

## ON MINIMAL GEODETIC DOMINATION IN GRAPHS

HEARTY M. NUENAY<sup>1</sup>

AND

FERDINAND P. JAMIL<sup>2</sup>

*Department of Mathematics and Statistics*  
*MSU-Iligan Institute of Technology*  
*Iligan City, Philippines*

**e-mail:** ferdinand.jamil@g.msuiit.edu.ph

### Abstract

Let  $G$  be a connected graph. For two vertices  $u$  and  $v$  in  $G$ , a  $u$ - $v$  geodesic is any shortest path joining  $u$  and  $v$ . The closed geodesic interval  $I_G[u, v]$  consists of all vertices of  $G$  lying on any  $u$ - $v$  geodesic. For  $S \subseteq V(G)$ ,  $S$  is a geodesic set in  $G$  if  $\bigcup_{u, v \in S} I_G[u, v] = V(G)$ .

Vertices  $u$  and  $v$  of  $G$  are neighbors if  $u$  and  $v$  are adjacent. The closed neighborhood  $N_G[v]$  of vertex  $v$  consists of  $v$  and all neighbors of  $v$ . For  $S \subseteq V(G)$ ,  $S$  is a dominating set in  $G$  if  $\bigcup_{u \in S} N_G[u] = V(G)$ . A geodesic dominating set in  $G$  is any geodesic set in  $G$  which is at the same time a dominating set in  $G$ . A geodesic dominating set in  $G$  is a minimal geodesic dominating set if it does not have a proper subset which is itself a geodesic dominating set in  $G$ . The maximum cardinality of a minimal geodesic dominating set in  $G$  is the upper geodesic domination number of  $G$ . This paper initiates the study of minimal geodesic dominating sets and upper geodesic domination numbers of connected graphs.

**Keywords:** minimal geodesic dominating set, upper geodesic domination number.

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<sup>2</sup>Corresponding author.

## 1. INTRODUCTION

Throughout this paper we consider only finite connected graphs with no loops or multiple edges. All basic graph theoretic terminologies and notations adapted here are taken from [11].

Let  $G$  and  $H$  be graphs with disjoint vertex sets. The *join*  $G + H$  of  $G$  and  $H$  is the graph with vertex set  $V(G + H) = V(G) \cup V(H)$  and edge set  $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$ . The *composition* (or *lexicographic product*)  $G[H]$  of  $G$  and  $H$  is the graph with vertex set  $V(G[H]) = V(G) \times V(H)$  and  $(u, v)(u', v') \in E(G[H])$  if and only if either  $uu' \in E(G)$  or  $u = u'$  and  $vv' \in E(H)$ .

Let  $G$  be a connected graph. For any two vertices  $u$  and  $v$  in  $G$ , a  $u$ - $v$  *geodesic* refers to any shortest path in  $G$  joining  $u$  and  $v$ . The length of a  $u$ - $v$  geodesic is called the *distance* between  $u$  and  $v$ , and is denoted by  $d_G(u, v)$ . The *eccentricity*  $e_G(v)$  of a vertex  $v$  is defined by  $e_G(v) = \max\{d_G(u, v) : u \in V(G)\}$  and the *diameter* of  $G$  is the number  $\text{diam}(G) = \max\{d_G(u, v) : u, v \in V(G)\}$ . The *closed geodesic interval*  $I_G[u, v]$  is the set of all vertices lying on any  $u$ - $v$  geodesic. For a subset  $S$  of the vertex set  $V(G)$  of  $G$ , the *geodesic closure* of  $S$  is the set  $I_G[S] = \bigcup_{u, v \in S} I_G[u, v]$ . Various concepts inspired by geodesic closures are introduced in [7, 11]. A *geodesic set* in  $G$  is any set  $S$  of vertices in  $G$  satisfying  $I_G[S] = V(G)$ . The minimum cardinality  $g(G)$  of a geodesic set is the *geodesic number* of  $G$ . Geodesic sets and geodesic numbers are studied in [1, 2, 3, 4, 5, 6]. A geodesic set  $S$  in  $G$  is a *minimal geodesic set* if  $S$  does not have a proper subset that is itself a geodesic set in  $G$ . The maximum cardinality of a minimal geodesic set in  $G$  is denoted by  $g^+(G)$ . Zhang *et al.* investigated a minimal geodesic set in a connected graph in [4].

We also define  $I_G(u, v) = I_G[u, v] \setminus \{u, v\}$  and  $I_G(S) = \bigcup_{u, v \in S} I_G(u, v)$ . We call  $S$  a *2-path closure absorbing set* if for each  $x \in V(G) \setminus S$ , there exist  $u, v \in S$  such that  $d_G(u, v) = 2$  and  $x \in I_G(u, v)$ . The minimum cardinality of a 2-path closure absorbing set in  $G$  is denoted by  $\rho_2(G)$ . Since a 2-path closure absorbing set is always a geodesic set,  $g(G) \leq \rho_2(G)$  for all connected graphs  $G$ . In [6], the geodesic numbers of some classes of graphs are described in terms of 2-path closure absorbing sets. A 2-path closure absorbing set  $S$  is a *minimal 2-path closure absorbing set* if  $S$  does not contain a proper subset that is itself 2-path closure absorbing. The maximum cardinality of a minimal 2-path closure absorbing set in  $G$  is denoted by  $\rho_2^+(G)$ .

The *open neighborhood* of a vertex  $v$  in  $G$  is the set  $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ . The *degree*,  $\text{deg}_G(v)$ , of a vertex  $v$  refers to the value  $|N_G(v)|$ , and we define  $\Delta(G) = \max\{\text{deg}_G(v) : v \in V(G)\}$ . The *closed neighborhood* of  $v$  is the set  $N_G[v] = N_G(v) \cup \{v\}$ . A vertex  $v$  is an *extreme vertex* if the induced subgraph  $\langle N_G[v] \rangle$  is a complete graph. The symbol  $\text{Ext}(G)$  denotes the

set of all extreme vertices in  $G$ . For  $S \subseteq V(G)$ , we define  $N_G(S) = \bigcup_{v \in S} N_G(v)$  and  $N_G[S] = N_G(S) \cup S$ . If  $N_G[S] = V(G)$ , then  $S$  is a *dominating set* in  $G$ . The minimum cardinality among dominating sets in  $G$  is called the *domination number* of  $G$ , and is denoted by  $\gamma(G)$ . A considerable number of studies have been dedicated in obtaining variations of the concept (see [12, 13, 14, 15]). The authors in [9] cited over 75 variations of domination and listed over 1,200 papers related to domination in graphs. An application to electrical power networks is being studied in [10].

A subset  $S$  of  $V(G)$  is a *geodetic dominating set* in  $G$  if  $S$  is a geodetic set and at the same time a dominating set in  $G$ . The minimum cardinality of a geodetic dominating set is called the *geodetic domination number* of  $G$ , and is denoted by  $\gamma_g(G)$ . The study of geodetic domination was initiated by Escuardo, Gera, Hansberg, Jafari Rad and Volkman [8] in 2011. Some other interesting results can also be found in [16].

Customarily or as used in several literatures, the symbols like  $g$ -set,  $\rho_2$ -set,  $\rho_2^+$ -set,  $\gamma$ -set, and  $\gamma_g$ -set in a graph  $G$  would refer to a geodetic set of cardinality  $g(G)$ , a 2-path closure absorbing set with cardinality  $\rho_2(G)$ , a minimal 2-path closure absorbing set with cardinality  $\rho_2^+(G)$ , a dominating set with cardinality  $\gamma(G)$ , and a geodetic dominating set with cardinality  $\gamma_g(G)$ , respectively.

Since a 2-path closure absorbing set is also a geodetic dominating set,  $g(G) \leq \gamma_g(G) \leq \rho_2(G)$  for all connected graphs  $G$  of order  $n \geq 2$ . In particular, if  $\text{diam}(G) = 2$ , then  $\gamma_g(G) = \rho_2(G)$ .

