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ON UNIQUE MINIMUM DOMINATING SETS IN SOME CARTESIAN PRODUCT GRAPHS

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Abstract

Unique minimum vertex dominating sets in the Cartesian product of a graph with a complete graph are considered. We first give properties of such sets when they exist. We then show that when the first factor of the product is a tree, consideration of the tree alone is sufficient to determine if the product has a unique minimum dominating set.

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1. Introduction

In this paper, we show that if T is a nontrivial tree, then $T \square K_n$ has a unique minimum dominating set if and only if T has a minimum dominating set D such that each vertex in D has at least $n+1$ external private neighbors with respect to D. The study of unique minimum vertex dominating sets began with Gunther, Hartnell, Markus and Rall in [12] where the authors established a method for recognizing unique γ-sets in trees, and provided a characterization of those trees which have a unique γ-set. Their work was later expanded upon by Fischermann in [3] where block graphs were considered, and by Fischermann and Volkmann in [8] where cactus graphs were considered. The maximum number of edges contained in graphs with unique γ-sets was studied in [5] and [11], and complexity results concerning unique γ-sets can be found in [6]. Uniqueness of other types of dominating sets has also been studied. For example, edge domination was studied in [17] and [7]. Distance k domination was analyzed in [7]. Total domination was first studied in [14] and later in [4]. Mixed domination was considered in

[8], and paired domination was studied in [1]. Connections between unique minimum dominating sets and unique irredundant and independent dominating sets was studied in [10], while connections between maximum independent sets and unique upper dominating sets can be found in [9]. Finally, properties of unique domination were used in [16] and [15] to study properties of Roman dominating sets.

In the work to follow, we consider unique minimum dominating sets in graphs $G \,\square\, K_n$ where G is a connected, fnite, simple, nontrivial graph and K_n is the complete graph on n vertices. A characterization of the unique γ-sets in such graphs is considered in Section 3. Using this characterization, we then generalize a main result of [12] in Section 4, giving a method for recognizing a γ-set as unique when the first factor G is a tree. In Section 5, we consider the ways two such graphs, each having a unique minimum dominating set, can be combined while preserving a unique γ-set. Finally, in Section 6, we present the proof of our main result and characterize those trees whose Cartesian product with a complete graph has a unique γ-set.

2. Notation and Definitions

Let G be a finite, simple graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex u in G, the *open neighborhood* of u is the set $N(u)$ defined by $N(u) = \{v : uv \in E(G)\}$, and the *closed neighborhood* of u, denoted $N[u]$, is the set $N(u) \cup \{u\}$. If S is a subset of $V(G)$, then the open neighborhood of S is $\bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $S \cup N(S)$; these are denoted by $N(S)$ and $N[S]$, respectively. Any subset D of $V(G)$ with the property that $N[D] = V(G)$ is called a *dominating set* of G. A dominating set of G of minimum cardinality is called a *minimum dominating set* or a *γ-set* of G, and its cardinality is denoted by $\gamma(G)$. If D is a dominating set of G and $x \in D$, then a *private neighbor of x with respect to D* (or just a *private neighbor* if the dominating set is clear from the context) is any vertex u that belongs to $N[x] - N[D - \{x\}]$. If $u \neq x$, then u is also called an *external* private neighbor of x with respect to D. We let $epn(x, D)$ denote the set of external private neighbors of x with respect to D. A vertex in a dominating set need not have a private neighbor, but if the dominating set is minimal with respect to set inclusion, then each of its vertices has a private neighbor.

The Cartesian product of two graphs G_1 and G_2 is the graph $G_1 \,\square\, G_2$ whose vertex set is the Cartesian product of the sets $V(G_1)$ and $V(G_2)$ with two vertices. Two vertices (a_1, a_2) and (b_1, b_2) in $G_1 \,\square\, G_2$ adjacent if either $a_1 = b_1$ and $a_2 b_2 \in E(G_2)$, or $a_2 = b_2$ and $a_1 b_1 \in E(G_1)$. For $i = 1, 2$ we define the projections $\pi_{G_i} : G_1 \,\square\, G_2 \to G_i$ by $\pi_{G_i}((u_1, u_2)) = u_i$. Additionally, for $(u_1, u_2) \in V(G_1 \,\square\, G_2)$, we define the G_i-layer through (u_1, u_2) to be the induced subgraph

$$G_i^{(u_1,u_2)} = \langle \{(v_1, v_2) : \pi_{G_{3-i}}((v_1, v_2)) = \pi_{G_{3-i}}((u_1, u_2))\} \rangle .$$

We note that if A is a dominating set of $G_1 \,\square\, G_2$, then $\pi_{G_i}(A)$ dominates G_i for $i = 1$ and $i = 2$. For other graph product terminology, we follow [13].

