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Note

ON THE HYPERCOMPETITION NUMBERS OF HYPERGRAPHS WITH MAXIMUM DEGREE AT MOST TWO

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Abstract

In this note, we give an easy and short proof for the theorem by Park and Kim stating that the hypercompetition numbers of hypergraphs with maximum degree at most two is at most two.

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1. Introduction

Throughout this note, we assume that hypergraphs have no loops (i.e., all the hyperedges of hypergraphs contain at least two vertices) and no multiple hyperedges. The notion of competition hypergraphs was introduced by Sonntag and Teichert [3] in 2004 as a generalization of the notion of competition graphs. Let D be a digraph. The *competition hypergraph* of D is the hypergraph $C\mathcal{H}(D)$ whose vertex set is the same as D such that $e \subseteq V(D)$ is a hyperedge of $C\mathcal{H}(D)$ if and only if $|e| \geq 2$ and e is the in-neighborhood of a vertex of D.

Given a hypergraph $\mathcal{H}$, it is easy to observe that $\mathcal{H}$ together with sufficiently many isolated vertices is the competition hypergraph of an acyclic digraph. Indeed, the digraph D defined by $V(D) = V(\mathcal{H}) \cup \{z_e \mid e \in E(\mathcal{H})\}$ and $A(D) = \{(x, z_e) \mid x \in e\}$ is acyclic and its competition hypergraph is $\mathcal{H} \cup \{z_e \mid e \in E(\mathcal{H})\}$, where z_e is a new vertex for each $e \in E(\mathcal{H})$. The *hypercompetition number* of a hypergraph $\mathcal{H}$, denoted by $\mathrm{hk}(\mathcal{H})$, is defined to be the minimum nonnegative integer k such that $\mathcal{H}$ together with k isolated vertices is the competition hypergraph of an acyclic digraph.

In 2012, Park and Kim [1] showed the following theorem. Here, recall that the *degree* of a vertex v in a hypergraph $\mathcal{H}$ is the number of hyperedges of $\mathcal{H}$ containing v and is denoted by $\deg_{\mathcal{H}}(v)$.

Theorem 1 [1]**.** *For a hypergraph $\mathcal{H}$, if $\deg_{\mathcal{H}}(v) \leq 2$ for any vertex v of $\mathcal{H}$, then $\mathrm{hk}(\mathcal{H}) \leq 2$ and the equality holds if and only if $\deg_{\mathcal{H}}(v) = 2$ for each vertex v of $\mathcal{H}$.*

The purpose of this note is to give an easy and short proof for the above theorem.

2. A Short Proof for Theorem 1

Lemma 2. *Let $\mathcal{H}$ be a hypergraph in which all the vertices have degree at most two. If $\mathcal{H}$ has a vertex of degree at most one, then $\mathrm{hk}(\mathcal{H}) \leq 1$.*

Proof. We prove the lemma by induction on the number of hyperedges of a hypergraph. If $|E(\mathcal{H})| = 0$, then $\mathrm{hk}(\mathcal{H}) = 0$ and so the statement holds. Assume that the statement holds for any hypergraph $\mathcal{H}$ with $|E(\mathcal{H})| \leq k$, where k is a nonnegative integer. Consider a hypergraph $\mathcal{H}$ with $|E(\mathcal{H})| = k + 1$. By assumption, there exists a vertex v such that $\deg_{\mathcal{H}}(v) \leq 1$. If $\deg_{\mathcal{H}}(v) = 0$, then let e be any hyperedge of $\mathcal{H}$. If $\deg_{\mathcal{H}}(v) = 1$, then let e be the hyperedge containing v. Consider the hypergraph $\mathcal{H}^*$ obtained from $\mathcal{H}$ by deleting the vertex v and the hyperedge e, i.e., $\mathcal{H}^* := (V(\mathcal{H}) \setminus \{v\}, E(\mathcal{H}) \setminus \{e\})$. Then $|E(\mathcal{H}^*)| = k$ and $\deg_{\mathcal{H}^*}(w) \leq 1$. By the induction hypothesis, we have $\mathrm{hk}(\mathcal{H}^*) \leq 1$. So there exists an acyclic digraph D^* such that $C\mathcal{H}(D^*) = \mathcal{H}^* \cup \{v\}$. We define a digraph D by $V(D) = V(D^*) \cup \{z\}$ and $A(D) = A(D^*) \cup \{(x, z) \mid x \in e\}$, where z is a new vertex. Then D is acyclic and $C\mathcal{H}(D) = \mathcal{H} \cup \{z\}$. Thus $\mathrm{hk}(\mathcal{H}) \leq 1$. Hence the lemma holds. ■

Recall a result on the lower bound for the hypercompetition numbers of hypergraphs given by Park and Sano [2].

Lemma 3 [2]**.** *For any hypergraph $\mathcal{H}$, $\mathrm{hk}(\mathcal{H}) \geq \min_{v \in V(\mathcal{H})} \deg_{\mathcal{H}}(v)$.*

Now we are ready to give a proof of Theorem 1.

***Proof of Theorem* 1.** If there exists a vertex of degree at most one in a hypergraph $\mathcal{H}$, then the statement follows from Lemma 2. Suppose that all the vertices in $\mathcal{H}$ have degree two. Let e be any hyperedge of $\mathcal{H}$. Consider the hypergraph $\mathcal{H}'$ obtained from $\mathcal{H}$ by deleting the hyperedge e, i.e., $\mathcal{H}' := (V(\mathcal{H}), E(\mathcal{H}) \setminus \{e\})$. Note that $\deg_{\mathcal{H}'}(x) \leq 1$ for any vertex x which is contained in e. By Lemma 2, we have $\text{hk}(\mathcal{H}') \leq 1$. So there exists an acyclic digraph D' such that $C\mathcal{H}(D') = \mathcal{H}' \cup \{z_1\}$, where z_1 is a new vertex. Define a digraph D by $V(D) = V(D') \cup \{z_2\}$ and $A(D) = A(D') \cup \{(x, z_2) \mid x \in e\}$, where z_2 is a new vertex. Then D is acyclic and $C\mathcal{H}(D) = \mathcal{H} \cup \{z_1, z_2\}$. Therefore, $\text{hk}(\mathcal{H}) \leq 2$. Since all the vertices in $\mathcal{H}$ have degree two, it follows from Lemma 3 that $\text{hk}(\mathcal{H}) \geq 2$. Thus $\text{hk}(\mathcal{H}) = 2$. Hence the theorem holds. ■

3. A Remark on the Hypercompetition Numbers of Hypergraphs with Bounded Degree

Theorem 1 says that, if the degrees of the vertices in a hypergraph are bounded above by two, then the hypercompetition number of the hypergraph is also bounded above by two. From this result, one might ask the following question: *Are the hypercompetition numbers of hypergraphs bounded when the degrees of the vertices are bounded?* However, the answer of this question is no even if the degrees of the vertices are bounded above by three. To see this, we recall a result by Park and Sano [2] on the hypercompetition numbers of connected graphs (2-uniform hypergraphs).

Lemma 4 [2]**.** *For any connected graph $\mathcal{H}$ (having at least two vertices), we have* $\text{hk}(\mathcal{H}) = |E(\mathcal{H})| - |V(\mathcal{H})| + 2$.

By Lemma 4, the hypercompetitiom number of a 3-regular connected graph with n vertices is equal to $\frac{1}{2}n + 2$. Thus, the hypercompetition numbers of hypergraphs in which all the vertices have degree at most three are unbounded.

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