The following is found in [8].

**Theorem 1.1** [8]. *Let  $G$  be a connected graph of order  $n \geq 2$ . Then*

- (i)  $\gamma_g(G) = 2$  if and only if there exists a geodetic set  $S = \{u, v\}$  such that  $d_G(u, v) \leq 3$ .
- (ii)  $\gamma_g(G) = n$  if and only if  $G$  is the complete graph on  $n$  vertices.
- (iii)  $\gamma_g(G) = n - 1$  if and only if there is a vertex  $v$  in  $G$  such that  $v$  is adjacent to every other vertex of  $G$  and  $G - v$  is the union of at least two complete graphs.

## 2. MINIMAL GEODETIC DOMINATION

A geodetic dominating set  $S$  in a connected graph  $G$  of order  $n \geq 2$  is a *minimal geodetic dominating set* in  $G$  if  $S$  does not have a proper subset which is itself a geodetic dominating set in  $G$ . The maximum cardinality of a minimal geodetic dominating set in  $G$  is the *upper geodetic domination number* of  $G$ , and is denoted by  $\gamma_g^+(G)$ . A minimal geodetic dominating set with cardinality  $\gamma_g^+(G)$  is also called a  $\gamma_g^+$ -set.

**Example 2.1.** (i) *If  $m, n \geq 2$  and  $U$  and  $W$  are the partite sets of the complete bipartite graph  $K_{m,n}$ , then the minimal geodetic dominating sets in  $K_{m,n}$  are  $U$  and  $W$  and all sets of the form  $S = \{u, v, x, y\}$ , where  $u, v \in U$  and  $x, y \in W$ . More precisely,*

$$\gamma_g^+(K_{m,n}) = \begin{cases} 4, & \text{if } m = n = 3, \\ \max\{m, n\}, & \text{otherwise.} \end{cases}$$

(ii) *For  $2 \leq n \leq 4$ ,  $\gamma_g^+(P_n) = 2$ , and for  $n \geq 5$ ,*

$$\gamma_g^+(P_n) = \begin{cases} 2 \lfloor \frac{n}{4} \rfloor + 1, & \text{if } n \equiv 1 \pmod{4}, \\ 2 \lceil \frac{n}{4} \rceil, & \text{otherwise.} \end{cases}$$

Suppose that  $n \equiv 1 \pmod{4}$ , and  $P_n = [u_1, u_2, \dots, u_n]$ . Since the set  $\{u_1, u_2, u_5, u_6, \dots, u_{4k-3}, u_{4k-2}, u_{4k+1}\}$  is a minimal geodetic dominating set in  $P_n$ ,  $\gamma_g^+(P_n) \geq 2 \lfloor \frac{n}{4} \rfloor + 1$ . Let  $S \subseteq V(P_n)$  be a minimal geodetic dominating set in  $P_n$ . For every  $j = 1, 2, \dots, n - 3$ ,  $S$  contains at most two of the vertices  $u_j, u_{j+1}, u_{j+2}$  and  $u_{j+3}$ . Thus,  $|S| \leq 2 \lfloor \frac{n}{4} \rfloor + 1$ . Since  $S$  is arbitrary,  $\gamma_g^+(P_n) \leq 2 \lfloor \frac{n}{4} \rfloor + 1$ . Now, suppose that  $n > 4$  but  $n \neq 4a + 1$  for all positive integers  $a$ . Let  $k$  be the largest positive integer for which  $4k + 1 < n$ . Since the set of vertices  $\{u_1, u_2, u_5, u_6, \dots, u_{4k+1}, u_n\}$  is a minimal geodetic dominating set in  $P_n$ ,  $\gamma_g^+(P_n) \geq 2 \lceil \frac{n}{4} \rceil$ . Using similar arguments, if  $S \subseteq V(P_n)$  is a minimal geodetic dominating set in  $P_n$ , then  $|S| \leq 2 \lceil \frac{n}{4} \rceil$ . This means that  $\gamma_g^+(P_n) \leq 2 \lceil \frac{n}{4} \rceil$ .

**Theorem 2.2.** *Let  $G$  be a connected graph of order  $n \geq 2$ . Then*

- (i)  $\gamma_g^+(G) = 2$  *if and only if  $G$  is one of the following graphs:  $P_2, C_4, \overline{K_2} + H$  where  $H$  is connected and either  $H = K_{n-2}$  or  $\rho_2^+(H) = 2$ ,  $G$  has a  $g$ -set  $\{u, v\}$  with  $u, v \in Ext(G)$  and  $d_G(u, v) = 3$ .*
- (ii)  $\gamma_g^+(G) = n$  *if and only if  $G = K_n$ .*
- (iii) *For  $n \geq 3$ ,  $\gamma_g^+(G) = n - 1$  if and only if  $G = K_1 + \bigcup_{j=1}^t K_{r_j}$ , where  $t \geq 2$ .*

**Proof.** (i) Suppose that  $\gamma_g^+(G) = 2$ , and let  $\{u, v\}$  be a  $\gamma_g^+$ -set in  $G$ . Let  $S = V(G) \setminus \{u, v\}$ . Then  $w \in I_G[u, v]$  for all  $w \in S$  and  $1 \leq d_G(u, v) \leq 3$  by Theorem 1.1. If  $d_G(u, v) = 1$ , then  $S = \emptyset$  and  $G = P_2$ . Suppose that  $d_G(u, v) = 2$ . Then  $G = \langle \{u, v\} \rangle + \langle S \rangle = \overline{K_2} + \langle S \rangle$ . If  $|S| = 1$ , then  $G = P_3 = \overline{K_2} + K_1$ . If  $|S| = 2$ , then either  $G = C_4$  or  $G = \overline{K_2} + K_2$ . Suppose that  $|S| \geq 3$ . Then  $\langle S \rangle$  is connected. If  $\langle S \rangle$  is the complete graph  $K_{n-2}$ , then  $G = \overline{K_2} + K_{n-2}$ . Suppose that  $H = \langle S \rangle$  is not complete, and let  $T$  be a  $\rho_2^+$ -set in  $H$ . Then  $T$  is a  $\gamma_g^+$ -set in  $G$ . Thus  $|T| = 2$ . Hence,  $\rho_2^+(H) = 2$ . Finally, suppose that  $d_G(u, v) = 3$ . For each  $x \in S$ , either  $ux \in E(G)$  or  $xv \in E(G)$ . Suppose that there exist  $x, y \in N_G[u]$  with  $d_G(x, y) = 2$ . Consider  $W = N_G(u) \cup \{v\}$ . Let  $z \in V(G) \setminus W$ . If  $z = u$ , then  $[x, z, y]$  is an  $x$ - $y$  geodesic in  $G$  so that  $z \in N_G[W]$  and  $z \in I_G[W]$ .

Suppose that  $z \neq u$ . Since  $z \in I_G[u, v]$ , there exist  $a, b \in V(G)$  such that  $z$  lies on the  $u$ - $v$  geodesic  $[u, a, b, v]$ . This means that  $a \in N_G(u)$  and  $z = b$ . Thus  $z \in N_G[W]$  and  $z \in I_G[W]$ . Accordingly,  $W$  is a geodesic dominating set in  $G$ . Let  $T \subseteq W$  be a minimal geodesic dominating set in  $G$ . Since  $v \notin N_G[N_G(u)]$ ,  $v \in T$ . Moreover, if  $|T \cap N_G(u)| = 1$ , then  $u \notin I_G[T]$ , a contradiction. Thus,  $|T \cap N_G(u)| \geq 2$  so that  $\gamma_g^+(G) \geq |T| \geq 3$ , a contradiction. Therefore,  $\langle N_G[u] \rangle$  is complete and  $u \in Ext(G)$ . Similarly,  $v \in Ext(G)$ .