We consider graphs $G \,\square\, K_n$ where G is a connected, finite, simple graph. We assume that the vertex set of K_n is $\{1, 2, \ldots, n\}$ which we will denote by $[n]$. For $u \in V(G)$ and for $k \in [n]$, we denote the G-layer through (u, k) as G^k for notational convenience. We let $\mathcal{U}$ denote the class of all finite simple graphs that have a unique minimum dominating set. If $G \in \mathcal{U}$, then we let $UD(G)$ denote the unique γ-set for G.

Our main theorem, proven in Section 6, is as follows.

Theorem 1. *Let n be a positive integer and let T be a nontrivial tree. The graph $T \,\square\, K_n \in \mathcal{U}$ if and only if T has a minimum dominating set D such that for all $v \in D$, $|epn(v, D)| \geq n + 1$.*

3. Basic Structure

Suppose that $G \,\square\, K_n \in \mathcal{U}$. What can we say about $UD(G \,\square\, K_n)$? We begin with the following observation.

Lemma 2. *If $G \,\square\, K_n \in \mathcal{U}$, then there exists $S \subseteq V(G)$ such that $UD(G \,\square\, K_n) = S \times [n]$.*

Proof. Denote $UD(G \,\square\, K_n)$ by D. Without loss of generality, suppose that $(v, 1) \in D$ but $(v, 2) \notin D$. Let

$$D' = \{(x, 1) : (x, 2) \in D\} \cup \{(y, 2) : (y, 1) \in D\} \cup \{(w, j) : (w, j) \in D, 3 \leq j \leq n\}.$$

We claim that D' is also a γ-set for $G \,\square\, K_n$.

- If $x \in \pi_G(D)$, then by the definition of D', it follows that the K_n-layer through $(x, 1)$ is contained in $N[D']$.
- If $x \notin \pi_G(D)$, then for $1 \leq j \leq n$, each (x, j) is dominated by some (v_j, j) in D. Thus, $(x, 1)$ is dominated by $(v_2, 1)$ in D', $(x, 2)$ is dominated by $(v_1, 2)$ in D', and (x, j) is dominated by (v_j, j) in D' for $3 \leq j \leq n$. Hence, every vertex in the K_n-layer through $(x, 1)$ is contained in $N[D']$.

Thus, we see that D' is a γ-set of $G \,\square\, K_n$ distinct from D, proving our result. ■

Corollary 3. *If $G \,\square\, K_n \in \mathcal{U}$, then $\gamma(G \,\square\, K_n)$ is a multiple of n.*

Any subset A of $V(G \,\square\, K_n)$ such that $A = S \times [n]$ for some subset S of $V(G)$ is said to have the *stacked property.* Before proceeding to our next result, we recall the following lemma from [12].

Lemma 4 [12]**.** *Let G be a graph with a unique γ-set D. Let $[u, v]$ be any edge in G other than an edge connecting a vertex in D to one of its private neighbors. Let G^- be the graph obtained from G by deleting the edge $[u, v]$. Then G^- has D as the unique γ-set.*

We now consider the following consequence of Lemma 2.

Proposition 5. *If $G \,\square\, K_n \in \mathcal{U}$, then $G \in \mathcal{U}$. Moreover, $G \,\square\, K_m \in \mathcal{U}$ for $1 \leq m \leq n$.*

Proof. Denote $UD(G \,\square\, K_n)$ by D. By Lemma 2, there exists $S \subseteq V(G)$ such that $D = S \times [n]$. Thus, for any $(x, i) \in D$, the external private neighbors of (x, i) with respect to D all belong to G^i. Define H to be the graph

$$G \,\square\, K_n - \{(v, n)(v, j) : v \in V(G), 1 \leq j \leq n-1\}.$$

We see that H is isomorphic to $(G \,\square\, K_{n-1}) \cup G$. By Lemma 4, D is still the unique γ-set for H. The proposition follows by induction. ∎

Suppose that $A \subseteq V(G \,\square\, K_n)$ has the stacked property and that $\{v\} \times [n] \subseteq A$. If $(u, j) \in epn((v, j), A)$ for some j, then $(u, i) \in epn((v, i), A)$ for $1 \leq i \leq n$. Bearing this in mind, suppose that D is a γ-set of $G \,\square\, K_n$ with the stacked property. Additionally, suppose that $(v, 1) \in D$ has $epn((v, 1), D) = \{(u_1, 1), (u_2, 1), \ldots, (u_j, 1)\}$ for some $j \leq n$. This implies that $epn((v, i), D) = \{(u_1, i), (u_2, i), \ldots, (u_j, i)\}$ for $2 \leq i \leq n$. The set D' defined by

$$D' = (D - \{(v, 1), (v, 2), \ldots, (v, j)\}) \cup \{(u_1, 1), (u_2, 2), \ldots, (u_j, j)\}$$

is a γ-set of $G \,\square\, K_n$ distinct from D. Thus, we have the following.

