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CRITICAL GRAPHS FOR $R(P_n, P_m)$ AND THE STAR-CRITICAL RAMSEY NUMBER FOR PATHS

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Abstract

The *graph Ramsey number* $R(G, H)$ is the smallest integer r such that every 2-coloring of the edges of K_r contains either a red copy of G or a blue copy of H. The *star-critical Ramsey number* $r_*(G, H)$ is the smallest integer k such that every 2-coloring of the edges of $K_r - K_{1,r-1-k}$ contains either a red copy of G or a blue copy of H. We will classify the *critical graphs*, 2-colorings of the complete graph on $R(G, H) - 1$ vertices with no red G or blue H, for the path-path Ramsey number. This classification will be used in the proof of $r_*(P_n, P_m)$.

Keywords: Ramsey number, critical graph, star-critical Ramsey number, path.

2010 Mathematics Subject Classification: 05C55, 05C38.

1. Introduction

For graphs G and H, the *graph Ramsey number* $R(G, H)$ is the smallest integer r such that every 2-coloring of the edges of K_r contains either a red copy of G or a blue copy of H. Furthermore, there exists a (G, H)*-free coloring* of K_{r-1}, a 2-coloring of the edges that does not contain a red copy of G or a blue copy of H. A (G, H)-free coloring of K_{r-1} is known as the *critical graph* for $R(G, H)$. The following section is devoted to classifying the critical graphs for $R(P_n, P_m)$.

The *star-critical Ramsey number* $r_*(G, H)$, introduced in [3], is the smallest integer k such that every 2-coloring of the edges of $K_r - K_{1,r-1-k}$ contains either a red copy of G or a blue copy of H. In other words, this is the largest star that can be removed from K_r yet every 2-coloring is still forced to contain either a red G or a blue H. To emphasize the size of the star, we can view the graph $K_r - K_{1,r-1-k}$ as $K_{r-1} \sqcup K_{1,k}$, the union of K_{r-1} and $K_{1,k}$ such that the k vertices of the star are vertices of K_{r-1}.

The graph $G + H$ is the *disjoint union* of G and a graph H, whereas the graph $G + \{v\}$ is the disjoint union of G and a vertex v. The deletion of edges of a subgraph H from G will be denoted as $G - H$ and the deletion of a vertex as $G - \{v\}$. The *join* of G and H, denoted by $G \vee H$, is the graph $G + H$ with the addition of the edges $\{xy : x \in V(G) \text{ and } y \in V(H)\}$. For a graph G with a 2-coloring of the edges, the *red subgraph* will be denoted as G^ρ which consists of all the vertices of G along with the red edge set. Similarly, G^β will denote the *blue subgraph*. The figures of the critical graphs contain solid red edges and dashed blue edges. The thick edges between cliques denote all edges between the sets of vertices. The following facts can be easily checked.

Observation 1.1. *The Ramsey number $R(G,H) = R(H,G)$ and the critical graphs for $R(H,G)$ are the critical graphs for $R(G,H)$ with the colors red and blue interchanged.*

Observation 1.2. *Let G' be a subgraph of G and H' be a subgraph of H. If $R(G',H') = R(G,H)$, then the class of critical graphs for $R(G',H')$ is a subset of the class of critical graphs for $R(G,H)$.*

In the following sections, we will consider the path-path graph Ramsey number. A path on n vertices is denoted by P_n and the length of the path refers to the number of vertices on the path. The critical graphs for $R(P_n,P_m)$ will be classified using a sequence of lemmas that depend on the parity of the length of the path. We will see that the (P_n,P_m)-free colorings of $K_{R(P_n,P_m)-1}$ must belong to an infinite family of graphs. The star-critical Ramsey number $r_*(P_n,P_m)$ will be determined using this classification.

2. The Critical Graphs for $R(P_n, P_m)$

The path-path graph Ramsey number was obtained by Gerencsér and Gyárfás [1] in 1967 and is stated as Theorem 2.1. The graphs G_1 and G_2 in Definition 2.4 are presented in [1] as examples to establish the lower bound of $R(P_n,P_m)$. A complete list of the critical graphs for the path-path Ramsey number is described in Definition 2.4 and classified in Proposition 2.6.

Theorem 2.1 [1]. *For all $n \geq m \geq 2$, $R(P_n,P_m) = n + \lfloor \frac{m}{2} \rfloor - 1$.*

Before classifying the critical graphs for $R(P_n,P_m)$ with $m \geq 4$, the critical graphs for $m = 2$ and $m = 3$ are as follows.

Proposition 2.2. *For given $n \geq m$, let $r = R(P_n, P_m) = n + \lfloor \frac{m}{2} \rfloor - 1$ and c be a (P_n, P_m)-free coloring of K_{r-1}.*

If $m = 2$ and $n > 2$, then the resulting graph is

$$H_1 : \begin{cases} H_1^{\rho} = K_{n-1}, \\ H_1^{\beta} = (n-1)K_1. \end{cases}$$

If $m = 3$ and $n > 3$, then for any $i \in \{0, 1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$ the resulting graph is

$$H_2 : \begin{cases} H_2^{\rho} = K_{n-1} - iK_2, \\ H_2^{\beta} = iK_2. \end{cases}$$

If $n = m = 2$, then the graph is a single vertex. If $n = m = 3$, then the graph is a single edge colored red or blue.