Conversely, if  $G$  is  $P_2$  or  $C_4$  or  $\overline{K_2} + K_{n-2}$ , then  $\gamma_g^+(G) = 2$ . Suppose that  $G = \overline{K_2} + H$ , where  $H$  is connected and noncomplete with  $\rho_2^+(H) = 2$ . Then  $diam(G) = 2$  and  $T = V(\overline{K_2})$  is a minimal geodesic dominating set in  $G$ . Put  $T = \{u, v\}$ , and let  $Z$  be a minimal geodesic dominating set in  $G$  distinct from  $T$ . Then  $|Z \cap T| \leq 1$ . Suppose that  $Z \cap T = \{u\}$ . Since  $ux \in E(G)$  for all  $x \in V(H)$ ,  $v \in I_G[Z \setminus \{u\}]$  and  $V(H) \subseteq I_G[Z \setminus \{u\}]$  so that  $Z \setminus \{u\}$  is a geodesic dominating set in  $G$ , a contradiction. Thus  $Z \subseteq V(H)$  and, consequently,  $Z$  is a minimal 2-path closure absorbing set in  $H$ . Thus,  $2 \leq |Z| \leq \rho_2^+(H) = 2$  so that  $|Z| = 2$ . Since  $Z$  is arbitrary,  $\gamma_g^+(G) = |Z| = 2$ . Now, let  $G$  have a  $g$ -set  $\{u, v\}$  with  $d_G(u, v) = 3$  and where the induced subgraphs  $\langle N_G[u] \rangle$  and  $\langle N_G[v] \rangle$  are complete. Then  $Ext(G) = \{u, v\}$ , which is the unique  $\gamma_g^+$ -set in  $G$ . The conclusion follows.

(ii) If  $G = K_n$ , then  $\gamma_g(G) = n$ , by Theorem 1.1. Hence  $\gamma_g^+(G) = n$ . Suppose that  $\gamma_g^+(G) = n$ . Then each proper subset of  $V(G)$  is not a geodesic dominating set in  $G$ . Let  $v \in V(G)$ , and set  $S = V(G) \setminus \{v\}$ . Then  $v \notin N_G[S]$  or  $v \notin I_G[S]$ . If  $v \notin N_G[S]$ , then  $v$  is an isolated vertex, a contradiction. Thus  $v \notin I_G[S]$  so that  $v \in Ext(G)$ . Since  $v$  is arbitrary,  $V(G) = Ext(G)$  and  $G = K_n$ .

(iii) If  $G = K_1 + \bigcup_{j=1}^t K_{r_j}$  for some  $t \geq 2$ , then  $\gamma_g(G) = n-1$ , by Theorem 1.1. By statement (ii),  $\gamma_g^+(G) < n$ . Thus  $\gamma_g^+(G) = n-1$ . Suppose that  $\gamma_g^+(G) = n-1$ . Let  $S = V(G) \setminus \{v\}$ , where  $v \in V(G)$ , be a  $\gamma_g^+$ -set in  $G$ . We claim that  $uv \in E(G)$  for all  $u \in S$ . Since  $v$  is not an endvertex, there exist  $x, y \in S$  such that  $[x, v, y]$  is an  $x$ - $y$  geodesic in  $G$ . Suppose that, in the contrary, there exists  $u \in S$  with  $d_G(u, v) = 2$ . Let  $[u, w, v]$  be a  $u$ - $v$  geodesic in  $G$ . If  $x = w$  or  $y = w$ , then  $S \setminus \{w\}$  is a geodesic dominating set in  $G$ , which is impossible. Suppose that  $x \neq w$  and  $y \neq w$ . If  $uy \notin E(G)$ , then  $S \setminus \{w\}$  is a geodesic dominating set in  $G$ . If  $uy \in E(G)$  and  $wy \notin E(G)$ , then  $S \setminus \{u\}$  is a geodesic dominating set in  $G$ . If  $uy, wy \in E(G)$  and  $ux \in E(G)$ , then  $S \setminus \{u\}$  is a geodesic dominating set in  $G$ . If  $uy, wy \in E(G)$  and  $ux \notin E(G)$ , then  $S \setminus \{w\}$  is a geodesic dominating set in  $G$ . Any of the above cases yields a contradiction. This proves the claim. Therefore,  $G = K_1 + H$  for some graph  $H$ . Next, we show that  $H = \bigcup_{j=1}^t K_{r_j}$ , where  $t \geq 2$ . Suppose that  $H$  has a component  $K$  which is not a complete graph. Then  $K$ , consequently  $G$ , has a geodesic  $[x, y, z]$  of length 2. Then  $S \setminus \{y\}$  is a geodesic dominating set in  $G$ , a contradiction. Therefore,  $H = \bigcup_{j=1}^t K_{r_j}$ . Since

$G$  is not a complete graph,  $t \geq 2$ . ■

Now follows a Nordhaus-Gaddum-type result. Let the symbol  $\Xi$  denote the infinite collection of all connected graphs  $G$  such that  $\overline{G}$  is also connected.

**Theorem 2.3.** *For all  $G \in \Xi$  of order  $n \geq 4$ ,*

$$4 \leq \gamma_g^+(G) + \gamma_g^+(\overline{G}) \leq 2n - 4.$$

*In particular,  $\gamma_g^+(G) + \gamma_g^+(\overline{G}) = 4$  if and only if  $n = 4$ .*

**Proof.** Let  $G \in \Xi$  be of order  $n \geq 4$ . Note that if  $G$  is either  $K_n$  or  $K_1 + \bigcup_{j=1}^t K_{r_j}$  with  $t \geq 2$ , then  $\overline{G}$  is disconnected, a contradiction. In view of Theorem 2.2,

$$\gamma_g^+(G) + \gamma_g^+(\overline{G}) \leq (n - 2) + (n - 2) = 2n - 4.$$

The inequality at the left side is obvious.

In particular, if  $n = 4$ , then  $\gamma_g^+(G) + \gamma_g^+(\overline{G}) = 4$ . Conversely, suppose that  $\gamma_g^+(G) + \gamma_g^+(\overline{G}) = 4$ . Necessarily,  $\gamma_g^+(G) = 2$  and  $\gamma_g^+(\overline{G}) = 2$ . By Theorem 2.2,  $G$  has a  $g$ -set  $\{u, v\}$  with  $u, v \in \text{Ext}(G)$  and  $d_G(u, v) = 3$ . Similarly,  $\overline{G}$  has a  $g$ -set  $\{x, y\}$  with  $x, y \in \text{Ext}(\overline{G})$  and  $d_{\overline{G}}(x, y) = 3$ . Assume that  $x \in N_G[u]$ . Suppose that  $x = u$ . Note that  $N_{\overline{G}}(x) = N_G[v]$ , and  $\langle N_G[v] \rangle$  is not complete in  $\overline{G}$ . This means that  $x \notin \text{Ext}(\overline{G})$ , a contradiction. Suppose that  $xu \in E(G)$ . Since  $xy \notin E(\overline{G})$ ,  $xy \in E(G)$ . If  $vy \notin E(G)$ , then  $xv, vy \in E(\overline{G})$  so that  $[x, v, y]$  is a geodesic in  $\overline{G}$ , a contradiction. Thus  $[u, x, y, v]$  is a  $u$ - $v$  geodesic in  $G$ . Suppose that  $n \geq 5$ , and let  $z \in V(G)$  be distinct from  $u, x, y$  and  $v$ . Assume  $xz \in E(\overline{G})$ . Since  $x \in \text{Ext}(\overline{G})$  and  $xv \in E(\overline{G})$ ,  $zv \in E(\overline{G})$  and, consequently,  $zu \in E(G)$ . Since  $u \in \text{Ext}(G)$ ,  $xz \in E(G)$ , a contradiction. Therefore,  $n = 4$ . ■

**Corollary 2.4.** *If  $G \in \Xi$  is of order  $n \geq 4$ , then  $\gamma_g^+(G) + \gamma_g^+(\overline{G}) = 4$  if and only if  $G = P_4$ .*

Theorem 2.3 implies that if  $G \in \Xi$  of order  $n \geq 5$ , then

$$5 \leq \gamma_g^+(G) + \gamma_g^+(\overline{G}) \leq 2n - 4.$$

Since  $\gamma_g^+(C_5) = \gamma_g^+(\overline{C_5}) = 3$ , this upper bound is sharp. Consider the graph  $G$  obtained from the cycle  $C_4 = [v_1, v_2, v_3, v_4, v_1]$  by adding to  $C_4$  two vertices  $x$  and  $y$  and the edges  $xv_1, xv_4, yv_2$  and  $yv_3$ . For this  $G$ ,  $\gamma_g^+(G) + \gamma_g^+(\overline{G}) = 5$ , showing that the lower bound is sharp.

