STRONG $f$-STAR FACTORS OF GRAPHS

ZHENG YAN

Yangtze University, Jingzhou
Hubei, China

e-mail: yanzhenghubei@163.com

Abstract

Let $G$ be a graph and $f : V(G) \to \{2, 3, \ldots\}$. A spanning subgraph $F$ is called strong $f$-star of $G$ if each component of $F$ is a star whose center $x$ satisfies $\deg_F(x) \leq f(x)$ and $F$ is an induced subgraph of $G$. In this paper, we prove that $G$ has a strong $f$-star factor if and only if $\text{oddca}(G - S) \leq \sum_{x \in S} f(x)$ for all $S \subset V(G)$, where $\text{oddca}(G)$ denotes the number of odd complete-cacti of $G$.

Keywords: $f$-star factor, strong $f$-star factor, complete-cactus, factor of graph.

2010 Mathematics Subject Classification: 05C70.

1. Introduction

We consider simple graphs, which have neither loops nor multiple edges. For a graph $G$, let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of $G$, respectively. We write $|G|$ for the order of $G$ (i.e., $|G| = |V(G)|$). For a vertex $v$ of $G$, we denote by $\deg_G(v)$ the degree of $v$ in $G$. For a vertex set $S$ of $G$, let $G - S$ denote the subgraph of $G$ induced by $V(G) - S$. Let $\text{Iso}(G)$ and $\text{iso}(G)$ denote the set of isolated vertices and the number of isolated vertices of $G$, respectively. A graph $G$ is called a complete-cactus if $G$ is connected and every block of $G$ is a complete graph. A complete-cactus is called an odd complete-cactus if all its blocks are complete graphs of odd order. Note that $K_1$ is an odd complete-cactus.

For a set $S$ of connected graphs, a spanning subgraph $F$ of a graph $G$ is called an $S$-factor of $G$ if each component of $F$ is isomorphic to an element of $S$. A complete bipartite graph $K_{1,n}$ is called a star, and its vertex of degree $n$ is called the center. For $K_{1,1}$, an arbitrarily chosen vertex is its center.

The following theorem was independently obtained by Las Vergnas [6] and by Amahashi and Kano [2].
**Theorem 1** [2, 6]. Let $n \geq 2$ be an integer. Then a graph $G$ has a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,n}\}$-factor if and only if $\text{iso}(G - S) \leq n|S|$ for all $S \subset V(G)$.

Let $G$ be a graph and let $f : V(G) \to \{2, 3, 4, \ldots\}$ be a function defined on $V(G)$. Then a spanning subgraph $F$ is called an $f$-star factor of $G$ if each component of $F$ is a star and its center $x$ satisfies $\deg_F(x) \leq f(x)$. The following theorem gives a criterion for a graph to have an $f$-star factor.

**Theorem 2** [3]. Let $G$ be a graph and let $f : V(G) \to \{2, 3, \ldots\}$ be a function. Then $G$ has an $f$-star factor if and only if $\text{iso}(G - S) \leq \sum_{x \in S} f(x)$ for all $S \subset V(G)$.

For a set $S$ of connected graphs, a subgraph $H$ of $G$ is called a strong $S$-subgraph if every component of $H$ is isomorphic to an element of $S$ and is an induced subgraph of $G$. A spanning strong $S$-subgraph is called a strong $S$-factor. A strong $\{K_{1,1}, K_{1,2}, \ldots\}$-factor is briefly called a strong star factor. Kelmans [7] and Saito and Watanabe [8] proved independently the following theorem.

**Theorem 3** [7, 8]. A connected graph $G$ has a strong star factor if and only if $G$ is not an odd complete-cactus.

For a graph $G$, let $\text{OddCa}(G)$ denote the set of components of $G$ that are odd complete-cacti, and let $\text{oddca}(G) = |\text{OddCa}(G)|$ denote the number of odd complete-cacti of $G$. Egawa, Kano and Kelmans [4] generalized the above theorem as follows by considering a strong $\{K_{1,1}, K_{1,2}, \ldots, K_{1,n}\}$-factor.

**Theorem 4** [4]. Let $n \geq 2$ be an integer. Then a graph $G$ has a strong $\{K_{1,1}, K_{1,2}, \ldots, K_{1,n}\}$-factor if and only if $\text{oddca}(G - S) \leq n|S|$ for all $S \subset V(G)$.

