AP, as well. Observe also that the first two graphs have circumference  $c(G) = n - 2$  and the other two have  $c(G) = n - 3$ .

It has to be noted that for  $n < 22$ , there are more graphs of order  $n$  and size greater than  $\binom{n-4}{2} + 12$  that are not AP. For example, the graph  $G = K_{(n-2)/2} \vee \overline{K}_{(n+2)/2}$  has no perfect matching, and its size  $\|G\| = \frac{1}{2}[\frac{n-2}{2}(n-1) + \frac{n+2}{2} \cdot \frac{n-2}{2}]$  is greater than  $\binom{n-4}{2} + 12$  for every even  $n = 10, \dots, 20$ . Another example is the graph  $G = K_{(n-3)/2} \vee \overline{K}_{(n+3)/2}$  which has no realization of the sequence  $(3, (2)^{\frac{n-3}{2}})$ , and its size  $\|G\| = \frac{1}{2}[\frac{n-3}{2}(n-1) + \frac{n+3}{2} \cdot \frac{n-3}{2}]$  is greater than  $\binom{n-4}{2} + 12$  for every odd  $n = 11, \dots, 17$ .

## 2. PRELIMINARY RESULTS

This section contains an initial stage of the proof of Theorem 8. We will make use of some classical sufficient conditions for the existence of long cycles in a graph.

**Theorem 9** (Erdős, Gallai [11]). *Let  $G$  be a graph of order  $n$ . If  $\|G\| > \frac{c}{2}(n-1)$ , then  $c(G) > c$ .*

Theorem 9 has been extended by Woodall.

**Theorem 10** (Woodall [20]). *Let  $G$  be a graph of order  $n = t(c-1) + p$ , where  $c \geq 2$ ,  $t \geq 0$  and  $1 \leq p \leq c$ . If*

$$\|G\| > t \binom{c}{2} + \binom{p}{2},$$

then  $c(G) > c$ .

Taking  $t = 1$ ,  $c = n - \delta$  and  $p = \delta + 1$ , we obtain the following

**Corollary 11.** *If  $n = |G|$ ,  $\delta = \delta(G)$  and*

$$\|G\| > \binom{n-\delta}{2} + \binom{\delta+1}{2},$$

then  $c(G) > n - \delta$ .

The next theorem is the well-known Erdős sufficient condition for hamiltonicity depending on the size and minimum degree.

**Theorem 12** (Erdős [10]). *Let  $G$  be a graph of order  $n$  and with minimum degree  $\delta$ . Denote*

$$f(n, \delta) = \max \left\{ \binom{n-\delta}{2} + \delta^2, \binom{n - \lfloor \frac{n-1}{2} \rfloor}{2} + \left\lfloor \frac{n-1}{2} \right\rfloor^2 \right\}.$$

*If  $\delta \geq \frac{n}{2}$  or  $\|G\| > f(n, \delta)$ , then  $G$  is Hamiltonian.*

We can use Theorem 12 for traceability as follows. Let  $H = G \vee K_1$ . Then  $H$  is Hamiltonian if and only if  $G$  is traceable. Denote  $g(n, \delta) = f(n+1, \delta+1) - n$ . Thus

$$g(n, \delta) = \max \left\{ \binom{n-\delta}{2} + (\delta+1)^2 - n, \binom{n+1 - \lfloor \frac{n}{2} \rfloor}{2} + \left\lfloor \frac{n}{2} \right\rfloor^2 - n \right\}.$$

As  $\binom{n-\delta}{2} + (\delta+1)^2 - n = \binom{n-\delta-1}{2} + \delta(\delta+1)$ , this justifies the following result.

**Corollary 13.** *Let  $G$  be a graph of order  $n$  and with minimum degree  $\delta$ . If  $\delta \geq \frac{n-1}{2}$  or*

$$\|G\| > \max \left\{ \binom{n-\delta-1}{2} + \delta(\delta+1), \binom{n+1 - \lfloor \frac{n}{2} \rfloor}{2} + \left\lfloor \frac{n}{2} \right\rfloor^2 - n \right\},$$

*then  $G$  is traceable, and hence AP.*

Suppose  $G$  is a graph with minimum degree  $\delta$  and with  $\|G\| > \binom{n-4}{2} + 12$ . It follows from Corollary 13 that  $G$  is traceable whenever  $\delta \geq \frac{n-1}{2}$  or  $g(n, \delta) \leq g(n, 3)$ . Observe that  $\binom{n-\delta-1}{2} + \delta(\delta+1)$  is a quadratic polynomial with respect to  $\delta$ , so the latter inequality holds unless  $g(n, \delta) = \binom{n+1 - \lfloor \frac{n}{2} \rfloor}{2} + \left\lfloor \frac{n}{2} \right\rfloor^2 - n$  and

$$\binom{n-4}{2} + 12 < \binom{n+1 - \lfloor \frac{n}{2} \rfloor}{2} + \left\lfloor \frac{n}{2} \right\rfloor^2 - n.$$

We solve this inequality regarding to the parity of the order  $n$  of  $G$ . If  $n$  is even, then the inequality is equivalent to  $n^2 - 30n + 176 < 0$ , so it holds only if  $9 \leq n \leq 21$ . If  $n$  is odd, then we have  $n^2 - 24n + 175 < 0$ , and this does not hold for any  $n$ .

Obviously, every connected graph  $G$  with  $c(G) = n - 1$  is traceable, and hence AP.

Thus, Corollary 11 and Corollary 13 for  $\delta = 3$  imply that for the proof of our main result we are left with the following situation

$$n \geq 22, \quad \|G\| > \binom{n-4}{2} + 12, \quad 1 \leq \delta(G) \leq 2, \quad \text{and } n-3 \leq c(G) \leq n-2.$$

The rest of our proof is divided into two parts corresponding to  $c(G) = n - 2$  (Section 3) and  $c(G) = n - 3$  (Section 4).