Lemma 6. *If G is a connected, nontrivial graph such that $G \,\square\, K_n \in \mathcal{U}$, then for each element $v \in UD(G \,\square\, K_n)$,*

$$|epn(v, UD(G \,\square\, K_n))| \geq n + 1.$$

The graph $K_{1,n+1} \,\square\, K_n$ demonstrates that this "bound" is sharp. The family of graphs $K_m \,\square\, K_n$, $m \geq n$, demonstrates that no condition on the number of external private neighbors for vertices in a minimum dominating set is, by itself, sufficient to force the product with K_n to have a unique γ-set. For use in the proof of Theorem 13 to follow, we note here the following.

Observation 7. *If $v \in V(G)$ has at least $n+1$ leaf neighbors, then $\{v\} \times [n]$ is contained in every γ-set of $G \,\Box\, K_n$.*

In [12], the authors prove the following lemma.

Lemma 8 [12]**.** *Let D be a γ-set of a graph G. If for every $x \in D$, $\gamma(G - x) > \gamma(G)$, then D is the unique γ-set of G.*

The following statement is a generalization of this result to our setting.

If $G \,\Box\, K_n$ has a γ-set D satisfying the stacked property such that for every $v \in \pi_G(D)$, $\gamma(G \,\Box\, K_n - (\{v\} \times [n])) > \gamma(G \,\Box\, K_n)$, then D is the unique γ-set of $G \,\Box\, K_n$.

This statement, however, does not hold for a general product $G \,\Box\, K_n$. The graph G illustrated in Figure 1 provides a counterexample. Define H to be the graph $G \,\Box\, K_2$. The set D defined by $D = \{1, 2, 3, 4, 5, 6\} \times \{1, 2\}$ is a γ-set satisfying the stacked property such that for every $v \in \pi_G(D)$, $\gamma(H - \{(v, 1), (v, 2)\}) > \gamma(H)$. However, D is not a unique γ-set since the set $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2), (4, 1), (5, 1), (6, 1), (10, 2), (14, 2), (18, 2)\}$ is also a γ-set of H.

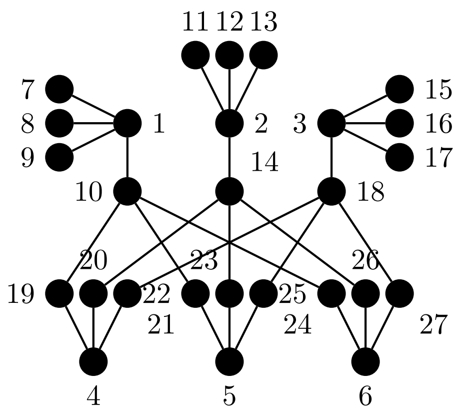

Figure 1

In the next section, we will show that if G is a tree, then the conditions above do imply that $G \,\Box\, K_n \in \mathcal{U}$. The following lemma will be used in the proof.

Lemma 9. *If $G \,\Box\, K_n$ has a γ-set D satisfying the stacked property such that for every $v \in \pi_G(D)$, $\gamma(G \,\Box\, K_n - (\{v\} \times [n])) > \gamma(G \,\Box\, K_n)$, then for all $y \in D$, $|epn(y, D)| \geq n + 1$.*

Proof. Let $v \in \pi_G(D)$. Suppose for some $j \leq n$ that

$$epn((v, 1), D) = \{(u_1, 1), (u_2, 1), \ldots, (u_j, 1)\}.$$

Since D satisfies the stacked property,

$$epn((v,i),D) = \{(u_1,i),(u_2,i),\ldots,(u_j,i)\}$$

for $1 \le i \le n$. The set

$$(D - (\{v\} \times [n])) \cup \{(u_1,1),(u_2,2),\ldots(u_j,j),(u_j,j+1),\ldots,(u_j,n)\}$$

is a dominating set of $G \square K_n - (\{v\} \times [n])$ of cardinality equal to $|D|$, a contradiction. Thus, our result follows. ■

Before we proceed to our first theorem, we need the following two lemmas, which are generalizations of Lemmas 3 and 4 from [12].