A subgraph $H$ is called a strong $f$-star subgraph of $G$ if each component of $H$ is a star, whose center $x$ satisfies $\deg_H(x) \leq f(x)$, and $H$ is an induced subgraph of $G$. A spanning $f$-star subgraph of $G$ is called a strong $f$-star factor of $G$. Obviously, if $f(x) = n$ for all $x \in V(G)$, then a strong $f$-star factor of $G$ is a strong $\{K_{1,1}, K_{1,2}, \ldots, K_{1,n}\}$-factor. In this paper, we obtain the following result which is a generalization of Theorem 4.

**Theorem 5.** Let $G$ be a graph and let $f : V(G) \to \{2, 3, \ldots\}$ be a function. Then $G$ has a strong $f$-star factor if and only if

$$\text{oddca}(G - S) \leq \sum_{x \in S} f(x), \text{ for all } S \subset V(G).$$

A strong $f$-star subgraph $H$ of a graph $G$ is said to be maximum if $G$ has no strong $f$-star subgraph $H'$ such that $|H'| > |H|$. A formula for the order of a maximum strong $f$-star subgraph of a graph is easily obtained as a maximum matching, which is given in the following theorem.
Theorem 6. Let $G$ be a graph and let $f : V(G) \to \{2, 3, 4, \ldots \}$ be a function. Then the order of a maximum strong $f$-star subgraph $H$ of $G$ is given by

$$
|H| = |G| - \max_{X \subset V(G)} \left\{ \text{oddca}(G - X) - \sum_{x \in X} f(x) \right\}.
$$

Finally, we consider a problem of covering a given vertex subset with a strong $f$-star subgraph. The condition for the existence of such a subgraph, which is given in the following theorem, is a natural extension of the criterion for the existence of a strong $f$-star factor.

Theorem 7. Let $G$ be a graph and let $f : V(G) \to \{2, 3, 4, \ldots \}$ be a function. Let $W$ be a subset of $V(G)$. Then $G$ has a strong $f$-star subgraph covering $W$ if and only if

$$
\text{oddca}(G - S[W]) \leq \sum_{x \in S} f(x), \text{ for all } S \subset V(G),
$$

where $\text{oddca}(G - S[W])$ denotes the number of odd complete-cacti of $G - S$ contained in $W$.

2. Proof of the Results

We need some other notations. For two sets $X$ and $Y$, $X \subset Y$ means that $X$ is a proper subset of $Y$. Let $G$ be a graph. For two vertices $x$ and $y$ of $G$, we write $xy$ or $yx$ for an edge joining $x$ to $y$. For a vertex $v$ of $G$, we denote by $N_G(v)$ the neighborhood of $v$. For a subset $S$ of $V(G)$, we define $N_G(S) := \bigcup_{x \in S} N_G(x)$. For convenience, we briefly call a complete-cactus a cactus in the following proofs. Analogously, an odd complete-cactus is called an odd cactus. Every block of a cactus is a complete graph, and we call it an odd block or even block according to its order.

In order to prove Theorem 5, we need the following lemmas.

Lemma 8 [4]. (i) Let $G$ be an odd complete-cactus. Then for every vertex $v$ of $G$, $G - v$ has a 1-factor.

(ii) An odd complete-cactus does not have a strong star factor.

Lemma 9 [5]. Let $G$ be a bipartite graph with bipartition $(A, B)$, and let $g, f : V(G) \to \mathbb{Z}$ be functions such that $g(x) \leq f(x)$ for all $x \in V(G)$. Then $G$ has a $(g, f)$-factor if and only if

$$
\gamma^*(X, Y) = \sum_{x \in X} f(x) + \sum_{x \in Y} (\deg_G(x) - g(x)) - e_G(X, Y) \geq 0,
$$

and

$$
\gamma^*(Y, X) = \sum_{x \in Y} f(x) + \sum_{x \in X} (\deg_G(x) - g(x)) - e_G(Y, X) \geq 0,
$$

for all subsets $X \subseteq A$ and $Y \subseteq B$. 