Let us state yet a lemma that follows the approach in [17] and will be used in both sections. First, we introduce some notation. If  $C$  is a cycle in a graph  $G = (V, E)$ , then each vertex of  $C$  adjacent to a vertex outside  $C$  is called an *attachment vertex*. Fix an orientation of  $C$ . For two vertices  $x, y \in V(C)$  we denote by  $C[x, y]$  the path of  $C$  from  $x$  to  $y$  along this orientation, and by  $\overleftarrow{C}[x, y]$  the path from  $x$  to  $y$  along the reverse orientation of  $C$ . For a vertex  $x \in V(C)$  we denote by  $x^+, x^-$  its successor and its predecessor along the orientation of  $C$ . We also denote  $d_C(x) = |N(x) \cap V(C)|$ . For two sets  $A, B \subset V$ , let  $E(A, B) = \{xy \in E : x \in A, y \in B\}$ .

**Lemma 14.** *Let  $G = (V, E)$  be a connected graph of order  $n \geq 22$  with  $\delta(G) \leq 2$  and  $\|G\| > \binom{n-4}{2} + 12$ . Let  $C$  be a longest cycle in  $G$  such that the set  $V \setminus V(C)$  is not a clique.*

- (1) *If  $c(G) = n - 3$ , then each vertex outside  $C$  is of degree one.*
- (2) *If  $c(G) = n - 2$ , then each vertex outside  $C$  has at most three neighbors on  $C$ .*

**Proof.** Let  $k = k(G) = \max\{d_C(u) : u \in V \setminus V(C)\}$ , and let  $u$  be a vertex outside  $C$  with  $d_C(u) = k$  and  $N(u) \cap V(C) = \{u_1, \dots, u_k\}$ . Fix an orientation of  $C$ . Clearly,  $k \leq \frac{c(G)}{2}$  and the set  $X = \{u_1^+, \dots, u_k^+\}$  is independent since  $C$  is a longest cycle in  $G$ . Moreover, for any pair  $u_i^+, u_j^+$  with  $i \neq j$  and any  $z \in C[u_i^{++}, u_j]$  we have  $u_i^+ z^+ \notin E$  or  $z u_j^+ \notin E$ , otherwise  $C$  would not be a longest cycle. Let  $C_1 = C[u_i^{++}, u_j], C_2 = C[u_j^{++}, u_i]$ . Then a classical counting argument (cf. [8]) shows that

$$\begin{aligned} d_C(u_i^+) + d_C(u_j^+) &= d_{C_1}(u_i^+) + d_{C_1}(u_j^+) + d_{C_2}(u_i^+) + d_{C_2}(u_j^+) \\ &\leq |V(C_1)| + 1 + |V(C_2)| + 1 = |V(C)|. \end{aligned}$$

Summing up this inequality for all  $\binom{k}{2}$  possible pairs of vertices and dividing by  $k - 1$  we obtain

$$\sum_{i=1}^k d_C(u_i^+) \leq \frac{k}{2} c(G).$$

Now we want to estimate  $|\bar{E}(C)|$ , i.e., the number of edges within  $C$  that are missing in  $G$ . Since  $X$  is independent, all edges incident to vertices from  $X$  are contained in  $E(X, V(C) \setminus X)$ . Hence  $|\bar{E}(C)| \geq |\bar{E}(X, V(C) \setminus X)| + |\bar{E}(G[X])| \geq k(c(G) - k) - \frac{k}{2}c(G) + \binom{k}{2} = \frac{k}{2}(c(G) - k - 1)$ . As  $V \setminus V(C)$  is not a clique and each vertex of  $V \setminus V(C)$  is connected to  $C$  by at most  $k$  edges, the number  $f(k) = \|\bar{G}\|$  of edges missing in the graph  $G$  satisfies the inequality

$$f(k) \geq 1 + (n - c(G))(c(G) - k) + \frac{k}{2}(c(G) - k - 1).$$

However, it is not difficult to see that if  $k = k(G)$  and  $f(k) = 1 + (n - c(G))(c(G) - k) + \frac{k}{2}(c(G) - k - 1)$  for a graph  $G$ , then  $\delta(G) = d(u) = k + n - c(G) - 2$  where  $u$  is a vertex outside  $C$ . Note that  $k + n - c(G) - 2 \geq k + 1$  if  $c(G) \leq n - 3$ . But  $\delta(G) \leq 2$  by assumption, hence we have to increase  $f(k)$  by  $k - 1$ , so actually

$$f(k) \geq (n - c(G))(c(G) - k) + \frac{k}{2}(c(G) - k + 1).$$

Note that  $\binom{n}{2} - \binom{n-4}{2} - 12 = 4n - 22$ , hence  $f(k) \leq 4n - 23$  since  $\|G\| > \binom{n-4}{2} + 12$ .

Consider first the case  $c(G) = n - 3$ . Then  $f(k) \leq 3(n - k - 3) + \frac{k}{2}(n - k - 2)$ . Suppose, contrary to the claim, that  $k \geq 2$ . We search for the smallest value of  $f(k)$ . The derivative  $f'(k) = \frac{n}{2} - k - 4$  is nonnegative for  $2 \leq k \leq \frac{n}{2} - 4$ . Hence  $f(k)$  is increasing for  $2 \leq k \leq \frac{n}{2} - 4$ , and decreasing for  $\frac{n}{2} - 4 \leq k \leq \frac{n-3}{2}$ . We have  $f(2) = 4n - 19 > 4n - 23$ . Also,  $f(\frac{n-3}{2}) = \frac{n^2 + 8n - 33}{8}$ , and  $f(\frac{n-3}{2}) - (4n - 23) = \frac{1}{8}(n^2 - 24n + 151) > 0$  for any  $n \geq 14$ . Thus  $f(k) > 4n - 23$  for every  $k$  with  $2 \leq k \leq \frac{n-3}{2}$ , a contradiction.