Proof. A (P_n, P_2)-free coloring of K_{n-1} does not have any blue edges and so the graph must be a red K_{n-1} for $n > 2$ and a single vertex if $n = 2$. A (P_n, P_3)-free coloring of K_{n-1} does not have a blue P_3 which implies that the blue edges must form a matching. Thus, if $n > 3$, the graph has red subgraph $K_{n-1} - iK_2$ and blue subgraph iK_2 for any $i \in \{0, 1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$. If $n = 3$, the graph consists of a single edge which may be colored red or blue. ■

The following fact will be frequently used throughout this paper. It simply states that a path can be created between two sets of vertices by alternating between the sets. The length of the path is twice the size of the smaller set of vertices plus one vertex.

Observation 2.3. *Let $G = A \vee B$. If $|V(A)| \geq k$ and $|V(B)| \geq k+1$, then there is a path P_{2k+1} that alternates between vertices in A and vertices in B starting and ending with a vertex in B. See Figure 2.*

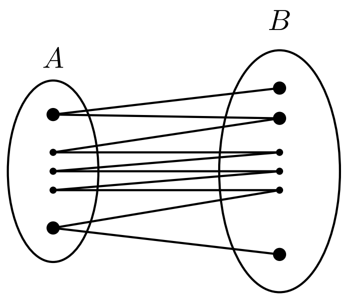

Figure 1. A path P_{2k+1} between A and B.

Definition 2.4. For given n and m with $n \geq m \geq 4$, let $r = R(P_n, P_m) = n + \lfloor \frac{m}{2} \rfloor - 1$ and A_k be any 2-coloring of the graph K_k. Define the class of graphs $\mathcal{G}$, shown in Figure 2, to consist of the five families of K_{r-1} listed below.

$$\text{If } n \geq m,\ G_1 : \begin{cases} G_1^{\rho} = K_{n-1} + A^{\rho}_{\lfloor \frac{m}{2} \rfloor -1}, \\ G_1^{\beta} = A^{\beta}_{\lfloor \frac{m}{2} \rfloor -1} \vee (n-1)K_1. \end{cases}$$

$$\text{If } n \geq m \text{ with } m \text{ odd},\ G_2 : \begin{cases} G_2^{\rho} = (K_{n-1} - K_2) + A^{\rho}_{\lfloor \frac{m}{2} \rfloor -1}, \\ G_2^{\beta} = A^{\beta}_{\lfloor \frac{m}{2} \rfloor -1} \vee (K_2 + (n-3)K_1). \end{cases}$$

$$\text{If } n = m,\ G_3 : \begin{cases} G_3^{\rho} = A^{\rho}_{\lfloor \frac{m}{2} \rfloor -1} \vee (n-1)K_1, \\ G_3^{\beta} = K_{n-1} + A^{\beta}_{\lfloor \frac{m}{2} \rfloor -1}. \end{cases}$$

$$\text{If } n = m \text{ with } m \text{ odd},\ G_4 : \begin{cases} G_4^{\rho} = A^{\rho}_{\lfloor \frac{m}{2} \rfloor -1} \vee (K_2 + (n-3)K_1), \\ G_4^{\beta} = (K_{n-1} - K_2) + A^{\beta}_{\lfloor \frac{m}{2} \rfloor -1}. \end{cases}$$

$$\text{If } n = m+1 \text{ with } m \text{ odd},\ G_5 : \begin{cases} G_5^{\rho} = A^{\rho}_{\frac{n}{2} -1} \vee (m-1)K_1, \\ G_5^{\beta} = K_{m-1} + A^{\beta}_{\frac{n}{2} -1}. \end{cases}$$

The graphs in Definition 2.4 (see Figure 2) are (P_n, P_m)-free colorings. The red subgraphs of G_1 and G_2 do not contain a red path on n vertices since each component has size at most $n-1$. A longest path in the blue subgraph of G_1 is a blue P_{m-1} by Observation 2.3 using blue edges between K^{ρ}_{n-1} and $A_{\lfloor \frac{m}{2} \rfloor -1}$. A longest path in the blue subgraph of G_2 is also a blue P_{m-1} using a longest blue P_{m-2} by Observation 2.3 between $(K_{n-1} - K_2)^{\rho}$ and $A_{\lfloor \frac{m}{2} \rfloor -1}$ and the single blue edge. The graphs G_3 and G_4 are (P_n, P_m)-free colorings since they are the graphs G_1 and G_2 with the colors red and blue interchanged. The blue subgraph of G_5 does not contain a blue path on m vertices since each component has size at most $m-1$. A longest path in the red subgraph of G_5 is a red P_{n-1} by Observation 2.3 using the red edges between K^{β}_{m-1} and $A_{\frac{n}{2}-1}$.

Remark 2.5. In Proposition 2.2, the (P_n, P_m)-free coloring when $m = 2$, the graph H_1, belongs to the family of graphs G_1 in $\mathcal{G}$ of Definition 2.4, but when $m = 3$ the graph H_2 does not belong to $\mathcal{G}$.