3. REALIZATION PROBLEMS

For nontrivial connected graphs  $G$ ,

$$2 \leq \gamma_g(G) \leq \gamma_g^+(G) \leq \rho_2^+(G).$$

In particular,  $\gamma_g^+(K_{n,n}) = \rho_2^+(K_{n,n})$  for  $n \geq 4$ .

**Theorem 3.1.** *For every pair of positive integers  $a$  and  $b$  with  $2 \leq a \leq b$ , there exists a connected graph  $G$  such that  $\gamma_g^+(G) = a$  and  $\rho_2^+(G) = b$ .*

**Proof.** If  $a = b$ , then we pick  $G = K_{a,a}$ . Suppose that  $b = a + 1$ . Obtain the graph  $G$  from  $P_3 = [v_1, v_2, v_3]$  by adding  $(a - 1)$  pendant edges  $v_3x_j$ ,  $j = 1, 2, \dots, a - 1$ . Then  $\gamma_g^+(G) = a$  and  $\rho_2^+(G) = b$ , which are determined by the sets  $\{v_1, x_1, x_2, \dots, x_{a-1}\}$  and  $\{v_1, v_2, x_1, x_2, \dots, x_{a-1}\}$ , respectively.

Suppose that  $b = a + k$ , where  $k \geq 2$ . Write  $V(K_k) = \{u_1, u_2, \dots, u_k\}$ . Obtain  $G$  by joining  $P_2 = [v_1, v_2]$  and  $K_k + \overline{K_{a-1}}$  using new  $k$  edges  $v_2u_j$ ,  $j = 1, 2, \dots, k$ . Note that  $Ext(G) = \{v_1\} \cup V(\overline{K_{a-1}})$ , and is a  $\gamma_g^+$ -set in  $G$ . Thus,  $\gamma_g^+(G) = a$ . On the other hand, if  $k \geq 2$ , then  $\rho_2^+(G) = a + k = b$  and is determined by the set  $V(K_k) \cup Ext(G)$ . ■

**Corollary 3.2.** *For every pair of positive integers  $a$  and  $b$  with  $2 \leq a < b$ , the smallest possible order of a connected graph  $G$  for which  $\gamma_g^+(G) = a$  and  $\rho_2^+(G) = b$  is  $b + 1$ .*

**Theorem 3.3.** *For all positive integers  $a, b, c$  with  $2 \leq a \leq b < c$  and  $c \geq b + 2$ , there exists a connected graph  $G$  such that  $\gamma_g(G) = a$ ,  $\gamma_g^+(G) = b$  and  $|V(G)| = c$ .*

**Proof.** Suppose that  $c = b + 2$ . Write  $a = 2 + k$  and  $b = r + k$ ,  $r \geq 2$  and  $k = 0, 1, 2, \dots$ . If  $k = 0$ , then we take  $G = K_{2,r}$ . In this case,  $\gamma_g(G) = 2$  and  $\gamma_g^+(G) = r$ . Suppose that  $k \geq 1$ . Consider the graph  $G$  as in Figure 1.

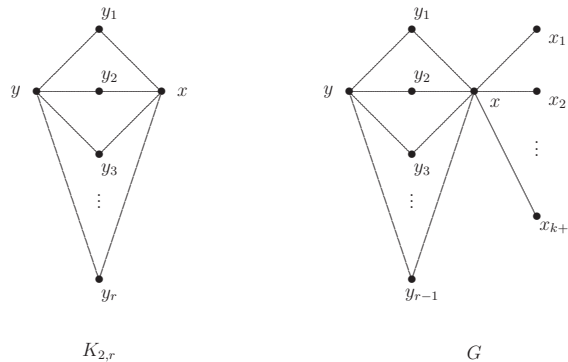


Figure 1

If  $r = 2$ ,  $G$  is obtained by adjoining to path  $[y, y_1, x]$   $(k + 1)$  pendant edges  $xx_j$ ,  $j = 1, 2, \dots, k + 1$ . Then  $Ext(G) = \{y, x_1, x_2, \dots, x_{k+1}\}$  is the unique minimal geodetic dominating set in  $G$ . Here we have  $\gamma_g(G) = \gamma_g^+(G) = 2 + k$ . Now, suppose that  $r \geq 3$ .  $G$  is obtained from  $K_{2,r-1}$  (with partite sets  $U = \{x, y\}$  and  $W = \{y_1, y_2, \dots, y_{r-1}\}$ ) by adding to  $K_{2,r-1}$   $(k + 1)$  pendant edges  $xx_j$ ,  $j = 1, 2, \dots, k + 1$ . The minimal geodetic dominating sets in  $G$  are  $\{y, x_1, x_2, \dots, x_{k+1}\}$  and  $W \cup \{x_1, x_2, \dots, x_{k+1}\}$ . Thus  $\gamma_g(G) = 2 + k$  and  $\gamma_g^+(G) = r + k$ .

Suppose that  $c = b + 3$ . Write  $a = 2 + k$  and  $b = r + k$ ,  $k = 0, 1, \dots$  and  $r \geq 2$ . Suppose that  $k = 0$ . Consider the graph  $G = G_1$  as in Figure 2.

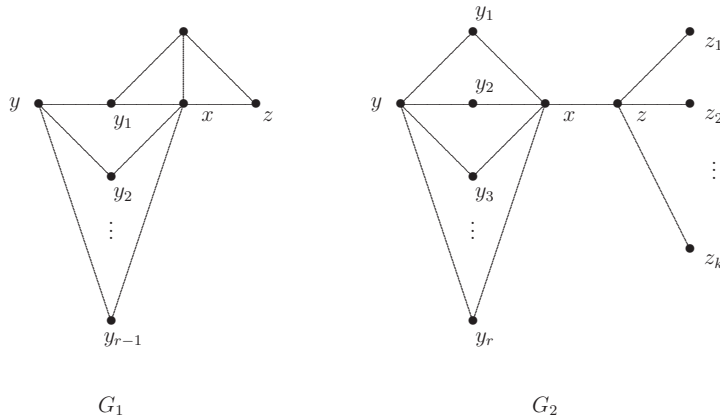


Figure 2

If  $r = 2$ , then  $\{y, z\}$  is the unique minimal geodetic dominating set in  $G$  so that  $\gamma_g(G) = \gamma_g^+(G) = 2$ . If  $r \geq 3$ , then  $\gamma_g(G) = 2$  and  $\gamma_g^+(G) = (r - 1) + 1 = r$ , the latter being determined by  $\{z, y_1, y_2, \dots, y_{r-1}\}$ . Suppose that  $k \geq 1$ . Obtain  $G$  as the graph  $G_2$  in Figure 2 by taking the union of  $K_{1,k+1}$  (with partite sets  $\{z\}$  and  $\{x, z_1, z_2, \dots, z_k\}$ ) and  $K_{2,r}$  (with partite sets  $\{x, y\}$  and  $\{y_1, y_2, \dots, y_r\}$ ). Note that  $\{z_1, z_2, \dots, z_k\}$  is always contained in a geodetic dominating set in  $G$ . Thus  $\gamma_g(G) = 2 + k$  and  $\gamma_g^+(G) = r + k$ .