Now, let  $c(G) = n - 2$ . Thus  $f(k) = 2(n - k - 2) + \frac{k}{2}(n - k - 1)$  and  $f'(k) = \frac{n-5}{2} - k$ . Hence  $f(k)$  is increasing for  $2 \leq k \leq \frac{n-5}{2}$ , and decreasing for  $\frac{n-5}{2} \leq k \leq \frac{n-2}{2}$ . Note that  $f(4) = \frac{4}{n} - 22 > 4n - 23$ , and  $f(\frac{n-2}{2}) = \frac{1}{8}(n^2 + 6n - 16) > 4n - 23$  because  $\frac{1}{8}(n^2 + 6n - 16) - (4n - 23) = \frac{1}{8}(n^2 - 26n + 168) > 0$ . Therefore  $k \leq 3$ . ■

In most cases considered in the next two sections, we apply the following strategy. To prove that a graph  $G = (V, E)$  satisfying certain conditions has no more than  $\binom{n-4}{2} + 12$  edges, we choose a graph  $G_0$  such that  $V(G_0) = V$ ,  $\|G_0\| \leq \binom{n-4}{2} + 12$ , and there exists an injective mapping of  $E \setminus E(G_0)$  into  $E(G_0) \setminus E$ , whence  $\|G\| \leq \|G_0\|$ .

### 3. PROOF FOR CIRCUMFERENCE $n - 2$

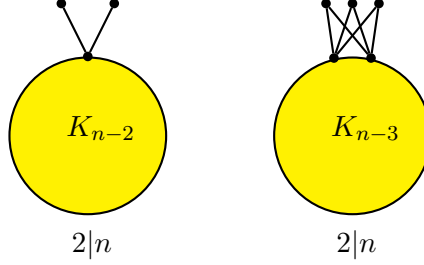
To prove that Theorem 8 holds for graphs with circumference  $n - 2$ , it is enough to justify the following.

**Proposition 15.** *If  $G = (V, E)$  is a connected graph of order  $n \geq 22$  with  $c(G) = n - 2$  and  $\|G\| > \binom{n-4}{2} + 12$ , then  $G$  is AP unless  $n$  is even and  $G$  is a spanning subgraph of one of two graphs of even order shown in Figure 5.*

**Proof.** Let  $C$  be a longest cycle in  $G$  and let  $u, v$  be the two vertices outside  $C$ . Clearly,  $G$  is traceable if  $uv \in E$ . Then assume  $uv \notin E$ .

First suppose that there is only one attachment vertex. If  $n$  is even, then the sequence  $(2)^{n/2}$  is not realizable,  $G$  is not AP and is a spanning subgraph of the first graph in Figure 5 when  $\|G\| > \binom{n-4}{2} + 12$  what is possible for  $n \geq 10$ .



Figure 5. Exceptional supergraphs with circumference  $n - 2$ .

If  $n$  is odd, then an admissible sequence contains an element  $n_i \geq 3$ . We take a part  $V_i$  containing  $u, v$  and their common neighbour, and the remaining graph is traceable, so  $G$  is AP.

Now assume that there are at least two attachment vertices. For every pair of independent edges  $uu', vv'$  with  $u', v' \in V(C)$ , the deletion of  $u', v'$  from  $C$  yields two paths of orders  $a$  and  $b$  such that  $a + b = n - 4$  and  $0 \leq a \leq b \leq n - 4$ . Thus  $\text{Sun}(a, b)$  is a spanning subgraph of  $G$ . By Theorem 5, the graph  $G$  is AP when at most one of the numbers  $a, b$  is odd (in particular when  $n$  is odd). Henceforth, we assume that  $n$  is even and both  $a$  and  $b$  are odd for any pair of independent edges  $uu', vv'$ . Again, Theorem 5 implies that to prove that  $G$  is AP, it suffices to show that the sequence  $(2)^{n/2}$  is realizable in  $G$ , i.e.,  $G$  admits a perfect matching. Choose edges  $uu', vv'$  such that  $a$  is as large as possible (and not greater than  $b$ ), and denote the vertices of  $C$  by  $u', x_1, \dots, x_a, v', y_1, \dots, y_b$  according to the orientation of  $C$ . Suppose that  $G$  is not AP.

*Case  $a = 1$ .* Suppose first that there are only two attachment vertices. Then  $d(u) \leq 2$  and  $d(v) \leq 2$ . Let  $d(x_1) = 2$ . If  $n$  is even, then  $G$  has no perfect matching and is a spanning subgraph of the second graph in Figure 5 whenever  $\|G\| > \binom{n-4}{2} + 12$ , and the latter inequality may hold for  $n \geq 12$ .

Then assume that  $d(x_1) \geq 3$ , i.e.,  $C$  has at least one chord incident to  $x_1$ . We will show that in this case there does not exist a non-AP graph satisfying our assumptions. Indeed, suppose there exists such a graph  $G$ . First observe that  $x_1$  cannot be adjacent to any vertex  $y_{2l-1}$  since otherwise  $G$  would have a perfect matching:  $\{uu', vv', x_1 y_{2l-1}\} \cup \{x_{2i-1} x_{2i} : i = 1, \dots, l-1\} \cup \{x_{2i} x_{2i+1} : i = l, \dots, \frac{b-1}{2}\}$ . Suppose  $l$  is the smallest positive integer such that  $x_1 y_{2l} \in E$ . Without loss of generality, we may assume that  $2l < \frac{b}{2}$  (we can change the orientation of  $C$ , if necessary), i.e.,  $l \leq \frac{n-4}{4}$ . For any  $i \leq l$  and  $j \geq l$ , an edge

$y_{2i-1}y_{2j+1}$  would give a perfect matching if it appeared in  $G$ . The number of these edges equals  $l(\frac{n-4}{2} - l) \geq l^2$ , and they are missing in  $G$ . Moreover, for every  $p > l$ , an edge  $x_1y_{2p} \in E$  creates a new missing edge  $y_{2p-1}y_{2p+1}$ , and an edge  $y_1y_{2p} \in E$ , except for  $p = \frac{n-6}{2}$ , creates another missing edge  $y_{2p-1}y_{2p+3}$ . Hence  $\|G\| \leq \binom{n-4}{2} + 8 + 2l - 2 - l(\frac{n-4}{2} - l) + 1 \leq \binom{n-4}{2} + 9 + 2l - l^2 \leq \binom{n-4}{2} + 9$ , a contradiction.