Lemma 10. *Let $G \square K_n \in \mathcal{U}$ and let $v \notin \pi_G(UD(G \square K_n))$. For any subset B of $\{v\} \times [n]$, $\gamma(G \square K_n - B) = \gamma(G \square K_n)$.*

Proof. Suppose that $\gamma(G \square K_n - B) < \gamma(G \square K_n)$. This implies that $G \square K_n - B$ is dominated by a set D' with $|D'| < |UD(G \square K_n)|$. However, for any $(v,i) \in B$, $D' \cup \{(v,i)\}$ is a dominating set of $G \square K_n$ distinct from $UD(G \square K_n)$ of cardinality less than or equal to $|UD(G \square K_n)|$, a contradiction. Thus, $\gamma(G \square K_n - B) \ge \gamma(G \square K_n)$. Since $UD(G \square K_n)$ dominates $G \square K_n - B$, we see that $\gamma(G \square K_n - B) = \gamma(G \square K_n)$. ■

Lemma 11. *Let G be a connected, nontrivial graph, let $G \square K_n \in \mathcal{U}$, and let $v \in \pi_G(UD(G \square K_n))$. For any subset B of $\{v\} \times [n]$, $\gamma(G \square K_n - B) \ge \gamma(G \square K_n)$.*

Proof. For the sake of contradiction, suppose that $\gamma(G \square K_n - B) < \gamma(G \square K_n)$ for some $B \subseteq \{v\} \times [n]$. If D' is a γ-set of $G \square K_n - B$, then $|D'| < |UD(G \square K_n)|$ and D' dominates all of the external private neighbors of the vertices in B with respect to $UD(G \square K_n)$. However, for any $(v,i) \in B$, $D' \cup \{(v,i)\}$ is a γ-set of $G \square K_n$ and $UD(G \square K_n) \ne D' \cup \{(v,i)\}$, a contradiction. ■

4. Trees

In this section, we restrict our attention to graphs $T \square K_n$ where T is a nontrivial tree. We prove a set of equivalences which can be used to determine whether a γ-set in $T \square K_n$ is unique. This result, formulated as Theorem 13 below, is a generalization of the following theorem from [12], and as such, the notation and proof structure are similar.

Theorem 12 [12]**.** *Let T be a tree of order at least 3. The following conditions are equivalent.*

(1) *T has a unique γ-set D.*

(2) *T has a γ-set D for which every vertex $x \in D$ has at least two private neighbors other than itself.*

(3) *T has a γ-set D for which every vertex $x \in D$ has the property that $\gamma(T-x) > \gamma(T)$.*

Theorem 13. *Let T be a nontrivial tree. The following conditions are equivalent.*

(1) $T \,\square\, K_n \in \mathcal{U}$.

(2) *$T \,\square\, K_n$ has a stacked γ-set D such that for all $v \in D$, $|epn(v, D)| \geq n+1$.*

(3) *$T \,\square\, K_n$ has a stacked γ-set A such that for every $v \in \pi_G(A)$, $\gamma(T \,\square\, K_n - (\{v\} \times [n])) > \gamma(T \,\square\, K_n)$.*

Proof. By Lemmas 2 and 6, we see that statement (1) implies statement (2). We first show that statement (2) implies statement (1). We proceed by induction on $|V(T)|$.

The base case is given by $T = K_{1,n+1}$ where the result holds. We note that for any other tree T on $n+2$ vertices, statement (2) does not hold for $T \,\square\, K_n$. Suppose then that the result has been shown whenever $|V(T)| < r$. Let T be a tree on r vertices for which there exists a subset $S \subseteq V(T)$ such that $S \times [n]$ is a γ-set for $T \,\square\, K_n$ and such that every element $v \in S \times [n]$ satisfies $|epn(v, S \times [n])| \geq n+1$. To simplify notation, we let $D = S \times [n]$ and $H = T \,\square\, K_n$. Suppose that $H - D$ contains two vertices $(u, 1)$, $(v, 1)$ which are connected by the edge $(u, 1)(v, 1)$. Let $H(u)$ be the component of $(T - uv) \,\square\, K_n$ containing $(u, 1)$, and let $H(v)$ be the component containing $(v, 1)$. Let $D(u) = D \cap V(H(u))$ and $D(v) = D \cap V(H(v))$. We first claim that $D(u)$ and $D(v)$ are γ-sets for $H(u)$ and $H(v)$ respectively. To see this, note that $D(u)$ and $D(v)$ dominate $H(u)$ and $H(v)$. Additionally, if $H(u)$, for example, had a γ-set A of cardinality smaller than $|D(u)|$, then $A \cup D(v)$ would be a dominating set of $T \,\square\, K_n$ smaller than D, a contradiction. Since all private neighbors with respect to D are preserved in the individual components, our induction hypothesis implies that $D(u)$ and $D(v)$ are the unique γ-sets for $H(u)$ and $H(v)$ respectively.

Assume now that D' is a γ-set of H distinct from D. If $D' \cap (\{u, v\} \times [n]) = \emptyset$ then $D' \cap V(H(u)) = D(u)$ and $D' \cap V(H(v)) = D(v)$, a contradiction. Thus, $D' \cap (\{u, v\} \times [n]) \neq \emptyset$.