Finally, suppose that  $c = b + d$ , where  $d \geq 4$ . Write  $a = 2 + k$  and  $b = r + k$ , where  $r \geq 2$  and  $k = 0, 1, 2, \dots$ , and put  $l = c - b - 3$ . For  $k = 0$ , we obtain  $G$  as the graph  $G_1^*$  in Figure 3 by joining  $K_{l+1} + \overline{K_2}$  and  $K_{r-1,2}$  using the common vertices  $x$  and  $y_1$ . Note that  $Ext(G) = \{z\}$ , and  $S = \{z, y\}$  is a minimal geodetic dominating in  $G$  and every minimal geodetic dominating set that contains  $y$  coincides  $S$ . Thus, aside from  $S$ , the other minimal geodetic dominating set in  $G$  is  $\{z, y_1, y_2, \dots, y_{r-1}\}$ . Consequently,  $\gamma_g(G) = 2 = a$  and  $\gamma_g^+(G) = r = b$ . Now, suppose that  $k \geq 1$ . Consider  $G$  as the graph  $G_2^*$  in Figure 3 obtained from  $G_1^*$  in Figure 3 by adjoining pendant edges  $zz_j$ .  $Ext(G) = \{z_1, z_2, \dots, z_k\}$  and if  $S$  is a  $\gamma_g^+$ -set that contains  $y$ , then  $|S| = k + 2$ . In this case,  $\gamma_g(G) = k + 2 = a$  and  $\gamma_g^+(G) = r + k = b$ . ■



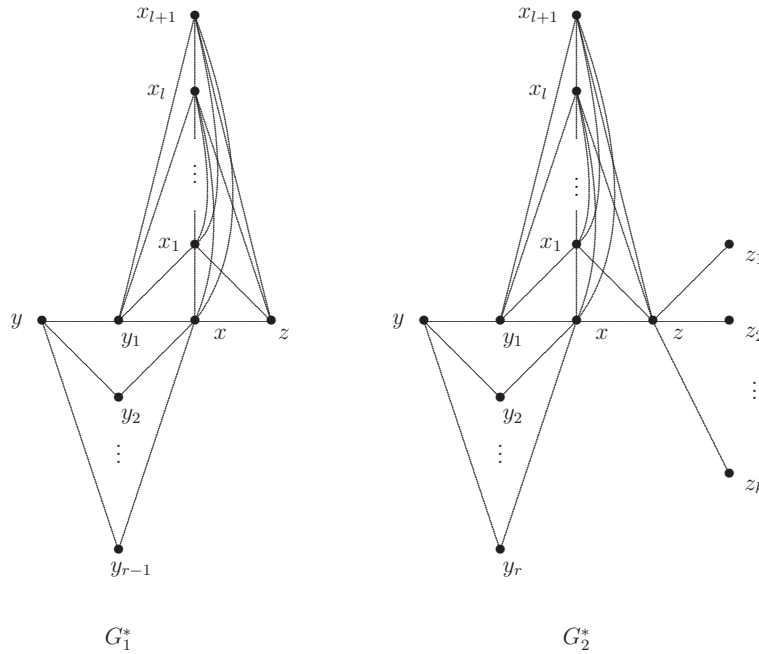


Figure 3

The next corollary follows from Theorem 2.2 and the existence proof of Theorem 3.3.

**Corollary 3.4.** *For every pair of positive integers  $a$  and  $b$  with  $2 \leq a < b$ , the minimum order of a connected graph  $G$  for which  $\gamma_g(G) = a$  and  $\gamma_g^+(G) = b$  is  $b + 2$ .*

#### 4. $\rho_3$ -SETS

Let  $G$  be a connected graph of order  $n \geq 2$  and  $S \subseteq V(G)$ .  $S$  is said to be a  $\rho_3$ -set in  $G$  if for every  $w \in V(G) \setminus S$  there exist  $u, v \in S$  such that  $d_G(u, v) \leq 3$  and  $w \in I_G[u, v]$ . We denote by  $\rho_3(G)$  the minimum cardinality of a  $\rho_3$ -set in  $G$ . Since every 2-path closure absorbing set is a  $\rho_3$ -set,  $\rho_3(G) \leq \rho_2(G)$ . In particular, if  $\text{diam}(G) = 2$ , then  $\rho_2(G) = \rho_3(G)$ . Since a  $\rho_3$ -set is a geodetic dominating set,  $\gamma_g(G) \leq \rho_3(G)$ . If  $\text{diam}(G) \leq 3$ , then  $\gamma_g(G) = \rho_3(G)$ . However, in general,  $\gamma_g(G)$  and  $\rho_3(G)$  are not necessarily equal.

Graph  $G$  is said to be  $K_3$ -free (resp.  $C_4$ -free) if  $G$  does not contain  $K_3$  (resp.  $C_4$ ) as a subgraph.

**Theorem 4.1.** *Let  $G$  be a connected graph of order  $n \geq 2$ , and let  $S \subseteq V(G)$ .*

- (i) *If  $S$  is a  $\rho_3$ -set in  $G$ , then for all  $v \in S$ ,  $\min\{d_G(u, v) : u \in S\} \leq 3$ .*

- (ii) If  $G$  is  $K_3$ -free and  $C_4$ -free and  $S$  is a geodetic dominating set in  $G$ , then  $S$  is a  $\rho_3$ -set in  $G$ .

**Proof.** The conclusions in statements (i) and (ii) are trivially satisfied for cases where  $n = 2$ ,  $n = 3$  and  $n = 4$ . Assume that  $n \geq 5$ , and let  $S \subseteq V(G)$ . To prove statement (i), suppose that  $S$  is a  $\rho_3$ -set in  $G$ , and suppose that there is  $v \in S$  such that  $d_G(v) = \min\{d_G(u, v) : u \in S\} \geq 4$ . Let  $w \in S$  be such that  $d_G(w, v) = d_G(v)$ . Then there exists  $u \in V(G) \setminus S$  lying on a  $w$ - $v$  geodesic with  $d_G(u, v) = 2$ . Since  $S$  is dominating in  $G$ , there exists  $z \in S$  such that  $uz \in E(G)$ . Observe that  $d_G(z, v) \leq 3$ , a contradiction. Therefore,  $d_G(v) \leq 3$  for all  $v \in S$ .

Next, we prove statement (ii). Suppose that  $G$  is  $K_3$ -free and  $C_4$ -free, and let  $v \in V(G) \setminus S$ . If  $S$  is a dominating set in  $G$ , then there exists  $x \in S$  such that  $xv \in E(G)$ . If  $S$  is a geodetic set, then  $v$  is not an endvertex of  $G$ . Pick  $u \in N_G(v)$  with  $u \neq x$ . Since  $G$  is  $K_3$ -free,  $[x, v, u]$  is an  $x$ - $u$  geodesic in  $G$ . If  $u \in S$ , then  $x$  and  $u$  are the desired vertices in  $S$  for  $v$ . Suppose that  $u \notin S$ . Pick  $y \in S$  such that  $uy \in E(G)$ . Since  $G$  is  $K_3$ -free and  $C_4$ -free,  $xy, vy \notin E(G)$ . Consequently,  $[x, v, u, y]$  is an  $x$ - $y$  geodesic in  $G$  with  $d_G(x, y) = 3$ . ■

If  $G$  is a connected graph of order  $n \geq 2$  which is  $K_3$ -free and  $C_4$ -free, then  $\rho_3(G) = \gamma_g(G)$ . In particular, if  $T$  is a tree of order  $n \geq 2$ , then  $\rho_3(T) = \gamma_g(T)$ .