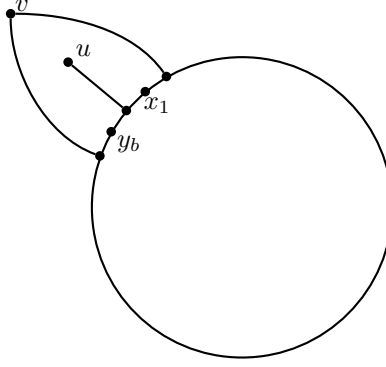


Figure 6. Three attachment vertices for  $a = 1$ .

It is easily seen that the number of attachment vertices can be at most three as  $a = 1$  was chosen greatest possible. Then one of the vertices outside  $C$ , say  $u$ , is a pendant vertex and  $v$  is adjacent to  $y_{b-1}$  (see Figure 6). Suppose that  $G$  satisfies our assumptions and has no perfect matching. Then clearly,  $x_1y_b$ , as well as  $x_1y_{2i+1}$  and  $y_by_{2i+1}$  cannot belong to  $E$ . Consider a graph  $G_0$  of size  $\binom{n-4}{2} + 10$  such that  $V(G_0) = V$ , the set  $V \setminus \{u, v, x_1, y_b\}$  is a clique, and  $E(G_0)$  contains also the edges  $uu', vv', vv', vy_{b-1}, v'x_1, x_1u', u'y_b, y_by_{b-1}, x_1y_{b-1}, ybv'$ . For every  $l = 1, \dots, \frac{b-3}{2}$ , whenever  $x_1y_{2l}$  belonged to  $E$ , the edge  $y_{2l-1}y_{2l+1}$  would create a perfect matching in  $G$ , thus it is missing in  $G$ , and whenever  $y_by_{2l} \in E$  (except  $2l = b - 3$ ), then  $y_{2l-1}y_{2l+3}$  is missing in  $G$ . Therefore  $\|G\| \leq \|G_0\| + 1 = \binom{n-4}{2} + 11 < \binom{n-4}{2} + 12$ , a contradiction.

*Case  $a \geq 3$ .* It follows from Lemma 14 that the vertices  $u, v$  are of degree at most three, since the total number of vertices incident to them cannot be greater than six. Let  $G_1$  be a graph of size  $\binom{n-4}{2} + 12$  containing these six edges, six edges  $x_1x_2, x_1u', x_1v', x_ax_2, x_a u', x_a v'$  and  $\binom{n-4}{2}$  edges of the clique  $V \setminus \{u, v, x_1, x_a\}$ . If  $a = 3$ , we set  $G_0 = G_1$ . For  $a \geq 5$ , we define  $G_0$  as follows. We add to  $G_1$  the edges  $x_1x_{2j}, x_ax_{2j}$ ,  $j = 2, \dots, \frac{a-1}{2}$ , and delete the edges  $x_{2i-1}y_{2j-1}$ ,  $i = 2, \dots, \frac{a-1}{2}$ ,  $j = 1, \dots, \frac{b+1}{2}$ . Thus the number of added edges equals  $a - 3$ , and the number of deleted ones equals  $\frac{a-1}{2} \cdot \frac{b+1}{2}$  and is not smaller than  $\frac{a^2-1}{4}$  since  $b \geq a$ . Therefore  $\|G_0\| < \|G_1\|$ . Observe that the set  $\{x_{2i} : i =$

$1, \dots, \frac{a-1}{2}\} \cup \{y_i : i = 1, \dots, b\} \cup \{u', v'\}$  forms a clique in  $G_0$ , and the only edges that may appear in  $G$  and not in  $G_0$  are of the form  $x_\nu x_{2j-1}$  or  $x_\nu y_{2l}$ , where  $\nu \in \{1, a\}$ .

For any  $i = 1, \dots, \frac{a-1}{2}$ , if  $x_1 x_{2i+1} \in E$ , then  $y_1 x_{2i} \notin E$ , and if  $x_a x_{2i-1} \in E$ , then  $y_b x_{2i} \notin E$ , otherwise  $G$  has a perfect matching. For any  $j = 1, \dots, \frac{b-1}{2}$ , if  $x_1 y_{2j} \in E$ , then  $y_1 y_{2j+1} \notin E$ , and if  $x_a y_{2j} \in E$ , then  $y_b y_{2j-1} \notin E$ . It is easy to see that we have just defined an injective mapping of  $E \setminus E(G_0)$  into  $E(G_0) \setminus E$  unless the edge  $y_1 y_b$  was counted twice as a missing edge in  $E$ . This means that either  $\|G\| \leq \|G_0\| \leq \binom{n-4}{2} + 12$  or  $\|G\| = \|G_0\| + 1 = \binom{n-4}{2} + 13$ . But it is easy to see that in the latter case  $\delta(G) = 3$ , so  $G$  is traceable by Corollary 13. We thus obtained a contradiction in both cases. ■

#### 4. PROOF FOR CIRCUMFERENCE $n - 3$

In this section we accomplish the proof of our main result by showing that Theorem 8 holds for graphs with circumference  $n - 3$ . Let us introduce some additional notation first.