Lemma 10. Let $G$ be a bipartite graph with bipartition $(A, B)$. Let $f : V(G) \to \{1, 2, 3, \ldots \}$ be a function such that $f(x) \geq 2$ for all $x \in A$, and $f(x) = 1$ for all $x \in B$. Then $G$ has a $(1, f)$-factor if and only if

$$|N_G(X)| \geq |X| \text{ for all } X \subseteq A, \text{ and}$$

$$\sum_{x \in N_G(Y)} f(x) \geq |Y| \text{ for all } Y \subseteq B. \tag{4}$$

Proof. If $G$ has a $(1, f)$-factor $F$, then (4) follows from

$$|N_G(X)| \geq |N_F(X)| \geq |X| \text{ and } \sum_{x \in N_G(Y)} f(x) \geq \sum_{x \in N_F(Y)} f(x) \geq |Y|.$$ 

Conversely, assume that (4) holds. We may assume that $G$ is connected, since otherwise each component satisfies (4) and has a $(1, f)$-factor by induction, and hence $G$ itself has a $(1, f)$-factor. For any subsets $X \subseteq A$ and $Y \subseteq B$, it follows from (4) that

$$\gamma^*(X, Y) = \sum_{x \in X} f(x) + \sum_{y \in Y} (\deg_{G-X}(y) - 1)$$

$$\geq \sum_{x \in X} f(x) - |S| \geq \sum_{x \in N_G(S)} f(x) - |S| \geq 0,$$

where $S = \text{Iso}(G - X) \cap Y \subseteq B$, $N_G(S) \subseteq X$.

$$\gamma^*(Y, X) = \sum_{y \in Y} f(y) + \sum_{x \in X} (\deg_{G-Y}(x) - 1)$$

$$\geq |Y| - |T| \geq |N_G(T)| - |T| \geq 0,$$

where $T = \text{Iso}(G - Y) \cap X \subseteq A$, $N_G(T) \subseteq Y$.

Therefore by Lemma 9, $G$ has the desired $(1, f)$-factor. \hfill $\blacksquare$

Proof of Theorem 5. Suppose that $G$ has a strong $f$-star factor $F$. Let $\emptyset \neq S \subset V(G)$. Since every odd cactus $D$ of $G - S$ does not have a strong $f$-star by Lemma 8, $F$ has an edge joining $D$ to $S$. It is obvious that for every vertex $s \in S$, $F$ has at most $f(x)$ edges joining $x$ to odd cacti in $G - S$. Hence $\text{oddca}(G - S) \leq \sum_{x \in S} f(x)$.

We shall prove the sufficiency of Theorem 5 by induction on $\sum_{x \in V(G)} f(x)$. We may assume that $|G| \geq 3$ and $G$ is connected, since otherwise by applying the induction hypothesis to each component, we can obtain the desired strong star-factor of $G$. By taking $S = \emptyset$, it follows that $G$ is not an odd cactus.

Obviously, $\sum_{x \in V(G)} f(x) \geq 2|G|$, since $f(x) \geq 2$ for all $x \in V(G)$. If $\sum_{x \in V(G)} f(x) = 2|G|$, then $f(x) = 2$ for all $x \in V(G)$. Thus the condition (1) becomes

$$\text{oddca}(G - S) \leq 2|S|, \text{ for all } S \subset V(G).$$
By Theorem 4, $G$ has a strong $\{K_{1,1}, K_{1,2}\}$-factor, which is the desired strong $f$-factor of $G$. So we may assume that $\sum_{x \in V(G)} f(x) \geq 2|G| + 1$. Then there exists a vertex $w \in V(G)$ such that $f(w) \geq 3$.

Let us define the number $\beta$ by

$$
\beta = \min_{\emptyset \neq X \subset V(G)} \left\{ \sum_{x \in X} f(x) - \text{oddca}(G - X) \right\}.
$$

Then $\beta \geq 0$ by (1), and it follows from the definition of $\beta$ that

$$
\text{oddca}(G - Y) \leq \sum_{x \in Y} f(x) - \beta, \quad \text{for all } \emptyset \neq Y \subset V(G).
$$

Take a maximal subset $S$ of $V(G)$ such that

$$
\sum_{x \in S} f(x) - \text{oddca}(G - S) = \beta.
$$

Claim 1. $\beta = 0$.

**Proof.** Suppose that $\beta \geq 1$. Define $f^* : V(G) \to \{2, 3, 4, \ldots\}$ by

$$
f^*(x) = \begin{cases} 
  f(x) - 1 & \text{if } x = w, \\
  f(x) & \text{otherwise}.
\end{cases}
$$

Let $\emptyset \neq X \subset V(G)$. Then we have

$$
\text{oddca}(G - X) \leq \sum_{x \in X} f(x) - \beta \leq \sum_{x \in X} f^*(x) - 1 \leq \sum_{x \in X} f^*(x).
$$

Hence, $G$ has a strong star-factor $F^*$ with respect to $f^*$ by induction, which is also the strong $f$-star factor of $G$. \hfill \square

Hereafter we assume $\beta = 0$.