Proposition 2.6. *For given n and m with $n \geq m \geq 4$, let $r = R(P_n, P_m) = n + \lfloor \frac{m}{2} \rfloor - 1$. If c is a (P_n, P_m)-free coloring of K_{r-1}, then the resulting graph must belong to the class of graphs $\mathcal{G}$ in Definition* 2.4.

The following sequence of lemmas will prove Proposition 2.6. We will first classify the critical graphs for odd $n \geq m+1$ in Lemma 2.9 and for odd $n = m$ in Lemma 2.11. Then we will proceed with even $n \geq m+1$ with m odd in Lemma 2.14 and m even in Lemma 2.15. Finally, Lemma 2.16 will classify the critical graphs for even $n = m$. Note that the order of the lemmas is necessary as Lemma 2.16 invokes Lemma 2.15 which invokes Lemma 2.14. Also, Lemma 2.9 is used in the proof of Lemma 2.14. Within the proofs of the lemmas, the cycle-path and

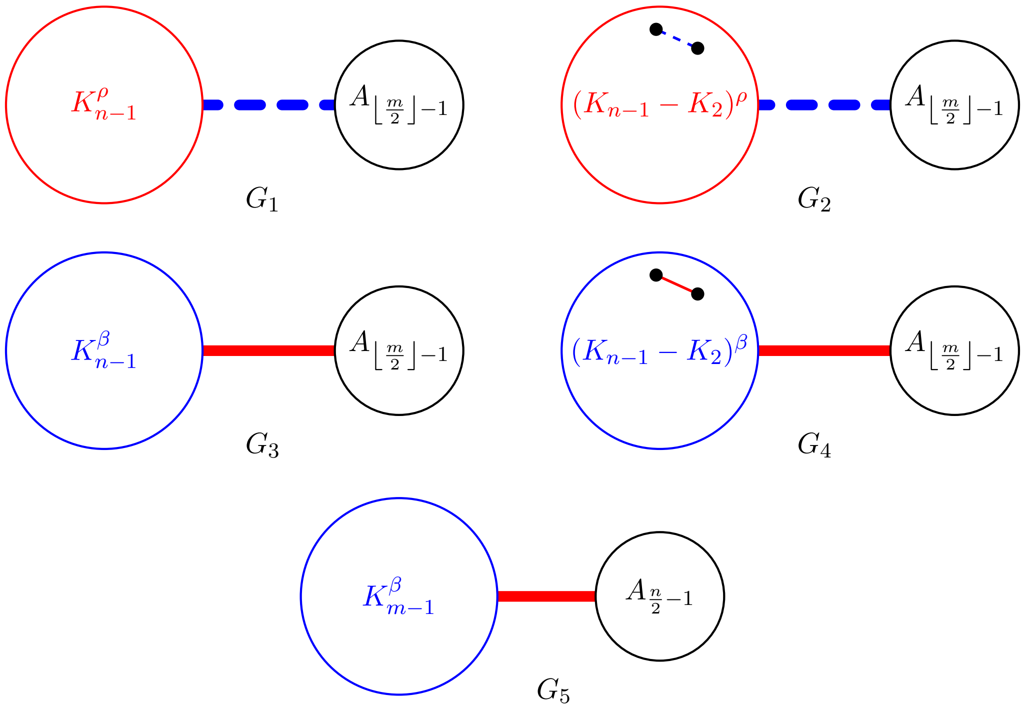

Figure 2. Critical graphs for $R(P_n, P_m)$. The graph G_1 is for all $n \geq m$. The graphs G_2 and G_4 are when m is odd. The graphs G_3 and G_4 are when $n = m$. The graph G_5 is for $n = m + 1$ with m odd.

cycle-cycle Ramsey numbers (Theorems 2.7 and 2.8) will be used in addition to Observation 2.3 and the observation that the graph of a clique K_{n-1} and an edge adjacent to any vertex of the clique contains a P_n.

Theorem 2.7 [4]**.** *For $n \geq m \geq 2$,*

$$R(C_n, P_m) = \begin{cases} \max\{n + \lfloor \frac{m}{2} \rfloor - 1, 2m - 1\} & \text{for } n \text{ odd}, \\ n + \lfloor \frac{m}{2} \rfloor - 1 & \text{for } n \text{ even}. \end{cases}$$

Theorem 2.8 [5]**.** *For $n \geq m \geq 3$,*

$$R(C_n, C_m) = \begin{cases} 2n - 1 & \text{for } m \text{ odd}, (n, m) \neq (3, 3), \\ n + \frac{m}{2} - 1 & \text{for } m, n \text{ both even}, (n, m) \neq (4, 4), \\ \max\{n + \frac{m}{2} - 1, 2m - 1\} & \text{for } m \text{ even and } n \text{ odd}. \end{cases}$$

Lemma 2.9. *For given n and m with n odd and $n \geq m + 1 \geq 5$, let $r = R(P_n, P_m) = n + \lfloor \frac{m}{2} \rfloor - 1$. If c is a (P_n, P_m)-free coloring of K_{r-1}, then the resulting graph is either G_1 or G_2 as in Definition 2.4.*