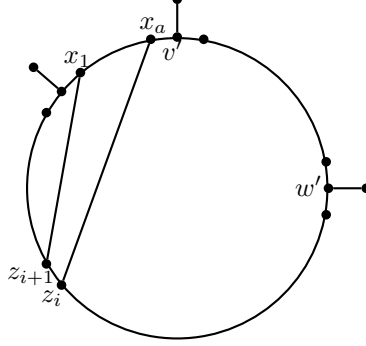
For any two vertices  $x$  and  $y$  of a cycle  $C$  of a sun  $S$  with a fixed orientation, we denote by  $xCy$  the caterpillar consisting of a path  $C[x, y]$  together with the leaves of  $S$  if the corresponding attachment vertex belongs to  $C[x, y]$ . By  $y\overleftarrow{C}x$  we denote the same caterpillar but in the reverse order.

Let  $G = (V, E)$  be a connected graph of size  $\|G\| > \binom{n-4}{2} + 12$ , and let  $C$  be a longest cycle of  $G$  of length  $n - 3$ . Lemma 14 states that each of three vertices  $u, v, w$  outside  $C$  has at most one neighbor on  $C$ . The attachment vertices of  $C$  adjacent to  $u, v, w$  are denoted by  $u', v', w'$ , respectively (some of the vertices  $u', v', w'$  may coincide or do not exist if there are less than three attachment vertices). If  $C$  has three attachment vertices, denote the vertices of  $C$  by  $u', x_1, \dots, x_a, v', y_1, \dots, y_b, w', z_1, \dots, z_c$  according to a fixed orientation of  $C$ . Let  $X = \{x_i : i = 1, \dots, a\}$ ,  $Y = \{y_i : i = 1, \dots, b\}$ ,  $Z = \{z_i : i = 1, \dots, c\}$ . If  $C$  has only two attachment vertices, then we assume that  $Z$  is empty, and  $C$  is the sequence  $u', x_1, \dots, x_a, v', y_1, \dots, y_b$ .

If  $a \geq 1$  and  $b \geq 2$ , then two edges of the form  $x_1 y_{i+1}, x_a y_i$  are said to be a *good couple* from  $X$  to  $Y$ . The case  $a = 1$  is allowed. Analogously we define good couples from  $Y$  to  $X$ , from  $X$  to  $Z$  and so on (see Figure 7).

**Lemma 16.** *Let  $C$  be a longest cycle of a connected graph  $G = (V, E)$  of size  $\|G\| > \binom{n-4}{2} + 12$  and circumference  $c(G) = n - 3$ . If the number of attachment vertices of  $C$  is two and their distance on  $C$  is at least three, then there exists a good couple of edges in  $G$ .*

**Proof.** Using the notation from the beginning of this section, we may assume without loss of generality that  $w$  is a vertex adjacent to  $v$  or  $v'$ . Then  $G - w$  is

Figure 7. A good couple of edges from  $X$  to  $Z$ .

spanned by  $\text{Sun}(a, b)$  where  $2 \leq a \leq b$  and  $a + b = n - 5$ . There are three or four edges outside  $C$ , therefore the number of chords of  $C$  missing in the graph  $G$  is less than  $n + 12$ . Indeed,  $\binom{n-3}{2} + 4 - \binom{n-4}{2} - 12 = n - 12$ .

Suppose that there is no good couple of edges in  $G$ . Then for every  $i = 1, \dots, b - 1$ , if  $x_1 y_{i+1} \in E$ , then  $x_a y_i \notin E$ . It follows that the number of missing edges between  $\{x_1, x_a\}$  and  $Y$  is at least  $b - 1$ . Moreover, for every  $i = 2, \dots, a - 1$ , if  $y_1 x_{i+1} \in E$ , then  $y_b x_i \notin E$ . Therefore the total number of missing chords between  $X$  and  $Y$  is at least  $a - 3 + b - 1 = n - 9 > n - 12$ , a contradiction. ■

**Lemma 17.** *Let  $C$  be a longest cycle of a connected graph  $G = (V, E)$  of size  $\|G\| > \binom{n-4}{2} + 12$  and circumference  $c(G) = n - 3$ . If  $C$  has three attachment vertices and no two of them are consecutive vertices on  $C$ , then there exists a good couple of edges in  $G$ .*

**Proof.** By assumptions, the graph  $G$  is spanned by  $\text{Sun}(a, b, c)$  where  $1 \leq a \leq b \leq c$  and  $a + b + c = n - 6$ . Lemma 14 implies that there are exactly three edges outside  $C$ . Hence, there are less than  $n - 12$  chords of  $C$  missing in  $G$ . Assume that  $G$  has no good couple of edges.

Suppose first that  $a = b = 1$ . Then  $x_1 z_{i+1} \in E$  implies  $x_1 z_i \notin E$ ,  $i = 1, \dots, c - 1$ , otherwise these two edges would be a good couple. Therefore, there are at least  $\frac{c-1}{2}$  missing edges from  $x_1$  to  $Z$ . Analogously, the number of missing edges between  $y_1$  and  $Z$  is not less than  $\frac{c-1}{2}$ . Altogether, we get at least  $c - 1 = n - 9 > n - 12$  missing chords of  $C$ , a contradiction.

Suppose now that  $a = 1$  and  $b \geq 2$ . We analogously infer that there are at least  $\frac{c-1}{2}$  missing edges from  $x_1$  to  $Z$ . For any  $i = 1, \dots, c - 1$ , whenever  $y_1 z_{i+1}$  is an edge in  $G$ , then  $y_b z_i$  is not, for, otherwise these two edges would be a good couple. Thus there are at least  $c - 1$  chords of  $C$  between  $\{y_1, y_b\}$  and  $Z$  missing in  $G$ . Furthermore,  $z_1 y_{i+1} \in E$  implies  $z_c y_i \notin E$  for  $i = 2, \dots, b - 1$ , so we get additional  $b - 3$  missing edges between  $Y$  and  $Z$ . Hence there are at least

$\frac{c-1}{2} + c - 1 + b - 3 \leq \frac{3}{2}c + b - 5 = n - 12 + \frac{c}{2} > n - 12$ , again a contradiction.