If $D' \cap (\{u\} \times [n]) \neq \emptyset$, then $D' \cap V(H(u))$ dominates $H(u)$ in which case $|D' \cap V(H(u))| > |D(u)|$. Similarly, if $D' \cap (\{v\} \times [n]) \neq \emptyset$, then $|D' \cap V(H(v))| > |D(v)|$.

If $D' \cap (\{u\} \times [n]) = \emptyset$ but $D' \cap (\{v\} \times [n]) \neq \emptyset$, then certainly $D' \cap V(H(u))$ dominates $H(u) - (\{u\} \times [n])$ in which case by Lemma 10, $|D' \cap V(H(u))| \geq |D(u)|$. Similarly, if $D' \cap (\{v\} \times [n]) = \emptyset$ but $D' \cap (\{u\} \times [n]) \neq \emptyset$, then $|D' \cap V(H(v))| \geq |D(v)|$.

Thus, since $D' \cap (\{u, v\} \times [n]) \neq \emptyset$, we see that $|D'| = |D' \cap V(H(u))| + |D' \cap V(H(v))| > |D(u)| + |D(v)| = |D|$, a contradiction. Hence, in this case, D is the unique γ-set for H.

Our last case assumes there are no edges in H of the form $(u, 1)(v, 1)$ with $(u, 1), (v, 1) \in V(H) - D$. In this case, let $(x, i) \in D$. If (y, i) is an external private neighbor of (x, i) with respect to D, then y is a leaf of T. Hence, $x \in V(T)$ has at least $n + 1$ leaf neighbors. As observed above, this implies that $\{x\} \times [n]$ is contained in every γ-set of H. Since $(x, i) \in D$ was arbitrary, we see that D is the unique γ-set of H. Hence, we have now shown that (1) and (2) are equivalent.

Assume now that statement (3) holds. By Lemma 9, statement (2) holds. Our work above then implies that statement (1) also holds. Thus, we next prove that statement (1) implies statement (3).

Let $T \,\square\, K_n \in \mathcal{U}$. Let $D = UD(T \,\square\, K_n)$ and let $H = T \,\square\, K_n$. By Lemma 2, there exists $S \subseteq V(T)$ such that $D = S \times [n]$. Suppose that $\{v\} \times [n] \subseteq D$. Partition $N((v, 1)) \cap V(G^1)$ as $epn((v, 1), D) \cup Q((v, 1))$. Let

$$epn((v, 1), D) = \{(p_1, 1), (p_2, 1), \ldots, (p_m, 1)\}$$

and

$$Q((v, 1)) = \{(q_1, 1), (q_2, 1), \ldots, (q_k, 1)\}.$$

We know that $m \geq n + 1$ and that $k \geq 0$. Let $H(p_i)$, respectively $H(q_j)$, be the component of $H - (\{v\} \times [n])$ containing $(p_i, 1)$, respectively $(q_j, 1)$. For $1 \leq i \leq m$, let $D(p_i) = D \cap V(H(p_i))$ and define $D(q_j)$ similarly. Since T is a tree, we see that

$$\gamma(H) = |D| = n + \sum\nolimits_{i=1}^{m} |D(p_i)| + \sum\nolimits_{j=1}^{k} |D(q_j)|.$$

Since $H - (\{v\} \times [n])$ is the disjoint union

$$\left[\bigcup\nolimits_{i=1}^{m} H(p_i)\right] \cup \left[\bigcup\nolimits_{j=1}^{k} H(q_j)\right],$$

we can calculate $\gamma(H - (\{v\} \times [n]))$ by calculating $\gamma(H(p_i))$ and $\gamma(H(q_j))$ for each i and j and summing the results.

First, we consider $H(p_i)$. If $V(H(p_i)) = \{p_i\} \times [n]$, then $D(p_i) = \emptyset$. In this case, it is easy to see that $\gamma(H(p_i)) = 1 = |D(p_i)| + 1$.

If $V(H(p_i)) \neq \{p_i\} \times [n]$, then $D(p_i) \neq \emptyset$. Moreover, for each j such that $1 \leq j \leq n$, no neighbor of (p_i, j) in the graph $H(p_i)$ is in $D(p_i)$, since $(p_i, j) \in epn((v, j), D)$. Thus, $D(p_i)$ is not a γ-set for $H(p_i)$ since it does not dominate $(p_i, 1)$. Nevertheless, suppose that $\gamma(H(p_i)) = |D(p_i)|$, and let B be a γ-set of $H(p_i)$. It follows that $(D - D(p_i)) \cup B$ is a dominating set of H of cardinality equal to $|D|$, contradicting the uniqueness of D. Hence, $\gamma(H(p_i)) > |D(p_i)|$. Since $D(p_i) \cup \{(p_i, 1)\}$ dominates $H(p_i)$, we see, once again, that $\gamma(H(p_i)) = |D(p_i)| + 1$.