Proof. Let $r = R(P_n, P_m) = n + \lfloor \frac{m}{2} \rfloor - 1$ and c be a (P_n, P_m)-free coloring of K_{r-1}. Since $R(C_{n-1}, P_m) = n + \lfloor \frac{m}{2} \rfloor - 2 = r - 1$ and the graph does not have a blue P_m, the graph must have a red $C_{n-1} = (v_1 v_2 \dots v_{n-1})$. A red edge from any v_i to a vertex not on the cycle creates a red P_n. Thus, there cannot be any such red edges and the red C_{n-1} has all blue edges to the remaining $\lfloor \frac{m}{2} \rfloor - 1$ vertices. Using these blue edges and Observation 2.3, there is a blue

$$P_{2(\lfloor \frac{m}{2} \rfloor - 1)+1} = P_{2\lfloor \frac{m}{2} \rfloor - 1} = \begin{cases} P_{m-1} & \text{if } m \text{ is even,} \\ P_{m-2} & \text{if } m \text{ is odd.} \end{cases}$$

If m is even, then there is a blue P_{m-1} which can begin and end at any vertex on the red C_{n-1}. Thus, a single blue edge between two vertices on the red cycle would create a blue P_m. This implies that there cannot be any blue edges within the red cycle and the resulting graph is G_1. If m is odd, then there is a blue P_{m-2} that can begin and end at any vertex on the red C_{n-1}. Thus, a single blue edge between two vertices on the red cycle would extend this path to a blue P_{m-1}. Two blue edges between vertices on the red cycle may be either disjoint or share a vertex. In either case, the blue P_{m-2} could be extended to a blue P_m using these blue edges at the beginning and end of the blue P_{m-2} if they are disjoint, or at the end of the blue P_{m-2} if they share a vertex. This implies that there can be at most one blue edge between two vertices on the cycle and the resulting graph is either G_1 or G_2. ∎

In Lemma 2.11, we classify the critical graphs for odd $n = m$. Within the proof, we use the Ramsey number $R(C_{n-1}, C_{m-1})$ for $n = m \geq 7$. The following result will be used in the case when $n = m = 5$.

Proposition 2.10 [3]. *For a given $n \geq 3$, let $r = R(P_n, C_4) = n + 1$. If c is a (P_n, C_4)-free coloring of K_{r-1}, then the resulting graph must belong to the class of graphs H_i for $i = 0, \dots, \lfloor \frac{n-1}{2} \rfloor$ such that*

$$H_i^\rho = (K_{n-1} - iK_2) + K_1,$$
$$H_i^\beta = K_{1,n-1} \cup iK_2.$$

Note that the blue subgraph $H_i^\beta = K_{1,n-1} \cup iK_2$ is obtained by adding i disjoint edges to the complete bipartite graph. Thus, the vertices of iK_2 are vertices of $K_{1,n-1}$. This is illustrated in Figure 3.

Lemma 2.11. *For given n and m with $n = m \geq 5$ odd, let $r = R(P_n, P_m) = n + \lfloor \frac{m}{2} \rfloor - 1$. If c is a (P_n, P_m)-free coloring of K_{r-1}, then the resulting graph is G_i with $i = 1, 2, 3$ or 4 as in Definition 2.4.*

Proof. For given n and m with $n = m$ odd, let $r = R(P_n, P_m) = n + \lfloor \frac{m}{2} \rfloor - 1$ and c be a (P_n, P_m)-free coloring of K_{r-1}. For $n, m \geq 7$, $R(C_{n-1}, C_{m-1}) =$

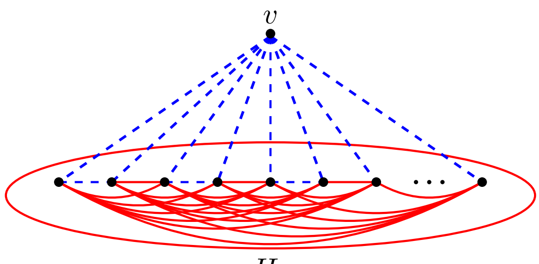

Figure 3. Critical graph H_3 for $R(P_n, C_4)$.

$(n-1) + \frac{m-1}{2} - 1 = n + \left\lfloor \frac{m}{2} \right\rfloor - 2 = r - 1$. Thus, the graph K_{r-1} has either a red C_{n-1} or a blue C_{m-1}. If there is a red C_{n-1}, then the same proof as in Lemma 2.9 holds and the resulting graph is either G_1 or G_2. If there is a blue C_{m-1}, then the proof in Lemma 2.9 holds with the colors red and blue interchanged and the resulting graph is either G_3 or G_4.

If $n = m = 5$, then the cycle-cycle Ramsey number $R(C_4, C_4) = 6 \neq 5 = r-1$. In this case, let c be a (P_5, P_5)-free coloring of K_5. Since $R(P_5, C_4) = 6$ and there is no red P_5, either the graph has a blue C_4 or the graph has a (P_5, C_4)-free coloring. If there is a blue C_4, then the same proof as in Lemma 2.9 holds and the resulting graph is either G_3 or G_4. Otherwise, the graph has a (P_5, C_4)-free coloring. Then, by Proposition 2.10, the graph must be H_0 or H_1. Note that H_2 contains a blue P_5. The graphs H_0 and H_1 are the graphs G_1 and G_2, respectively. ■

Next, we will consider the cases when n is even. In the proofs of the lemmas with even n, Proposition 2.12 will be used, which is stated as Exercise 7.2.38 in West [6], and Lemma 2.13.