Finally, let  $a \geq 2$ . Denote by  $\rho$  the number of edges joining  $x_a$  with the set  $Z$ . Then  $c - \rho$  edges between  $x_a$  and  $Z$  are missing. Moreover, for each edge  $x_a z_i$ , the edge  $x_1 z_{i+1}$  is missing. Therefore, at least  $\rho$  edges between  $x_1$  and  $Z$  are missing. So, since there is no good couple from  $X$  to  $Z$ , at least  $(c - \rho) + \rho = c$  edges joining the vertices  $x_1$  and  $x_a$  with  $Z$  are missing. Analogously, since there is no good couple from  $X$  to  $Y$  as well as  $Y$  to  $Z$  we can show that there are at least  $b$  missing edges joining the vertices  $x_1$  and  $x_a$  with  $Y$  and at least  $c$  missing edges joining the vertices  $y_1$  and  $y_b$  with  $Z$ . Therefore, there are at least  $c + (b + c)$  missing edges, and since  $b + c \geq \frac{2}{3}(n - 6)$  and  $c \geq \frac{1}{3}(n - 6)$ , we have at least  $n - 6 > n - 12$  missing edges, a contradiction. ■

Let  $C$  have only one attachment vertex  $u'$ . If the subgraph  $G[\{u, v, w\}]$  induced by  $u, v, w$  is traceable, then  $G$  is traceable itself. If all three vertices  $u, v, w$  are pendant in  $G$ , then  $G$  is a spanning subgraph of the third graph in Figure 4 and is not AP for any  $n$  because either  $(2)^{n/2}$  or  $(3, (2)^{(n-3)/2})$  is an admissible and nonrealizable sequence. Otherwise,  $G[\{u, v, w\}]$  has exactly one edge, say  $uv$ , and  $G$  is a spanning subgraph of the fourth graph in Figure 4. Then  $G$  is not AP if and only if the order  $n$  of  $G$  is a multiple of three since the sequence  $(3)^{n/3}$  cannot be realized. For any other  $n$ , every admissible sequence  $(n_1, \dots, n_k)$  either has an element  $n_i = 2$  or  $n_i \geq 4$ . If  $n_i = 2$  we take a corresponding part  $V_i = \{u, v\}$  and if  $n_i \geq 4$  we take  $V_i \supseteq \{u, v, w, u'\}$ . Then  $G - V_i$  is traceable, and hence AP.

Suppose that  $C$  has two attachment vertices  $u', v'$  with  $uu', vv' \in E$ . As before, we assume that  $w$  is adjacent  $v$  or  $v'$ . Observe that the subgraph  $G' = G - w$  of size  $\|G'\| \geq \binom{n-4}{2} + 11$  is spanned by a sun with two rays  $\text{Sun}(a, b)$  with  $0 \leq a \leq b$ . We will first show that  $G'$  is traceable. This is clear for  $a = 0$ . If  $a \geq 2$  then  $G$  has a good couple of edges. Without loss of generality, we may assume that  $x_1 y_{i+1}, x_a y_i$  is a good couple. Then  $vv' y_1 \cdots y_i x_a \cdots x_1 y_{i+1} \cdots y_b u' u$  is a Hamiltonian path of  $G'$ . If  $a = 1$ , suppose that  $G'$  is not traceable and consider the graph  $G_0$  such that  $V(G_0) = V(G')$  and  $E(G_0)$  consists of  $\binom{n-4}{2} + 4$  edges:  $uu', vv', u'x_1, v'x_1$  and all edges of the clique induced by  $V(C) \setminus \{x_1\}$ . Hence  $G'$  has at least seven chords incident to  $x_1$ . However, if  $E(G')$  contained  $x_1 y_1$  or  $x_1 y_b$ , then it is easy to see that  $G'$  would be traceable. Moreover, for  $i = 2, \dots, b - 1$ , if  $x_1 y_i \in E(G')$  then  $y_{i-1} y_b \notin E(G')$  since otherwise  $vv' y_1 \cdots y_{i-1} y_b \cdots y_i x_1 u' u$  would be a Hamiltonian path of  $G'$ . Thus  $\|G'\| \leq \|G_0\| < \binom{n-4}{2} + 11$ , a contradiction.

It follows that  $G$  is traceable whenever  $vw \in E$ . Then assume  $vw \notin E$ . Let  $(n_1, \dots, n_k)$  be an admissible sequence for  $G$  ordered decreasingly:  $n_1 \geq \dots \geq n_k \geq 2$ . If  $n_1 \geq 3$ , then we put  $w, v, v'$  to  $V_1$  and continue a partition of  $V$  along the Hamiltonian path of  $G'$ . Otherwise,  $(n_1, \dots, n_k) = (2)^{n/2}$  and  $n$  is even.

Then  $G$  is a spanning subgraph of the second graph in Figure 4 without a perfect matching.

To end the proof of Theorem 8 it suffices to settle the case when a longest cycle  $C$  has three attachment vertices.

**Lemma 18.** *Let  $G = (V, E)$  be a graph of size  $\|G\| > \binom{n-4}{2} + 12$  and circumference  $c(G) = n - 3$ . If a longest cycle  $C$  has three attachment vertices, then  $G$  is AP.*

**Proof.** It follows from Lemma 14, that there are exactly three independent edges outside  $C$ , namely  $uu', vv', ww'$ , and  $G$  is spanned by  $\text{Sun}(a, b, c)$  where  $0 \leq a \leq b \leq c \leq n - 6$ . To show that  $G$  is AP, we consider three cases depending on admissible sequences.

*Case 1:* Sequence  $(2)^{n/2}$ . Suppose that the sequence  $(2)^{n/2}$  is admissible but not realizable in the graph  $G$ . Hence exactly two of the numbers  $a, b, c$  are odd, say  $a$  and  $b$  with  $a \leq b$ .