**Theorem 4.2.** Let  $G$  be a connected  $K_3$ -free graph of order  $n \geq 2$ . Then

$$\rho_3(G) \leq \gamma_g^+(G).$$

**Proof.** Suppose that  $Ext(G) \neq \emptyset$ . Put  $Ext(G) = \{x_1, x_2, \dots, x_k\}$  for some positive integer  $k$ . For each  $j = 1, 2, \dots, k$ , define  $S_j = \{x_1, x_2, \dots, x_j\}$ . If  $N_G[S_k] \neq V(G)$ , choose  $x_{k+1} \in V(G) \setminus N_G[S_k]$ , and put  $S_{k+1} = \{x_1, x_2, \dots, x_k, x_{k+1}\}$ . If  $N_G[S_{k+1}] \neq V(G)$ , then choose  $x_{k+2} \in V(G) \setminus N_G[S_{k+1}]$ , and put  $S_{k+2} = \{x_1, x_2, \dots, x_{k+1}, x_{k+2}\}$ . Continuing in this way, there is a smallest positive integer  $m$  such that  $N_G[S_m] = V(G)$ . If  $Ext(G) = \emptyset$ , then construct  $S_m = \{x_1, x_2, \dots, x_m\}$  by choosing any  $x_1 \in V(G)$  and put  $S_1 = \{x_1\}$ , and for  $j \geq 2$ ,  $x_j \in V(G) \setminus N_G[S_{j-1}]$ , where  $S_{j-1} = \{x_1, x_2, \dots, x_{j-1}\}$ . In any case, we claim that  $S = S_m$  is a minimal geodetic dominating set and at the same time a  $\rho_3$ -set in  $G$ . Clearly,  $S$  is a dominating set in  $G$ . Let  $u \in V(G) \setminus S$ . Then there exists  $w \in S$  such that  $uw \in E(G)$ . Since  $u \notin Ext(G)$  and  $G$  is  $K_3$ -free, there exists  $v \in V(G)$  such that  $[v, u, w]$  is a  $v$ - $w$  geodesic in  $G$ . Suppose that  $v \notin S$ . There exists  $z \in S$  such that  $zv \in E(G)$ . Since  $G$  is  $K_3$ -free,  $uz \notin E(G)$ . Also, by the construction of  $S$ ,  $zw \notin E(G)$ . Thus,  $[w, u, v, z]$  is a  $w$ - $z$  geodesic in  $G$ . Here,  $d_G(w, z) \leq 3$  and  $u \in I_G[w, z] \subseteq I_G[S]$ . Since  $u$  is arbitrary,  $S$  is a  $\rho_3$ -set and a geodetic dominating set in  $G$ . Now let  $S^* = S \setminus \{x_j\}$ ,  $j = 1, 2, \dots, m$ . We will show that  $S^*$  is not a dominating set in  $G$ . Suppose that  $Ext(G) \neq \emptyset$ . If  $j \leq k$ , then  $x_j \in Ext(G)$  and  $S^*$  is not a geodetic set in  $G$ . Suppose that  $j > k$ . Since  $x_j \notin Ext(G)$ , there exist

$u, v \in V(G)$  such that  $[u, x_j, v]$  is a  $u$ - $v$  geodesic in  $G$ . Since  $x_j \in S$ ,  $u, v \notin S$ . In fact,  $x \notin S$  for all  $x \in N_G[x_j]$ . Thus  $x_j \notin N_G[S^*]$ , and  $S^*$  is not a dominating set in  $G$ . The case where  $Ext(G) = \emptyset$  is handled similarly. Since  $j$  is arbitrary,  $S$  is a minimal geodetic dominating set in  $G$ . Therefore,  $\rho_3(G) \leq |S| \leq \gamma_g^+(G)$ . ■

It is easy to verify that  $\rho_3(P_5) = 3 = \gamma_g^+(P_5)$ . Hence the bound given in Theorem 4.2 is sharp.

5. JOIN AND COMPOSITION OF GRAPHS

For connected graphs  $G$  and  $H$ , if  $S \subseteq V(G)$  is a 2-path closure absorbing set in  $G$ , then  $S$  is a geodetic dominating set in  $G + H$ .

**Theorem 5.1.** *For noncomplete connected graphs  $G$ ,  $\gamma_g^+(G + K_n) = \rho_2^+(G)$ .*

**Proof.** First, we claim that if  $S \subseteq V(G + K_n)$  is a geodetic dominating set in  $G + K_n$ , then  $A = S \cap V(G)$  is a 2-path closure absorbing set in  $G$ . Let  $S \subseteq V(G + K_n)$  be a geodetic dominating set in  $G + K_n$ . Let  $x \in V(G) \setminus A$ , and let  $u, v \in S$  such that  $x \in I_G[u, v]$ . Necessarily,  $u, v \in V(G)$ . Since  $diam(G + K_n)$  is 2,  $[u, x, v]$  is a  $u$ - $v$  geodesic in  $G$ . Thus  $d_G(u, v) = 2$  and  $A$  is a 2-path closure absorbing set in  $G$ .

Now let  $S \subseteq V(G + K_n)$  be a minimal geodetic dominating set in  $G + K_n$ . The above result implies that  $A = S \cap V(G)$  is a 2-path closure absorbing set in  $G$ , and consequently,  $A$  is a geodetic dominating set in  $G + K_n$ . Since  $S$  is a minimal geodetic dominating set,  $S = A$  so that  $S$  is a minimal 2-path closure absorbing set in  $G$ . Since  $S$  is arbitrary,  $\gamma_g^+(G + K_n) \leq \rho_2^+(G)$ .

Conversely, let  $S \subseteq V(G)$  be a  $\rho_2^+$ -set in  $G$ . Then  $S$  is a geodetic dominating set in  $G + K_n$ . That  $S$  is, in fact, a minimal geodetic dominating set in  $G + K_n$  follows from the claim above. This yields  $\rho_2^+(G) \leq \gamma_g^+(G + K_n)$ . ■

**Theorem 5.2.** *For all noncomplete connected graphs  $G$  and  $H$ ,*

$$\gamma_g^+(G + H) = \max\{4, \rho_2^+(G), \rho_2^+(H)\}.$$

**Proof.** Let  $S \subseteq V(G + H)$  be a minimal geodetic dominating set in  $G + H$ . If  $S \subseteq V(G)$ , then  $S$  is a minimal 2-path closure absorbing set in  $G$  since  $diam(G + H) = 2$ . This means that  $|S| \leq \rho_2^+(G)$ . Similarly, if  $S \subseteq V(H)$ , then  $|S| \leq \rho_2^+(H)$ . Suppose that  $A = S \cap V(G) \neq \emptyset$  and  $B = S \cap V(H) \neq \emptyset$ . Then  $|A| \geq 2$  and  $|B| \geq 2$ , and  $V(H) \subseteq I_{G+H}[A]$  and  $V(G) \subseteq I_{G+H}[B]$ . The minimality of  $S$  implies that  $|A| = |B| = 2$  and  $|S| = 4$ . Hence  $\gamma_g^+(G + H) \leq \max\{4, \rho_2^+(G), \rho_2^+(H)\}$ .

To prove the other inequality, note that if  $S \subseteq V(G)$ , then  $S$  is a minimal geodetic dominating set in  $G + H$  if and only if  $S$  is a minimal 2-path closure

absorbing set in  $G$ . This means that  $\max\{\rho_2^+(G), \rho_2^+(H)\} \leq \gamma_g^+(G + H)$ . Since  $G$  and  $H$  are noncomplete, we can pick  $u, v \in V(G)$  and  $x, y \in V(H)$  such that  $d_G(u, v) = 2$  and  $d_H(x, y) = 2$ . Then  $\{u, v, x, y\}$  is a minimal geodetic dominating set in  $G + H$ . This means that  $4 \leq \gamma_g^+(G + H)$ . This completely establishes the desired inequality. ■

Next, we investigate the minimal geodetic domination in the composition of graphs  $G + K_n$ .