Proposition 2.12 [6]**.** *If G is connected with minimum degree $k \geq 2$ and G has more than $2k$ vertices, then G has a path on $2k+1$ vertices.*

Lemma 2.13. *Let $G = A \vee B$ with $|V(A)| \geq k$ and $|V(B)| \geq k + i$. If there is a path on i vertices in B, then G has a path P_{2k+i}.*

Proof. By Observation 2.3, there is a path P_{2k+1} that alternates between vertices in A and vertices in B starting and ending with a vertex in B. Since $G = A \vee B$, this path can begin at any vertex in B. Thus, we may begin the path of length $2k+1$ at an endpoint of the path on i vertices in B and the graph has a path of length $2k+i$. Note that there are at least k vertices in B not on the path of length i and so an endpoint of this P_i and k vertices not on the path P_i are sufficient to create the path P_{2k+i}. ■

Lemma 2.14. *For given n and m with even n and odd $m = 2k+1$ such that $n \geq m+1 \geq 6$, let $r = R(P_n, P_m) = n + \lfloor \frac{m}{2} \rfloor - 1$. If c is a (P_n, P_m)-free coloring of K_{r-1}, then the resulting graph is either G_1, G_2 or G_5 as in Definition* 2.4.

Proof. For given n and m with even n and odd $m = 2k+1$ such that $n \geq m+1 \geq 6$, let $r = R(P_n, P_m) = n + \lfloor \frac{m}{2} \rfloor - 1 = n + \lfloor \frac{2k+1}{2} \rfloor - 1 = n+k-1$ and c be a (P_n, P_m)-free coloring of K_{r-1}. If K_{r-1}^{β} is connected with the degree of each vertex at least k, then there is a blue path on at least $2k+1$ vertices by Proposition 2.12 and hence a blue P_m. Therefore, either there is a vertex in K_{r-1}^{β} of degree at most $k-1$ or K_{r-1}^{β} is disconnected.

Case I: There is a vertex in K_{r-1}^{β} of degree at most $k-1$. This implies that there is a vertex v in K_{r-1}^{ρ} of degree at least $(r-2)-(k-1) = (n+k-1-2)-(k-1) = n-2$. Note that $|V(K_{r-1}-\{v\})| = n+k-3$ and the cycle-cycle Ramsey number, $R(C_{n-2}, C_{m-1}) = n-2+\frac{m-1}{2}-1 = n-2+\frac{2k+1-1}{2}-1 = n+k-3$ (except when $n=6$ and $m=5$) by Theorem 2.8. Thus, $K_{r-1}-\{v\}$ has either a red C_{n-2} or a blue C_{m-1}.

Suppose that $K_{r-1}-\{v\}$ has a red C_{n-2}. If v has a red edge to a vertex on the red C_{n-2} and to a vertex not on the red C_{n-2}, then the graph contains a red P_n. Since K_{r-1} does not have a red P_n, the vertex v must have exactly $n-2$ red edges to the red C_{n-2} and $k-1$ blue edges to the remaining $k-1$ vertices. A red edge from a vertex of the red C_{n-2} to any of the $k-1$ vertices creates a red P_n. Hence, all edges from the red C_{n-2} must be blue (see Figure 4).

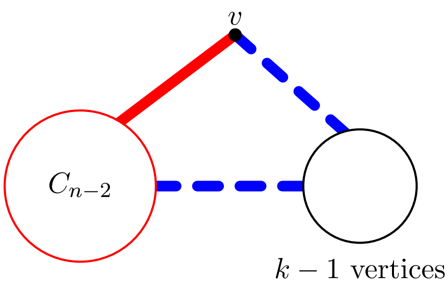

Figure 4. A red C_{n-2} with blue edges to remaining vertices.

By Observation 2.3, there is a blue $P_{2(k-1)+1} = P_{m-2}$ that can begin and end at any vertex on the red C_{n-1}. Note that this red cycle on $n-1$ vertices includes the vertex v. Thus, a single blue edge between two vertices on the red cycle would extend this path to a blue P_{m-1} and two blue edges between vertices on the red cycle would create a blue P_m. Thus, there can be at most one blue edge between two vertices on the red cycle and the resulting graph is either G_1 or G_2.

If $n = 6$ and $m = 5$, then the cycle-cycle Ramsey number $R(C_4, C_4) = 6 \neq 5 = n + k - 3$. In this case, let c be a (P_6, P_5)-free coloring of K_6. Since $R(P_6, C_4) = 7$ and the graph does not contain a red P_6, it has either a blue C_4 or a (P_6, C_4)-free coloring. If there is a blue C_4, then the same proof as below holds and the resulting graph is G_5. Otherwise, the graph has a (P_6, C_4)-free coloring. Then, by Proposition 2.10, the graph must be H_0 or H_1. Note that H_2 contains a blue P_5. The graphs H_0 and H_1 are the graphs G_1 and G_2, respectively.