Let  $a = 1$ . Consider a graph  $G_0$  of size  $\|G_0\| = \binom{n-4}{2} + 6$  containing all edges of the clique  $V \setminus \{u, v, w, x_1\}$  and the edges  $uu', vv', ww', u'x_1, x_1v', x_1w'$ . It follows that  $x_1$  is adjacent to at least seven vertices of  $Y \cup Z$ . However, any edge of the form  $x_1y_{2l-1}$  would give a perfect matching in  $G$ . Moreover, if  $x_1y_{2l} \in E$ , then  $y_1y_{2l+1} \notin E$ . Furthermore, if  $x_1z_{2l} \in E$ , then  $y_1z_{2l-1} \notin E$ , and if  $x_1z_{2l-1} \in E$ , then  $y_1z_{2l} \notin E$ , otherwise  $G$  would have a perfect matching. Therefore  $\|G\| \leq \|G_0\|$ , a contradiction.

Let  $a \geq 3$ . Here we argue similarly as in Section 3 for  $a \geq 3$ . Let  $G_1$  be a graph of size  $\binom{n-5}{2} + 11$  containing the edges  $uu', vv', ww', u'x_1, x_1x_2, x_1v', x_1w', x_ax_2, x_a u', x_a v', x_a w'$  and all edges of the clique formed by  $V \setminus \{u, v, w, x_1, x_a\}$ . If  $a = 3$ , we set  $G_0 = G_1$ . For  $a \geq 5$ , we define  $G_0$  as follows. We add to  $G_1$  the edges  $x_1x_{2j}, x_ax_{2j}$ ,  $j = 2, \dots, \frac{a-1}{2}$ , and delete the edges  $x_{2i-1}y_{2j-1}$ ,  $i = 2, \dots, \frac{a-1}{2}$ ,  $j = 1, \dots, \frac{b+1}{2}$ . Thus the number of added edges equals  $a - 3$ , and the number of deleted ones equals  $\frac{a-1}{2} \cdot \frac{b+1}{2}$  and is not smaller than  $\frac{a^2-1}{4}$  since  $b \geq a$ . Therefore  $\|G_0\| < \|G_1\|$ . To avoid a perfect matching, the only edges that may appear in  $G$  and are not in  $G_0$  are of the form  $x_\nu x_{2i-1}$  or  $x_\nu y_{2j}$  or  $x_\nu z_l$  where  $\nu \in \{1, a\}$ .

For any  $i = 1, \dots, \frac{a-1}{2}$ , if  $x_1x_{2i+1} \in E$ , then  $y_1x_{2i} \notin E$ , and if  $x_ax_{2i-1} \in E$ , then  $y_bx_{2i} \notin E$ , otherwise  $G$  admits a perfect matching. For any  $j = 1, \dots, \frac{b-1}{2}$ , if  $x_1y_{2j} \in E$ , then  $y_1y_{2j+1} \notin E$ , and if  $x_ay_{2j} \in E$ , then  $y_by_{2j-1} \notin E$  (here the edge  $y_1y_b$  may be counted twice as missing in  $E$ ). For any  $j = 1, \dots, \frac{c-1}{2}$ , if  $x_1z_{2j} \in E$ , then  $y_1z_{2j-1} \notin E$ , and if  $x_az_{2j} \in E$ , then  $y_bz_{2j-1} \notin E$ . Finally, for any  $j = 1, \dots, \frac{c+1}{2}$ , if  $x_1z_{2j-1} \in E$ , then  $y_1z_{2j} \notin E$ , and if  $x_az_{2j-1} \in E$ , then  $y_bz_{2j} \notin E$ . Whence,  $\|G\| \leq \|G_0\| + 1 \leq \binom{n-4}{2} + 12$ , a contradiction.

*Case 2:* Sequence  $(3)^{n/3}$ . Suppose that the sequence  $(3)^{n/3}$  is admissible but

is not realizable in  $G$ . It easily follows from Theorem 6 that either  $a \equiv b \equiv c \equiv 0 \pmod{3}$  or  $a \equiv b \equiv c \equiv 2 \pmod{3}$ .

Assume first that  $a \equiv b \equiv c \equiv 0 \pmod{3}$ . Let  $0 \leq a \leq b \leq c$ . Put  $V_1 = \{u, u', z_c\}$  and  $V_2 = \{w, w', z_1\}$ . Let  $G_0$  be a subgraph of  $G$  obtained by deleting all chords of  $C$  incident to  $v'$  except  $v'z_1$  and  $v'z_c$ . Then  $\|G_0\| \leq \binom{n-4}{2} + 7$ , and there are another chords of  $C$  incident to  $v'$  in  $G$ .

Suppose  $a = b = 0$ . If  $v'z_{3l+2}$  was an edge of  $G$  for some  $l \geq 0$ , then the sequence  $(3)^{n/3}$  would have a realization in  $G$ . Indeed, we put  $V_3 = \{v, v', z_{3k+2}\}$  and observe that the cycle  $C$  splits into at most four paths of order divisible by 3 after removing the vertices of  $V_1 \cup V_2 \cup V_3$ . Also, if  $v'z_{3l} \in E$  with  $1 \leq l \leq \frac{c}{3} - 1$ , then  $z_{3l-1}z_{3l+1} \notin E$  otherwise  $G$  has a realization of  $(3)^{n/3}$ . Similarly, if  $v'z_{3l+1} \in E$  with  $1 \leq l \leq \frac{c}{3} - 1$ , then  $z_{3l}z_{3l+2} \notin E$ .

Suppose  $b > 0$ . If  $v'y_{3k} \in E$ , then  $y_{3k-1}z_2 \notin E$ , since otherwise we would have a realization of  $(3)^{n/3}$  by taking  $V_3 = \{y_{3k-2}, y_{3k-1}, z_2\}$ . Analogously, if  $v'y_{3k+1} \in E$ , then  $y_{3k+2}z_2 \notin E$  because we could take  $V_3 = \{y_{3k+2}, y_{3k}, z_2\}$ . Again, if both edges  $v'y_{3k+2}$  and  $y_{3k+1}z_1$  appeared in  $G$ , then we could redefine  $V_2 = \{w, w', y_b\}$  and put  $V_3 = \{v, v', y_{3k+2}\}$ ,  $V_4 = \{y_{3k+1}, z_1, z_2\}$  to obtain a realization of  $(3)^{n/3}$ .