For  $A \subseteq V(G)$ , we define  $A^g = A \cap I_G(A)$ , and for  $S \subseteq V(G[H])$ ,  $S_G = \{u \in V(G) : (u, v) \in S \text{ for some } v \in V(H)\}$ .

It is known (see [16]) that if  $S \subseteq V(G[H])$  is a geodetic dominating set in  $G[H]$ , then  $S_G$  is a geodetic dominating set in  $G$ .

**Theorem 5.3.** [16] *Let  $G$  be a noncomplete connected graph and  $n \geq 2$ . Then  $S \subseteq V(G[K_n])$  is a geodetic dominating set in  $G[K_n]$  if and only if  $S = [(A \setminus A^g) \times V(K_n)] \cup T$ , where  $A = S_G$  and  $T_G = A^g$ .*

**Corollary 5.4.** *For all noncomplete connected graphs  $G$  and  $n \geq 2$ ,*

$$\gamma_g^+(G[K_n]) \geq \max\{n|A| - (n - 1)|A^g| : A \text{ is a minimal geodetic dominating set in } G\}.$$

**Proof.** Let

$$\alpha = \max\{n|A| - (n - 1)|A^g| : A \text{ is a minimal geodetic dominating set in } G\}.$$

Let  $A \subseteq V(G)$  be a minimal geodetic dominating set in  $G$ , and let  $S = [(A \setminus A^g) \times V(K_n)] \cup [A^g \times \{v\}]$ , where  $v \in V(K_n)$ . By Theorem 5.3,  $S$  is a geodetic dominating set in  $G[K_n]$ . Suppose that there exists  $S^* \subseteq S$  such that  $S^*$  is a geodetic dominating set in  $G[K_n]$ . By Theorem 5.3,  $S^* = [(B \setminus B^g) \times V(K_n)] \cup T$ , where  $B$  is a geodetic dominating set in  $G$  and  $T_G = B^g$ . Since  $S^* \subseteq S$ ,  $B \subseteq A$ . Since  $A$  is a minimal geodetic dominating set in  $G$ ,  $A = B$ . Therefore,  $S = S^*$  and  $S$  is a minimal geodetic dominating set in  $G[K_n]$ . Thus,  $\gamma_g^+(G[K_n]) \geq |S| = n|A| - (n - 1)|A^g|$ . Since  $A$  is arbitrary,  $\gamma_g^+(G[K_n]) \geq \alpha$ . ■

**Lemma 5.5.** *Let  $G$  be a noncomplete connected graph and  $n \geq 2$ .*

- (i) *If  $S \subseteq V(G)$  is a geodetic dominating set (respectively, minimal geodetic dominating set) in  $G$ , then  $\{u\} \times S$  is a geodetic dominating set (resp. minimal geodetic dominating set) in  $K_n[G]$  for all  $u \in V(K_n)$ .*
- (ii) *If  $S \subseteq V(G)$  is a geodetic set (resp. minimal geodetic set but not dominating) in  $G$ , then  $\{(w, z)\} \cup (\{u\} \times S)$  is a geodetic dominating set (respectively, minimal geodetic dominating set) in  $G$  for all  $z \in V(G)$  and for all distinct  $w, u \in V(K_n)$ .*

**Proof.** Let  $S$  be a geodetic dominating set in  $G$  and  $u \in V(K_n)$ . Let  $(x, y) \in V(K_n[G]) \setminus (\{u\} \times S)$ . Suppose that  $x \neq u$ . Then  $(x, y)(u, v) \in E(K_n[G])$  for all  $v \in S$ . Thus,  $(x, y) \in N_{K_n[G]}[\{u\} \times S]$ . Choose  $v_1, v_2 \in S$  such that  $d_G(v_1, v_2) \geq 2$ . Then  $(x, y) \in I_{K_n[G]}[(u, v_1), (u, v_2)] \subseteq I_{K_n[G]}[\{u\} \times S]$ . Suppose that  $x = u$ . Then  $y \notin S$ . Since  $S$  is a geodetic dominating set in  $G$ ,  $y \in N_G[S] \cap I_G[S]$ . Thus,  $(x, y) \in N_{K_n[G]}[\{u\} \times S]$  and  $(x, y) \in I_{K_n[G]}[\{u\} \times S]$ . This proves that  $\{u\} \times S$  is a geodetic dominating set in  $K_n[G]$ . Finally, let  $\{u\} \times T \subseteq \{u\} \times S$  be a geodetic dominating set in  $K_n[G]$ . Then  $T \subseteq S$  and  $T$  is a geodetic dominating set in  $G$ . If  $S$  is a minimal geodetic dominating set in  $G$ , then  $T = S$ , and this proves statement (i).

To prove statement (ii), let  $C = \{(w, z)\} \cup (\{u\} \times S)$ , where  $S \subseteq V(G)$  is a geodetic set in  $G$ ,  $z \in V(G)$  and  $u, w \in V(K_n)$  with  $u \neq w$ . Let  $(a, b) \in V(K_p[G]) \setminus C$ . Suppose that  $a = u$ . Then  $b \notin S$  and  $aw \in E(K_n)$  so that  $(a, b)(w, z) \in E(K_p[G])$ . Since  $S$  is a geodetic set in  $G$ , there exist  $x, y \in S$  such that  $b \in I_G[x, y]$ . Then  $(u, x), (u, y) \in C$  and  $(a, b) \in I_{K_p[G]}[(u, x), (u, y)]$ . Suppose that  $a \neq u$ . Then  $au \in E(K_n)$  and  $(a, b) \in N_{K_p[G]}[\{u\} \times S]$ . Choose  $x, y \in S$  such that  $d_G(x, y) \geq 2$ . Then  $(u, x), (u, y) \in C$  and  $(a, b) \in I_{K_p[G]}[(u, x), (u, y)]$ . In any case,  $(a, b) \in N_{K_p[G]}[C]$  and  $(a, b) \in I_{K_p[G]}[C]$ . Since  $(a, b)$  is arbitrary,  $C$  is a geodetic dominating set in  $K_p[G]$ . Suppose that  $S$  is a minimal geodetic set in  $G$  but not dominating. Let  $(a, b) \in C$ , and put  $C^* = C \setminus \{(a, b)\}$ . If  $a = u$ , then  $b \in S$  and  $S \setminus \{b\}$  is not a geodetic set in  $G$ . This case means that  $C^*$  is not a geodetic set in  $K_p[G]$ . On the other hand, if  $a \neq u$ , then  $(a, b) = (w, z)$  and  $C^*$  is not a dominating set in  $K_p[G]$ . Therefore,  $C$  is a minimal geodetic dominating set in  $K_p[G]$ . ■

**Theorem 5.6.** *Let  $G$  be a noncomplete connected graph and  $n \geq 2$ , and let  $C \subseteq V(K_n[G])$ . Then  $C$  is a minimal geodetic dominating set in  $K_n[G]$  if and only if one of the following is true:*

- (i)  $C = \{u\} \times S$  for some minimal geodetic dominating set in  $G$  and  $u \in V(K_n)$ ;
- (ii)  $C = \{(w, z)\} \cup (\{u\} \times S)$  for some nondominating but minimal geodetic set  $S$  in  $G$ , for some  $z \in V(G)$  and distinct  $w, u \in V(K_n)$ ;
- (iii)  $C = \{(u_1, v_1), (u_1, v_2), (u_2, w_1), (u_2, w_2)\}$  for some distinct  $u_1, u_2 \in V(K_n)$  and some  $v_1, w_1, v_2, w_2 \in V(G)$  with  $d_G(v_1, v_2) \geq 2$  and  $d_G(w_1, w_2) \geq 2$ .

**Proof.** By Lemma 5.5, if property (i) or property (ii) holds, then  $C$  is a minimal geodetic dominating set in  $K_n[G]$ . It can also be readily verified that if property (ii) holds, then  $C$  is a minimal geodetic dominating set.