Suppose that $K_{r-1} - \{v\}$ has a blue C_{m-1}. Since the graph does not have a blue P_m, there cannot be any blue edges between a vertex on the blue C_{m-1} and a vertex not on the blue C_{m-1}. Therefore, all edges between the vertices of the blue C_{m-1} and the remaining $n - k - 2$ vertices are red. We will now show that the only (P_n, P_m)-free coloring occurs when $n = m + 1$. Moreover, if $n \geq m + 3$ and the graph has a blue C_{m-1}, then every graph contains a red P_n.

If $n = m + 1$, then the graph has a red path $P_{2(n-k-2)+1} = P_{2(n-1)-m} = P_m = P_{n-1}$ with endpoints on the cycle and so there cannot be any red edges between vertices on the blue C_{m-1}. Thus, there is a blue K_{m-1} with all red edges to the remaining $n - k - 2$ vertices. Note that $n - k - 2 = n - \frac{m-1}{2} - 2 = \frac{2n-m-3}{2} = \frac{2n-(m+1)-2}{2} = \frac{n-2}{2} = \frac{n}{2} - 1$ and the resulting graph is G_5.

If $m + 3 \leq n \leq m + (k + 1)$, then $m - 1 \geq n - k - 2$. By Observation 2.3, the graph has a red $P_{2(n-k-2)} = P_{2n-2k-4} = P_{2n-2-(2k+1)-1} = P_{2(n-1)-(m+1)}$ and hence a red P_n.

If $m + (k + 2) \leq n \leq m + 2k$, then $m - 1 < n - k - 2$. By Observation 2.3, the graph has a red $P_{2(m-1)+1} = P_{m+2k}$ and hence a red P_n.

If $n \geq m + (2k + 1)$, then $m - 1 < n - k - 2$. By Observation 2.3, the graph has a red $P_{2(m-1)+1} = P_{m+2k}$. Let A denote the graph consisting of the $n - k - 2$ vertices not on the blue C_{m-1}.

For $m + (2k + 1) \leq n < m + 4k$, the Ramsey number

$$\begin{aligned} R(P_{n-(m+2k)+1}, P_m) &= R(P_m, P_{n-(m+2k)+1}) \\ &= m + \left\lfloor \frac{n - (m + 2k) + 1}{2} \right\rfloor - 1 = \frac{n}{2}. \end{aligned}$$

For $n \geq m + 4k$, the Ramsey number

$$\begin{aligned} R(P_{n-(m+2k)+1}, P_m) &= n - (m + 2k) + 1 + \left\lfloor \frac{m}{2} \right\rfloor - 1 \\ &= n - 3k - 1. \end{aligned}$$

Since $n - k - 2 > R(P_{n-(m+2k)+1}, P_m)$ and the graph does not have a blue P_m, there must be a red $P_{n-(m+2k)+1}$ in A. By Lemma 2.13, we can extend this path to a red P_n since $P_{2(m-1)+n-(m+2k)+1} = P_{m+n-2k-1} = P_{2k+1+n-2k-1} = P_n$. Note that since $n - k - 2 - (n - (m + 2k) + 1) = m + k - 3$, there are at least $3k - 2$ vertices in A not on the path that may be used to form the red P_{m+2k}.

Case II: The graph K^{β}_{r-1} is disconnected. Let X and Y be any partition of the graph K_{r-1} such that all edges between X and Y are red. If both X and Y have at least $\frac{n}{2}$ vertices, then there is a red P_n by Observation 2.3. Therefore, one of X or Y must have at most $\frac{n}{2} - 1$ vertices. Without loss of generality, assume that $|V(Y)| \leq \frac{n}{2} - 1$. For some $i = 0, 1, 2, \ldots, \frac{n}{2} - 2$, we have that $|V(X)| = (\frac{n}{2} + k - 1) + i$ and $|V(Y)| = (\frac{n}{2} - 1) - i$. Note that if there is a red $P_{2(i+1)}$ in X, then the graph contains a red P_n by Lemma 2.13.

If $n = m+1$, then $|V(X)| = 2k+i$. Since $R(P_{2(i+1)}, P_m) = R(P_m, P_{2(i+1)}) = m + i$, X either has a red $P_{2(i+1)}$, a blue P_m or X is the critical graph for $R(P_m, P_{2(i+1)})$ with m odd and the colors interchanged. Since the first two possibilities contradict the graph having a (P_n, P_m)-free coloring, X must be such a critical graph. By Lemma 2.9, X is the graph G_3, a blue K_{2k} with all edges to the remaining i vertices colored red. If we move the i vertices to Y, then the graph is a blue $K_{2k} = K_{m-1}$ with all red edges to the $\frac{n}{2} - 1$ remaining vertices. Thus, the resulting graph is G_5.