If also  $a > 0$ , then the same arguments as in the previous paragraph for  $b > 0$  can be applied to justify the assertion: every new chord from  $E \setminus E(G_0)$  causes the absence of another chord in  $G$ . It follows that  $\|G\| \leq \|G_0\| < \binom{n-4}{2} + 12$ .

Now, assume that  $a \equiv b \equiv c \equiv 2 \pmod{3}$ . Let  $V_1 = \{u, u', x_1\}$ ,  $V_2 = \{v, v', x_a\}$ ,  $V_3 = \{w, w', y_b\}$ . Remember that the number of chords of  $C$  missing in  $G$  is at most  $n - 12$ . For every  $i = 1, \dots, \frac{b}{3}$  and  $j = 1, \dots, \frac{c}{3}$ , the vertex  $y_{3i-2}$  cannot be a neighbor neither of  $z_{3j-2}$  nor of  $z_{3j-1}$  because then we could take  $V_4 = \{y_{3i-2}, z_{3j-2}, z_{3j-1}\}$ , and  $C$  would split into paths of orders being multiples of three after removing  $V_1 \cup V_2 \cup V_3 \cup V_4$ . Analogously,  $y_{3i-1}z_{3j-2}, y_{3i-1}z_{3j-1} \notin E$ . Thus, there are  $4 \cdot \frac{b}{3} \cdot \frac{c}{3}$  missing chords of  $C$ . As  $b + c \geq \frac{2}{3}(n - 6)$ , we have  $\frac{4}{9}bc \geq \frac{4}{81}(n - 6)^2 > n - 12$  for any  $n$ , a contradiction.

*Case 3:* Sequences different from  $(2)^{n/2}$  and  $(3)^{n/3}$ . Consider first the case  $a \geq 1$ . Then, we can apply Lemma 17. Without loss of generality, we may assume that the good couple is from  $X$  to  $Z$ , i.e., there is an  $i$  such that  $x_1z_{i+1}, x_az_i \in E$  and  $1 \leq i \leq c$ . Then, observe that the subgraph of  $G$  induced by the vertex sequence  $x_0 \overleftarrow{C} z_{i+1} x_1 C x_a z_i \overleftarrow{C} v'$  contains a caterpillar  $\text{Cat}(b+3)$ . So, by Theorem 4, we are able to realize all admissible sequences except, maybe, for sequences of the form  $(d)^{n/d}$  for  $d|(b+3)$ . If  $a = 1$ , then a part of such a sequence could be realized on the caterpillar  $\text{Cat}(2) = x_1 C y_b$  of order  $b+3$  because  $d \neq 2$  by assumption, and the rest of it on the path  $w w' C u' u$ . If  $a \geq 2$ , then a part of this sequence could be realized either on  $\text{Cat}(2) = x_1 C y_b$  or on  $\text{Cat}(2) = y_1 C z_1$ , and the rest of the sequence either on  $\text{Cat}(a) = x_{a-1} \overleftarrow{C} w'$  or on  $\text{Cat}(a+3) = v' \overleftarrow{C} z_2$ , respectively. If none of the two latter caterpillars admits a realization of the sequence  $(d)^{n/d}$

that means that  $d|a$  and  $d|(a+3)$ . This implies that  $d=3$ .

Let  $a=0$  and  $b \geq 1$ . Then  $G$  contains a caterpillar  $\text{Cat}(b+3) = vv'\overleftarrow{C}u'u$ . So, by Theorem 4, any admissible and nonrealizable sequence should be of the form  $(d)^{n/d}$  for  $d|(b+3)$ . As  $d \neq 2$ , then a part of such a sequence could be realized on  $\text{Cat}(2) = y_1Cz_1$ , and the rest of the sequence on  $\text{Cat}(3) = vv'\overleftarrow{C}z_2$ , except for the case where  $d=3$ .

If  $a=b=0$  then it is easy to see that  $G$  is spanned by a caterpillar  $\text{Cat}(3)$ , so only the sequence  $(3)^{n/3}$  may not be realizable. ■

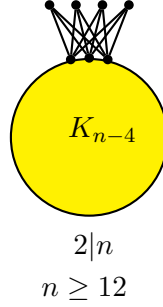


Figure 8. A non-AP graph of size  $\binom{n-4}{2} + 12$ .

## 5. FINAL REMARKS

The following is an easily seen consequence of Corollary 13.

**Proposition 19.** *If  $G$  is a connected graph of order  $n$  and size  $\|G\| > \binom{n-2}{2} + 2$ , then  $G$  is traceable.*

Clearly, the bound  $\binom{n-2}{2} + 2$  is sharp for every  $n \geq 4$  since the first graph shown in Figure 4 (a clique  $K_{n-2}$  with two pendant edges attached to it in one vertex) is not traceable. The difference between  $\binom{n-2}{2} + 2$  and the lower bound  $\binom{n-4}{2} + 12$  in our main result equals  $2n - 17$ .

Observe that there are quite many connected nontraceable graphs  $G$  with more than  $\binom{n-4}{2} + 12$  edges, which are AP by Theorem 8. In particular, if the order  $n$  of  $G$  is not divisible neither by two nor by three, then  $G$  is AP unless it is a spanning subgraph of the third graph in Figure 4 (a clique  $K_{n-3}$  with three pendant edges attached in one and the same vertex). Moreover, for every  $n$  if



$c(G) = n - 3$  and  $G$  has three independent pendant edges, then  $G$  is AP, and clearly nontraceable.

It has to be noted that if we decrease the bound  $\binom{n-4}{2} + 12$  even by one, then we obtain new exceptional graphs that are not AP. For example, the graph in Figure 8 has  $\binom{n-4}{2} + 12$  edges and is not AP for even  $n$ .

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