Claim 2. Every component of $G - S$ which is not an odd cactus has a strong $f$-star factor.

**Proof.** Let $D$ be a component of $G - S$ which is not an odd cactus, and let $\emptyset \neq X \subset V(D)$. Then by (5), we have

$$
\text{oddca}(G - S) + \text{oddca}(D - X) = \text{oddca}(G - S \cup X)
$$

$$
\leq \sum_{x \in S \cup X} f(x) = \sum_{x \in S} f(x) + \sum_{x \in X} f(x).
$$

Thus $\text{oddca}(D - X) \leq \sum_{x \in X} f(x)$ by (6), which implies that $D$ has a strong $f$-star factor by induction. \hfill \square
We construct a bipartite graph $B$ with bipartition $(S, \text{OddCa}(G - S))$ in which two vertices $x \in S$ and a component $C \in \text{OddCa}(G - S)$ are joined by an edge of $B$ if and only if $x$ is adjacent to $C$ in $G$.

**Claim 3.** For every $\emptyset \neq X \subseteq S$ and $\emptyset \neq Y \subseteq \text{OddCa}(G - S)$, it follows that $|N_{B}(X)| \geq |X|$ and $\sum_{x \in N_{B}(Y)} f(x) \geq |Y|$.

**Proof.** Let $\emptyset \neq X \subseteq S$. By (5) and $\beta = 0$, we obtain

$$\sum_{x \in S - X} f(x) \geq \text{oddca}(G - (S - X)) \geq \text{oddca}(G - S) - |N_{B}(X)|$$

$$\geq \sum_{x \in S} f(x) - |N_{B}(X)|,$$

which means $|N_{B}(X)| \leq \sum_{x \in X} f(x) \geq |X|$. Let $\emptyset \neq Y \subseteq \text{OddCa}(G - S)$. Then $N_{B}(Y) \subseteq S$, and by (1) we have

$$|Y| \leq \text{oddca}(G - N_{B}(Y)) \leq \sum_{x \in N_{B}(Y)} f(x).$$

Therefore Claim 3 holds. \hfill \square

By Claim 3, $B$ has a strong $f$-star factor $H$ given in Lemma 10, which is a $(1, f)$-factor with minimal edge set, and every component of $\text{OddCa}(G - S)$ has degree one in $H$. Consequently, by Lemma 8(i) and Claim 2, we can obtain a strong $f$-star factor of $G$ from $H$. \hfill \blacksquare

**Proof of Theorem 6.** Let $d = \max_{X \subseteq V(G)} \{\text{oddca}(G - X) - \sum_{x \in X} f(x)\}$.

Then $d \geq 0$ by considering the case $X = \emptyset$. Moreover, if $d = 0$, then (2) follows from Theorem 5. Hence we may assume $d \geq 1$. Let $S$ be a subset of $V(G)$ such that

$$\text{oddca}(G - S) - \sum_{x \in S} f(x) = d.$$

Then by considering $(S \cup \text{OddCa}(G - S))_{G}$, which is the subgraph of $G$ induced by $S \cup \text{OddCa}(G - S)$, we have that every strong $f$-star subgraph of $G$ cannot cover at least $\text{oddca}(G - S) - \sum_{x \in S} f(x)$ odd cacti of $\text{OddCa}(G - S)$. Hence $|H| \leq |G| - d$, when $H$ is a maximum strong $f$-star subgraph of $G$.

Next we prove the inverse inequality $|H| \geq |G| - d$ for a maximum strong $f$-star subgraph $H$ of $G$. Add $2d$ new vertices $\{v_{i}, u_{i} : 1 \leq i \leq d\}$ together with $d$ new edges $\{v_{i}u_{i} : 1 \leq i \leq d\}$ to $G$. Then join every $v_{i}$ to every vertex of $G$ by new edges. Denote the resulting graph by $G^{*}$, and define a function $f^{*} : V(G^{*}) \to \{2, 3, 4, \ldots\}$ by $f^{*}(u_{i}) = f^{*}(v_{i}) = 2$ for all $1 \leq i \leq d$, and $f^{*}(x) = f(x)$ for all $x \in V(G)$.