If $n \geq m+3$, then $|V(X)| \geq \frac{m+3}{2} + k - 1 + i = m + i$. For $i < k$, the Ramsey number $R(P_{2(i+1)}, P_m) = R(P_m, P_{2(i+1)}) = m + \left\lfloor \frac{2(i+1)}{2} \right\rfloor - 1 = m + i$. For $i \geq k$, the Ramsey number $R(P_{2(i+1)}, P_m) = 2(i+1) + \left\lfloor \frac{m}{2} \right\rfloor - 1 = 2i + k + 1 \geq m + k$. Thus, for all i, there must be a red $P_{2(i+1)}$ in X and the graph does not have a (P_n, P_m)-free coloring. ■

Lemma 2.15. *For given n and m both even such that $n \geq m+1 \geq 5$, let $r = R(P_n, P_m) = n + \left\lfloor \frac{m}{2} \right\rfloor - 1$. If c is a (P_n, P_m)-free coloring of K_{r-1}, then the resulting graph is G_1 as in Definition* 2.4.

Proof. For given n and m both even such that $n \geq m+1 \geq 5$, let $r = R(P_n, P_m) = n + \left\lfloor \frac{m}{2} \right\rfloor - 1$ and c be a (P_n, P_m)-free coloring of K_{r-1}. For m even, $R(P_n, P_m) = R(P_n, P_{m+1})$. By Lemma 2.14, the critical graphs for $R(P_n, P_{m+1})$ are G_1, G_2 or G_5. The critical graphs for $R(P_n, P_m)$ must be a subset of these graphs by Observation 1.2. Since both G_2 and G_5 contain a blue P_m, the only critical graph for $R(P_n, P_m)$ is G_1. ■

Lemma 2.16. *For given n and m with $n = m$ even, let $r = R(P_n, P_m) = n + \left\lfloor \frac{m}{2} \right\rfloor - 1$. If c is a (P_n, P_m)-free coloring of K_{r-1}, then the resulting graph is either G_1 or G_3 as in Definition* 2.4.

Proof. For given n and m with $n = m = 2k$ even, let $r = R(P_{2k}, P_{2k}) = 2k + \left\lfloor \frac{2k}{2} \right\rfloor - 1 = 3k - 1$ and c be a (P_{2k}, P_{2k})-free coloring of K_{r-1}. If K^{ρ}_{r-1} is connected with the degree of each vertex at least k, then there is a red P_{2k} by Proposition 2.12. Similarly, if K^{β}_{r-1} is connected with the degree of each vertex at least k, then there is a blue P_{2k} by Proposition 2.12. Therefore, either there is a vertex in both K_{r-1}^{ρ} and K_{r-1}^{β} of degree at most $k-1$ or at least one of K_{r-1}^{ρ} or K_{r-1}^{β} is disconnected.

Case I: There is a vertex in both K_{r-1}^{ρ} and K_{r-1}^{β} of degree at most $k-1$ and both subgraphs are connected. This implies that there is a vertex v in K_{r-1}^{ρ} of degree at least $2k-2$ and a vertex w in K_{r-1}^{β} of degree at least $2k-2$. Note that $|V(K_{r-1} - \{v, w\})| = 3k-4$ and the cycle-cycle Ramsey number $R(C_{2k-2}, C_{2k-2}) = 3k-4$ by Theorem 2.8 (except when $k=3$). Thus, $K_{r-1} - \{v, w\}$ has either a red C_{2k-2} or a blue C_{2k-2}.

Suppose that $K_{r-1} - \{v, w\}$ has a red C_{2k-2}. If v has a red edge to a vertex on the red C_{2k-2} and to a vertex not on the red C_{2k-2}, then the graph contains a red P_{2k}. Since K_{r-1} does not have a red P_{2k}, the vertex v must have exactly $2k-2$ red edges to the red C_{2k-2} and $k-1$ blue edges to the remaining $k-1$ vertices which include w. A red edge from a vertex of the red C_{2k-2} to any of the $k-1$ vertices creates a red P_{2k}. Hence, all edges from the red C_{2k-2} must be blue. Note the graph contains a blue P_{2k-1} which implies that the red C_{2k-2} along with the vertex v must be a red K_{2k-1}. Since the red graph is connected, the remaining vertices must form a blue clique and the graph belongs to the class of graphs G_1. A similar proof holds if we suppose that $K_{r-1} - \{v, w\}$ has a blue C_{2k-2} resulting with the graph G_3.

If $k = 3$, then the cycle-cycle Ramsey number $R(C_4, C_4) = 6 \neq 5 = 3k-4$. In this case, let c be a (P_6, P_6)-free coloring of K_7. Since $R(P_6, C_4) = 7$ and the graph does not contain a red P_6, there must be a blue C_4. The same proof as above holds for $K_7 - \{v, w\}$ with a blue C_4 and the resulting graph is G_3.

Case II: Either K_{r-1}^{ρ} or K_{r-1}^{β} is disconnected. Suppose that K_{r-1}^{ρ} is disconnected. Let X be a red component and $Y = K_{r-1} - X$. If both X and Y have at least $\frac{n}{2} = k$ vertices, then the graph contains a blue path P_n by Observation 2.3. Therefore, one of X or Y has at most $\frac{n}{2} - 1 = k-1$ vertices. This implies that $|V(X)| = (2k-1) + i$ and $|V(Y)| = (k-1) - i$ for some $i \in \{0, 1, 2, \ldots, \frac{n}{2} - 2 = k-2\}$. Note that if there is a blue $P_{2(i+1)}$ in X, then the graph contains a blue P_{2k} by Lemma 2.13. The Ramsey number $R(P_{2k}, P_{2(i+1)}) = 2k+i$ and so X either has a red P_{2k}, a blue $P_{2(i+1)}$ or X is the critical graph for $R(P_{2k}, P_{2(i+1)})$. Since the first two possibilities contradict the graph having a (P_{2k}, P_{2k})-free coloring, X must be such a critical graph. By Lemma 2.15, X must be G_1, a red K_{2k-1} with all edges to the remaining i vertices colored blue. Thus, the graph is G_1, a red K_{2k-1} with all blue edges to the remaining $i + |V(Y)| = i + (k-1) - i = k-1$ vertices. A similar proof holds if we suppose that K_{r-1}^{β} is disconnected resulting with the graph G_3. ■