Next, we consider $H(q_j)$. Since $(q_j, i) \notin epn((v,i), D)$ for $1 \leq i \leq n$, we see that $D(q_j)$ is a γ-set of $H(q_j)$. Moreover, for each $v \in D(q_j)$, $|epn(v, D(q_j))| \geq n+1$. Thus, $D(q_j)$ is the unique γ-set of $H(q_j)$, giving us that $\gamma(H(q_j)) = |D(q_j)|$.

Thus, we can now compute $\gamma(H - (\{v\} \times [n]))$:

$$\begin{aligned}
\gamma(H - (\{v\} \times [n])) &= \sum_{i=1}^{m} \gamma(H(p_i)) + \sum_{j=1}^{k} \gamma(H(q_j)) \\
&= \sum_{i=1}^{m} (|D(p_i)| + 1) + \sum_{j=1}^{k} |D(q_j)| \\
&= \gamma(H) + m - n \\
&\geq \gamma(H) + (n+1) - n \\
&= \gamma(H) + 1 > \gamma(H).
\end{aligned}$$

Thus, we see that statement (1) implies statement (3), and our proof is complete. ∎

In Section 6 to follow, we will use this result to show that finding a γ-set in $T \,\square\, K_n$ is not required to determine whether $T \,\square\, K_n \in \mathcal{U}$. We will show that analysis of a γ-set of T will suffice.

5. Combining Graphs with Unique γ-Sets

Suppose that $G_1 \,\square\, K_n$ and $G_2 \,\square\, K_n$ have unique minimum dominating sets. In this section, we consider the ways in which these two graphs can be combined to produce a new graph having a unique minimum dominating set. We discuss four operations. Throughout this section, $G_1 \,\square\, K_n$ and $G_2 \,\square\, K_n$, denoted H_1 and H_2 respectively, are nontrivial graphs in $\mathcal{U}$. Let D_1 and D_2 denote the sets $UD(G_1 \,\square\, K_n)$ and $UD(G_2 \,\square\, K_n)$ respectively.

Operation 1. If $x \notin \pi_{G_1}(D_1)$ and $y \notin \pi_{G_2}(D_2)$, then $((G_1 \cup G_2) + xy) \,\square\, K_n \in \mathcal{U}$ and $UD(((G_1 \cup G_2) + xy) \,\square\, K_n) = D_1 \cup D_2$.

Proof. Let H denote the graph $((G_1 \cup G_2) + xy) \,\square\, K_n$. First, we see that $D_1 \cup D_2$ dominates all of H. Let D be a γ-set for H. It follows that

$$|D| \leq |D_1 \cup D_2| = |D_1| + |D_2|.$$

Without loss of generality, suppose that $|D \cap V(H_1)| \leq |D_1|$. Since the only vertices of H_1 that could be dominated from outside of H_1 are elements of $\{x\} \times [n]$, we see that either $D \cap V(H_1)$ dominates all of H_1, or $D \cap V(H_1)$ fails to dominate a subset B of $\{x\} \times [n]$.

First, suppose that $D \cap V(H_1)$ dominates all of H_1. Since H_1 has a unique γ-set, and since we are assuming $|D \cap V(H_1)| \leq |D_1|$, we have that $D \cap V(H_1) =$

D_1. However, if $D \cap V(H_1) = D_1$, then we also have $D \cap V(H_2) = D_2$ since $x \notin \pi_{G_1}(D_1)$. Thus, in this case, we have that $D = D_1 \cup D_2$.

Now suppose that $D \cap V(H_1)$ fails to dominate a subset B of $\{x\} \times [n]$. By Lemma 10, we have that $|D \cap V(H_1)| \geq |D_1|$. Since $|D| \leq |D_1| + |D_2|$, we have that $|D \cap V(H_2)| \leq |D_2|$. Note, however, that $D \cap V(H_2)$ intersects $\{y\} \times [n]$, in which case we have a set of cardinality at most $|D_2|$ that is distinct from D_2 and dominates H_2. This contradicts the uniqueness of D_2. Our result now follows. ■

Operation 2. Let $x \in \pi_{G_1}(D_1)$ and $y \in \pi_{G_2}(D_2)$. If u is a new vertex in neither G_1 nor G_2, then $((G_1 \cup G_2) + \{ux, uy\}) \,\square\, K_n \in \mathcal{U}$ and $UD(((G_1 \cup G_2) + \{ux, uy\}) \,\square\, K_n) = D_1 \cup D_2$.