Let  $C \subseteq V(K_n[G])$  be a minimal geodetic dominating set in  $K_n[G]$ . Then  $C$  contains distinct vertices  $(u, v)$  and  $(u, y)$ . For if it were false and  $(u, v) \in C$ , then for all  $y \in V(G) \setminus \{v\}$ ,  $(u, y) \notin I_{K_n[G]}[C]$ , a contradiction. Moreover, since  $G$  is noncomplete, we may choose  $v$  and  $y$  such that  $d_G(v, y) \geq 2$ . Suppose that  $C = \{u\} \times S$  for some  $S \subseteq V(G)$ . Let  $z \in V(G) \setminus S$ . Since  $(u, z) \in N_{K_n[G]}[C]$ ,

$z \in N_G[S]$ . Similarly,  $z \in I_G[S]$ . Accordingly,  $S$  is a geodetic dominating set in  $G$ . Let  $T \subseteq S$  be a geodetic dominating set in  $G$ . Then  $\{u\} \times T \subseteq C$  and is a geodetic dominating set in  $K_n[G]$  by Lemma 5.5. By the definition of  $C$ ,  $T = S$  and  $S$  is a minimal geodetic dominating set in  $G$ . This establishes property (i).

Now suppose that  $C \neq \{u\} \times S$  for any  $S \subseteq V(G)$ . Let  $S = \{t \in V(G) : (u, t) \in C\}$ . Note that  $(a, b) \in N_{K_n[G]}[(u, v), (u, y)] \cap I_{K_n[G]}[(u, v), (u, y)]$  for all  $a \neq u$  and all  $b \in V(G)$ . Since  $C$  is a minimal geodetic dominating set in  $K_n[G]$ ,  $\{u\} \times S$  is not a geodetic dominating set in  $K_p[G]$ . Thus,  $N_G[S] \neq V(G)$  or  $I_G[S] \neq V(G)$ . Suppose that  $I_G[S] = V(G)$ . Then  $S$  is not a dominating set in  $G$ . Let  $b \in V(G) \setminus N_G[S]$ . There exists  $(w, z) \in C$  such that  $(u, b)(w, z) \in E(K_p[G])$ . Necessarily,  $w \neq u$ . Since  $\{(w, z)\} \cup (\{u\} \times S)$  is a geodetic dominating set in  $K_p[G]$ ,  $C = \{(w, z)\} \cup (\{u\} \times S)$ . In view of Lemma 5.5,  $S$  is a minimal geodetic set in  $G$ , and property (ii) is established. Finally suppose that  $I_G[S] \neq V(G)$ , and let  $b \in V(G) \setminus I_G[S]$ . Then there exists  $w \in V(K_n)$  distinct from  $u$  and some  $z, r \in V(G)$  with  $d_G(z, r) \geq 2$  such that  $(u, b) \in I_{K_n[G]}[(w, z), (w, r)]$ . Since  $\{(u, v), (u, y), (w, z), (w, r)\} \subseteq C$  is a geodetic dominating set,  $C = \{(u, v), (u, y), (w, z), (w, r)\}$ . ■

**Corollary 5.7.** *Let  $G$  be a noncomplete connected graph with  $g^+(G) < \gamma_g^+(G)$  and  $n \geq 2$ . Then*

$$\gamma_g^+(K_n[G]) = \max\{4, \gamma_g^+(G)\}.$$

### 6. $\gamma_g^+$ -SUBGRAPH

A graph  $H$  is a  $\gamma_g^+$ -subgraph if there exists a connected graph  $G$  containing  $H$  as an induced subgraph such that  $V(H)$  is a  $\gamma_g^+$ -set in  $G$ .

The idea of the following result is taken from [4].

**Theorem 6.1.** *Let  $H$  be a connected graph. Then  $H$  is a  $\gamma_g^+$ -subgraph if and only if either  $H$  is complete or  $H$  has no vertex  $v$  with  $e_H(v) = 1$ .*

**Proof.** Let there be a connected graph  $G$  containing  $H$  as an induced subgraph and such that  $V(H)$  is a  $\gamma_g^+$ -set in  $G$ . Suppose that  $H$  is noncomplete and suppose that  $v \in V(H)$  with  $e_H(v) = 1$ . We claim that  $S = V(H) \setminus \{v\}$  is a geodetic dominating set in  $G$ . Let  $w \in V(G) \setminus S$ . Suppose that  $w = v$ . Since  $H$  is noncomplete, there exist  $a, b \in V(H)$  such that  $d_G(a, b) = d_H(a, b) = 2$ . Necessarily,  $a \neq v$  and  $b \neq v$  so that  $a, b \in S$ . Since  $av, bv \in E(H) \subseteq E(G)$ ,  $w \in I_G[a, b] \subseteq I_G[S]$  and  $w \in N_G[a] \subseteq N_G[S]$ . Suppose that  $w \neq v$ . Since  $V(H)$  is a geodetic set in  $G$ , there exist  $a, b \in V(H)$  such that  $w \in I_G[a, b]$ . If  $a = v$  or  $b = v$ , then  $d_H(a, b) = d_G(a, b) = 1$ , a contradiction. Thus  $a, b \in S$ . Since  $av, bv \in E(G)$ ,  $d_G(a, b) = 2$  and  $aw, bw \in E(G)$ . Hence  $w \in I_G[S]$  and  $w \in N_G[S]$ . Thus,  $S$  is a geodetic dominating set in  $G$ . This is a contradiction

since  $V(H)$  is a minimal geodetic dominating set in  $G$  and  $S$  is a proper subset of  $V(H)$ .

By Theorem 2.2, if  $H$  is complete, then  $V(H)$  is the  $\gamma_g^+$ -set in  $G = H$ . Suppose that  $H$  is noncomplete having no vertex  $u$  with  $e_H(u) = 1$ . For each  $u \in V(H)$ , choose  $v \in V(H)$  such that  $d_H(u, v) = 2$ . Corresponding to each pair  $u$  and  $v$ , add to  $H$  the vertex  $x_{u,v}$  and the edges  $ux_{u,v}$  and  $vx_{u,v}$ . Let  $G$  be the resulting graph of minimum order obtained in this way. Then  $|V(G) \setminus V(H)| \leq |V(H)|$ . We claim that  $V(H)$  is a  $\gamma_g^+$ -set in  $G$ . Let  $x \in V(G) \setminus V(H)$ . Then  $x = x_{u,v}$  for some  $u, v \in V(H)$  with  $d_H(u, v) = d_G(u, v) = 2$ . More precisely,  $xu, xv \in E(G)$ . Thus,  $x \in I_G[u, v]$  and  $x \in N_G[u]$ . In other words,  $V(H)$  is a geodetic dominating set in  $G$ .

Let  $u \in V(H)$ , and let  $v \in V(H)$  such that  $d_H(u, v) = 2$ . Corresponding to  $u$  and  $v$  is a  $x_{u,v} \in V(G) \setminus V(H)$ . By its construction,  $x_{u,v} \notin I_G[V(H) \setminus \{u\}]$ . Since  $u$  is arbitrary,  $V(H)$  is a minimal geodetic dominating set in  $G$ . Finally, let  $S \subseteq V(G)$  be a minimal geodetic dominating set in  $G$ . For the triple  $u, v, x_{u,v}$ , if  $u, v \in S$ , then  $x_{u,v} \notin S$ , or, equivalently, if  $x_{u,v} \in S$ , then  $u \notin S$  or  $v \notin S$ . Thus

$$|S| = |S \setminus V(H)| + |V(H) \cap S| \leq |V(H) \setminus S| + |V(H) \cap S| = |V(H)|.$$

Since  $S$  is arbitrary,  $V(H)$  is a  $\gamma_g^+$ -set in  $G$ . ■

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