***Proof of Proposition* 2.6.** Let $r = R(P_n, P_m) = n + \lfloor \frac{m}{2} \rfloor - 1$ and c be a (P_n, P_m)-free coloring of K_{r-1}. For odd $n \geq m+1$, the critical graphs are G_1

and G_2 by Lemma 2.9. For odd $n = m$, the critical graphs are G_1, G_2, G_3 and G_4 by Lemma 2.11. For even $n \geq m + 1$, the critical graphs are G_1, G_2 and G_5 if m is odd by Lemma 2.14 or G_1 if m is even by Lemma 2.15. For even $n = m$, the critical graphs are G_1 and G_3 by Lemma 2.16. ■

3. Determination of $r_*(P_n, P_m)$

Theorem 3.1. *For all $n \geq m \geq 4$, $r_*(P_n, P_m) = \lceil \frac{m}{2} \rceil$.*

Proof. A (P_n, P_m)-free coloring of $K_{n+\lfloor \frac{m}{2} \rfloor -2} \sqcup K_{1,\lceil \frac{m}{2} \rceil -1}$ is the graph G_1 as in Definition 2.4 and a vertex v with all red edges to $A_{\lfloor \frac{m}{2} \rfloor -1}$. Hence, $r_*(P_n, P_m) \geq \lceil \frac{m}{2} \rceil$.

Consider a 2-coloring of $K_{n+\lfloor \frac{m}{2} \rfloor -1}$ and remove a vertex v. Then the underlying complete graph on $n + \lfloor \frac{m}{2} \rfloor - 2$ vertices must have the structure of the critical graphs as in Proposition 2.6.

Case I: The underlying graph is G_1 in Definition 2.4. The red subgraph of G_1 is $K_{n-1} + A^{\rho}_{\lfloor \frac{m}{2} \rfloor -1}$. If v has a red edge to the red K_{n-1}, then the graph contains a red P_n. If m is even, then the graph contains a blue P_{m-1} and a blue edge from v to the red K_{n-1} creates a blue P_m. Therefore, $\lceil \frac{m}{2} \rceil$ edges adjacent to v force v to have an edge to the red clique yielding either a red P_n or a blue P_m. If m is odd, then the graph contains a blue P_{m-2} and v can have one blue edge to the red K_{n-1} without creating a blue P_m. This implies that $\lfloor \frac{m}{2} \rfloor + 1 = \lceil \frac{m}{2} \rceil$ edges force the graph to have either a red P_n or a blue P_m.

Case II: The underlying graph is G_2 with m odd in Definition 2.4. The red subgraph of G_2 is $(K_{n-1} - K_2) + A^{\rho}_{\lfloor \frac{m}{2} \rfloor -1}$. If v has a red edge to the red K_{n-1}, then the graph contains a red P_n. The graph also contains a blue P_{m-1} which uses the blue edges between the red $(K_{n-1} - K_2)$ and $A_{\lfloor \frac{m}{2} \rfloor -1}$ and the single blue edge within the red clique. A blue edge from v to the red $(K_{n-1} - K_2)$ creates a blue P_m. Therefore, $\lceil \frac{m}{2} \rceil$ edges adjacent to v force v to have an edge to the red clique yielding either a red P_n or a blue P_m.

Case III: The underlying graph is G_3 with $n = m$ in Definition 2.4. A similar proof holds as in Case I by interchanging the colors red and blue.

Case IV: The underlying graph is G_4 with $n = m$ odd in Definition 2.4. A similar proof holds as in Case II by interchanging the colors red and blue.

Case V: The underlying graph is G_5 with $n = m+1$ and m odd in Definition 2.4. The blue subgraph of G_5 is $K_{m-1} + A^{\beta}_{\frac{n}{2}-1}$. If v has a blue edge to the blue K_{m-1}, then the graph contains a blue P_m. The graph contains a red P_{n-1} and a red edge from v to the blue K_{m-1} creates a red P_n. Therefore, $\frac{n}{2} = \frac{m+1}{2} = \lceil \frac{m}{2} \rceil$ edges adjacent to v force v to have an edge to the red clique yielding either a red P_n or a blue P_m. Thus, it follows that $r_*(P_n, P_m) = \lceil \frac{m}{2} \rceil$. ■

Other classifications of critical graphs and star-critical Ramsey numbers including trees versus complete graphs, multiple copies of K_2 and K_3, and paths versus C_4 can be found in [2] and [3]. The critical graphs for $R(P_n, C_m)$ and cycles versus K_3 and K_4 have been found as well as their star-critical Ramsey numbers.

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