Proof. Let H denote the graph $((G_1 \cup G_2) + \{ux, uy\}) \,\square\, K_n$. First, note that $D_1 \cup D_2$ dominates H. If D is a γ-set of H with $|D| < |D_1| + |D_2|$, then $D \cap (\{u\} \times [n]) \neq \emptyset$. Suppose that $\{(u, i_1), (u, i_2), \ldots, (u, i_k)\} \subseteq D$. Then $\{(x, i_1), (x, i_2), \ldots, (x, i_k)\}$ and $\{(y, i_1), (y, i_2), \ldots, (y, i_k)\}$ need not be dominated from H_1 and H_2 respectively. However, by Lemma 11, we know that $|D \cap V(H_1)| \geq |D_1|$ and that $|D \cap V(H_2)| \geq |D_2|$. Thus, $|D| \geq |D_1| + |D_2| + k > |D_1 \cup D_2|$. Thus, no γ-set of H intersects $\{u\} \times [n]$. Hence, any γ-set of H intersects each of $V(H_1)$ and $V(H_2)$ in a γ-set, in which case $D = D_1 \cup D_2$. ■

Before we discuss the next operation, we need the following lemma.

Lemma 14. *Let T be a tree, and let $T \,\square\, K_n \in \mathcal{U}$. If $(v, i) \notin UD(T \,\square\, K_n)$ is adjacent to at least two elements of $UD(T \,\square\, K_n)$, then $(T - v) \square K_n \in \mathcal{U}$ and $UD((T - v) \square K_n) = UD(T \square K_n)$.*

Proof. Let H' denote the graph $(T - v) \square K_n$ and let D denote the set $UD(T \square K_n)$. By Lemma 10, we know that $\gamma(H') = \gamma(T \,\square\, K_n)$. Thus, D is a γ-set for H'. We must show that D is the only γ-set for H'. Note that the removal of $(v, 1), (v, 2), \ldots, (v, n)$ from $T \,\square\, K_n$ breaks $T \,\square\, K_n$ into $k \geq 2$ components; call them $H_1, H_2, \ldots, H_k$.

We claim that for $i = 1, 2, \ldots k$, $D_i = D \cap V(H_i)$ is the unique γ-set for H_i. Without loss of generality, consider D_1. Clearly D_1 is a dominating set for H_1. If D_1' were a smaller dominating set of H_1, then $D_1' \cup D_2 \cup \cdots \cup D_k$ would be a smaller γ-set for $T \,\square\, K_n$. Thus, D_1 will be a γ-set of H_1. By the same logic, D_1 is the unique γ-set for H_1.

Thus, each H_i has D_i as its unique minimum dominating set, in which case H' has $D = D_1 \cup D_2 \cup \cdots \cup D_k$ as its unique γ-set. ■

Operation 3. Let G_2 be a tree. If $x \in \pi_{G_1}(D_1)$, $y \notin \pi_{G_2}(D_2)$, and y is a neighbor of at least two vertices in $\pi_{G_2}(D_2)$, then $((G_1 \cup G_2) + xy) \,\square\, K_n \in \mathcal{U}$ and $UD(((G_1 \cup G_2) + xy) \,\square\, K_n) = D_1 \cup D_2$.

Proof. Let H denote the graph $((G_1 \cup G_2) + xy) \,\square\, K_n$. Note that $D_1 \cup D_2$ dominates H. Let D be a γ-set of H. Suppose that $(\{y\} \times [n]) \cap D \neq \emptyset$. This implies that some subset of $\{x\} \times [n]$ will be dominated from outside of H_1. By Lemma 11, we still have that $|D \cap V(H_1)| \geq |D_1|$. Additionally, $D \cap V(H_2)$ dominates H_2, in which case $|D \cap V(H_2)| > |D_2|$ since D_2 is the unique γ-set for H_2 and $y \notin \pi_{G_2}(D_2)$. Thus, we have $|D| > |D_1 \cup D_2|$. This implies that no γ-set of H intersects $\{y\} \times [n]$. Hence, if D is a γ-set for H, then $D \cap V(H_1) = D_1$. Lemma 6 then implies that $D \cap V(H_2) = D_2$. Thus, $D_1 \cup D_2$ is the unique γ-set for H. ∎

Operation 4. Let G_1 and G_2 be trees. If $x \in \pi_{G_1}(D_1)$ and $y \in \pi_{G_2}(D_2)$, then $((G_1 \cup G_2) + \{xy\}) \,\square\, K_n \in \mathcal{U}$ and $UD(((G_1 \cup G_2) + \{xy\}) \,\square\, K_n) = D_1 \cup D_2$.

Proof. Once again, let H denote the graph $((G_1 \cup G_2) + \{xy\}) \,\square\, K_n$. Since $D_1 \cup D_2$ dominates H, we have that $\gamma(H) \leq |D_1 \cup D_2| = |D_1| + |D_2|$. Let A denote the set $\{x\} \times [n]$, let B denote the set $\{y\} \times [n]$, and suppose that D is a γ-set for H.

If $A \subseteq D$ and $B \subseteq D$, then $D \cap V(H_1)$ and $D \cap V(H_2)$ are γ-sets for H_1 and H_2, respectively, in which case $D = D_1 \cup D_2$.