Let $Y$ be a non-empty subset of $V(G^{*})$. We may assume that $Y$ contains no vertices of $\{u_{1}, \ldots, u_{d}\}$, when we estimate $\text{oddca}(G^{*} - Y)$. If $|\{v_{1}, \ldots, v_{d}\} \cap Y| < d$, then

$$\text{oddca}(G^{*} - Y) \leq |Y \cap \{v_{1}, \ldots, v_{d}\}| + 1 \leq \sum_{x \in Y} f(x).$$
If \( \{v_1, \ldots, v_d\} \subset Y \), then all the vertices of \( \{u_1, \ldots, u_d\} \) become isolated vertices of \( G^* - Y \), and so by the definition of \( d \), we obtain
\[
\text{oddca}(G^* - Y) \leq \text{oddca}(G - (Y \cap V(G))) + d
\leq \sum_{x \in Y \cap V(G)} f(x) + d = \sum_{x \in Y} f(x).
\]

Hence by Theorem 5, \( G^* \) has a strong \( f^* \)-star factor \( F^* \). Then \( H = F^* - \{u_i, v_i : 1 \leq i \leq d\} \) is a strong \( f^* \)-star subgraph of \( G \), which covers at least \( |G| - d \) vertices. Hence \( |H| \geq |G| - d \). Consequently, the theorem is proved.

**Proof of Theorem 7.** First suppose that \( G \) has a strong \( f^* \)-star subgraph \( F \) covering \( W \). Then for every odd cactus \( C \) of \( G - S \) contained in \( W \), \( F \) has at least one edge joining \( C \) to \( S \). Hence
\[
\text{oddca}(G - S|W) \leq \sum_{x \in S} \text{deg}_F(x).
\]

Next we assume that (3) holds. We may assume that \( G \) is connected, since otherwise, by applying the induction hypothesis to each component of \( G \), we can obtain the desired factor of \( G \). By Theorem 5, we may assume that \( W \) is a proper subset of \( V(G) \), and so \( V(G) - W \neq \emptyset \). Let \( n = |V(G) - W| \). We construct a new graph \( H \) from \( G \) by adding two new vertices \( w_1, w_2 \) and by joining \( w_i(i = 1, 2) \) to every vertex in \( V(G) - W \). Define \( f^* : V(H) \to \{2, 3, \ldots\} \) by
\[
f^*(x) = \begin{cases} f(x) & \text{if } x \in V(G), \\ \max\{2, n\} & \text{if } x \in \{w_1, w_2\}. \end{cases}
\]

It is easy to see that \( G \) has a strong \( f^* \)-star subgraph covering \( W \) if and only if \( H \) has a strong \( f^* \)-star factor.

Let \( X \subset V(H) \). If \( w_1, w_2 \in X \), let \( S = X - \{w_1, w_2\} \), then
\[
\text{oddca}(H - X) \leq \text{oddca}(G - S|W) + n \leq \sum_{x \in S} f(x) + n < \sum_{x \in X} f^*(x).
\]

If \( w_1 \in X \) and \( w_2 \notin X \), let \( S = X - \{w_1\} \), then
\[
\text{oddca}(H - X) \leq \text{oddca}(G - S|W) + 1 \leq \sum_{x \in S} f(x) + 1 < \sum_{x \in X} f^*(x).
\]

If \( w_1, w_2 \notin X \), then
\[
\text{oddca}(H - X) = \text{oddca}(G - X|W) \leq \sum_{x \in X} f(x).
\]

Therefore, by Theorem 5, \( H \) has a strong \( f^* \)-star factor, and thus \( G \) has the desired strong \( f^* \)-star subgraph which covers \( W \).
Acknowledgments

The author would like to thank Prof. Mikio Kano for introducing me to problems on factors of graph and for his valuable suggestions.

References

doi:10.1007/978-3-642-21919-1

doi:10.1016/0012-365X(82)90048-6

doi:10.1016/S1385-7258(78)80007-9

doi:10.1002/(SICI)1097-0118(199707)25:3.0.CO;2-H

doi:10.1016/S0021-9800(70)80006-0

doi:10.1016/0012-365X(78)90006-7


Received 27 December 2013
Revised 29 September 2014
Accepted 29 September 2014