Suppose that $D \cap A = \emptyset$. This implies that $D \cap V(H_1)$ dominates $H_1 - A$. However, by Theorem 13, we know that $\gamma(H_1 - A) > \gamma(H_1) = |D_1|$. Additionally, in this case $D \cap V(H_2)$ is a γ-set of H_2 implying that $D \cap H_2 = D_2$. Thus, we have $|D| > |D_1| + |D_2| = |D_1 \cup D_2|$. The same contradiction arises if $D \cap B = \emptyset$.

This leaves us with one case to consider. Without loss of generality, suppose that $0 < |D \cap A| < |A|$ and that $D \cap B \neq \emptyset$. Then $D \cap V(H_1)$ dominates H_1 and $D \cap V(H_2)$ dominates H_2. However, since D_1 and D_2 are the unique γ-sets for H_1 and H_2 respectively, and since $A \subseteq D_1$, we have that $|D \cap V(H_1)| > |D_1|$ and that $|D \cap V(H_2)| \geq |D_2|$. Thus, we have $|D| > |D_1 \cup D_2|$, a contradiction.

Thus, we have $D = D_1 \cup D_2$, which implies $D_1 \cup D_2$ is the unique γ-set for H. ∎

6. Main Result

We are now able to prove our main theorem, which we restate for your convenience.

Theorem 1. *Let n be a positive integer and let T be a nontrivial tree. The graph $T \,\square\, K_n \in \mathcal{U}$ if and only if T has a minimum dominating set D such that for all $v \in D$, $|epn(v, D)| \geq n + 1$.*

Proof. Suppose that $T \,\square\, K_n$ has a unique γ-set denoted UD. By Lemma 2, we know that UD satisfies the stacked property. Thus, there exists $S \subseteq V(T)$

such that $UD = S \times [n]$. Additionally, by Lemma 6, for every element $v \in UD$, $|epn(v, UD)| \geq n + 1$. This implies that for every element $w \in S$, $|epn(w, S)| \geq n + 1$. Finally, by the proof of Proposition 5, S is a γ-set for T.

Now suppose that T is a tree, and suppose that T has a γ-set D' for which every element in D' has at least $n+1$ external private neighbors with respect to D'. Define H to be the graph $T \,\Box\, K_n$, and let $D = D' \times [n]$. Clearly D is a dominating set for H. Furthermore, since every element of D' has at least $n+1$ external private neighbors with respect to D' in T, every element of D has at least $n+1$ external private neighbors with respect to D in H. Thus, if we can prove that D is a γ-set for H, Theorem 13 will imply that D is the unique γ-set for H. We do this by induction on the cardinality of T.

The base case is given by $T = K_{1,n+1}$ where the result holds. Thus, assume the result holds whenever $|V(T)| < r$. Let T be a tree on r vertices having a γ-set D' for which every element in D' has at least $n+1$ external private neighbors with respect to D'. By Theorem 12, D' is the unique γ-set for T. Let H be $T \,\Box\, K_n$, and let D be defined as above. Consider a diametral path in T, call it $v_1 v_2 \cdots v_k$. Note that v_k is a leaf and cannot be an element of D'. This implies that $v_{k-1} \in D'$. In order for v_{k-1} to have at least $n+1$ external private neighbors, v_{k-1} must be adjacent to at least $n-1$ other leaves. Let A be the set $\{v_{k-1}\} \cup epn(v_{k-1}, D')$ and let $B = \{v \in N(v_{k-1}) : v \notin D', |N(v) \cap D'| \geq 2\}$. We note that B equals either the empty set or $\{v_{k-2}\}$. By Theorem 12, $\{v_{k-1}\}$ and $D' - \{v_{k-1}\}$ are the unique minimum dominating sets for $T\langle A\rangle$ and $T - (A \cup B)$ respectively. By our induction hypothesis, $\{v_{k-1}\} \times [n]$ and $D - (\{v_{k-1}\} \times [n])$ are the unique minimum dominating sets for $T\langle A\rangle \,\Box\, K_n$ and $(T - (A \cup B)) \,\Box\, K_n$ respectively. Our original graph H can be reconstructed from $T\langle A\rangle \,\Box\, K_n$ and $(T - (A \cup B)) \,\Box\, K_n$ by performing at least one of the operations discussed in Section 5 above. Hence, D is the unique γ-set for H. ■

Theorem 1 implies that in order to determine whether $T \,\Box\, K_n \in \mathcal{U}$ it is sufficient to consider T alone. That is, one need only find a γ-set in T and count the number of external private neighbors for each vertex in the set. Since finding a minimum dominating set in a tree can be done in linear time (see [2]), we see that the problem of determining for which K_n, $T \,\Box\, K_n \in \mathcal{U}$ is polynomial.

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