

## EXTENDING THE MAX ALGORITHM FOR MAXIMUM INDEPENDENT SET

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### Abstract

The maximum independent set problem is an NP-hard problem. In this paper, we consider Algorithm MAX, which is a polynomial time algorithm for finding a maximal independent set in a graph  $G$ . We present a set of forbidden induced subgraphs such that Algorithm MAX always results in finding a maximum independent set of  $G$ . We also describe two modifications of Algorithm MAX and sets of forbidden induced subgraphs for the new algorithms.

**Keywords:** maximum independent set, stable set, stability number, independence number, reduction, graph transformation, MAX Algorithm, MIN Algorithm, Vertex Order Algorithm.

**2010 Mathematics Subject Classification:** 05C69, 05C85.

### 1. INTRODUCTION

In a simple graph  $G = (V, E)$ , a set of vertices is *independent* (or *stable*) if no two vertices in this set are adjacent. The cardinality of a maximum size independent set in  $G$  is called the *independence number* (or the *stability number*) of  $G$ , denoted by  $\alpha(G)$ . The problem of determining an independent set of maximum cardinality, so-called Maximum Independent Set (or MIS for short) problem,

finds important applications in various fields, some examples are computer vision and pattern recognition. It is well-known that the problem is generally NP-hard. Alekseev has shown in [1] that if a graph  $H$  has a connected component not of the form  $S_{i,j,k}$ , where  $S_{i,j,k}$  is the graph consisting of three induced paths of lengths  $i, j$  and  $k$ , with a common initial vertex, then the MIS problem is NP-hard for  $H$ -free graphs. It is shown that the MIS problem is solvable in polynomial time for  $S_{1,1,1}$ -free graphs (claw-free graphs) [28, 34] and for  $S_{0,1,2}$ -free graphs ( $P_4$ -free graphs) [10]. These results were generalized to  $S_{1,1,2}$ -free graphs (fork-free graphs) [2] and  $S_{0,2,2}$  ( $P_5$ -free graphs) [21]. Hence, the complexity status of the MIS problem in the graph classes defined by a single forbidden induced subgraph of the form  $S_{i,j,k}$  was solved for the case  $i + j + k \leq 4$ . For larger  $i + j + k$  ( $P_6, S_{1,2,2}, S_{2,2,2}$  and so on), it is still an open question and gained much effort to solve in some subclasses. Some examples are  $(P_n, K_{1,n})$ -free graphs [22],  $(P_6, \text{diamond})$ -free graphs [29],  $(P_6, K_{2,3})$ -free graphs [30],  $(S_{2,2,2}, \text{banner})$ -free graphs [12],  $(S_{2,2,2}, B_2, \text{domino}, M_3, K_{1,m})$ -free graphs [20],  $(S_{1,2,k}, \text{banner}, K_{1,n})$ -free graphs [19],  $(S_{1,2,5}, \text{banner})$ -free graphs [24],  $S_{1,2,k}$ -free planar graphs,  $S_{1,k,k}$ -free graphs of low degree [23], and  $S_{2,2,2}$ -free subcubic graphs [26].

Useful techniques for solving the problem are heuristic algorithms, i.e., the methods giving maximal (inclusion sense) independent sets in polynomial time. Three well-known algorithms are VO (Vertex Ordering) [27], MIN [31], and MAX [15]. Moreover, for some restricted graph classes, these algorithms always result in finding a maximum independent set of  $G$ , and hence yield the exact value of the independence number of  $G$  in polynomial time.

If  $H_1, H_2, \dots, H_k$  are graphs, then we say that  $G$  is  $\{H_1, H_2, \dots, H_k\}$ -free if  $G$  does not contain a copy of any of the graphs  $H_1, H_2, \dots, H_k$  as an induced subgraph. In [27], Mahadev and Reed characterized a class of graphs for which a maximum independent set can be obtained by VO, that is the  $\mathcal{F}_1$ -free graphs, where  $\mathcal{F}_1 = \{F_1, F_2, F_3, F_4, F_5, F_6\}$ .

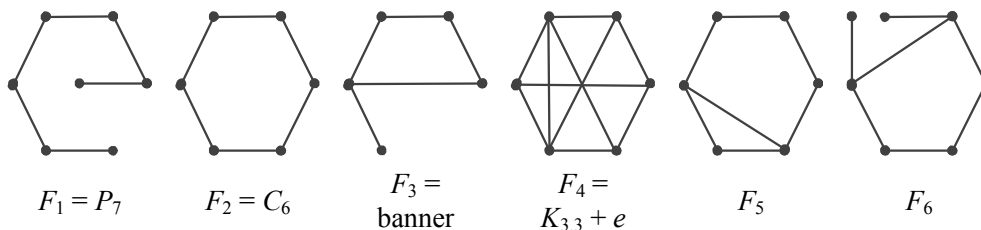


Figure 1. Forbidden subgraphs for Algorithm VO.

A set of forbidden induced subgraphs  $\mathcal{F}_2$ , under which Algorithm MIN always results in finding a maximum independent is given in [17] by Harant, Schiermeyer, and Ryjáček, where  $\mathcal{F}_2 = \{F_1, F_3, F_5, F_6, F_7, F_8, F_9, F_{10}, F_{11}, F_{12}, F_{13}\}$ .

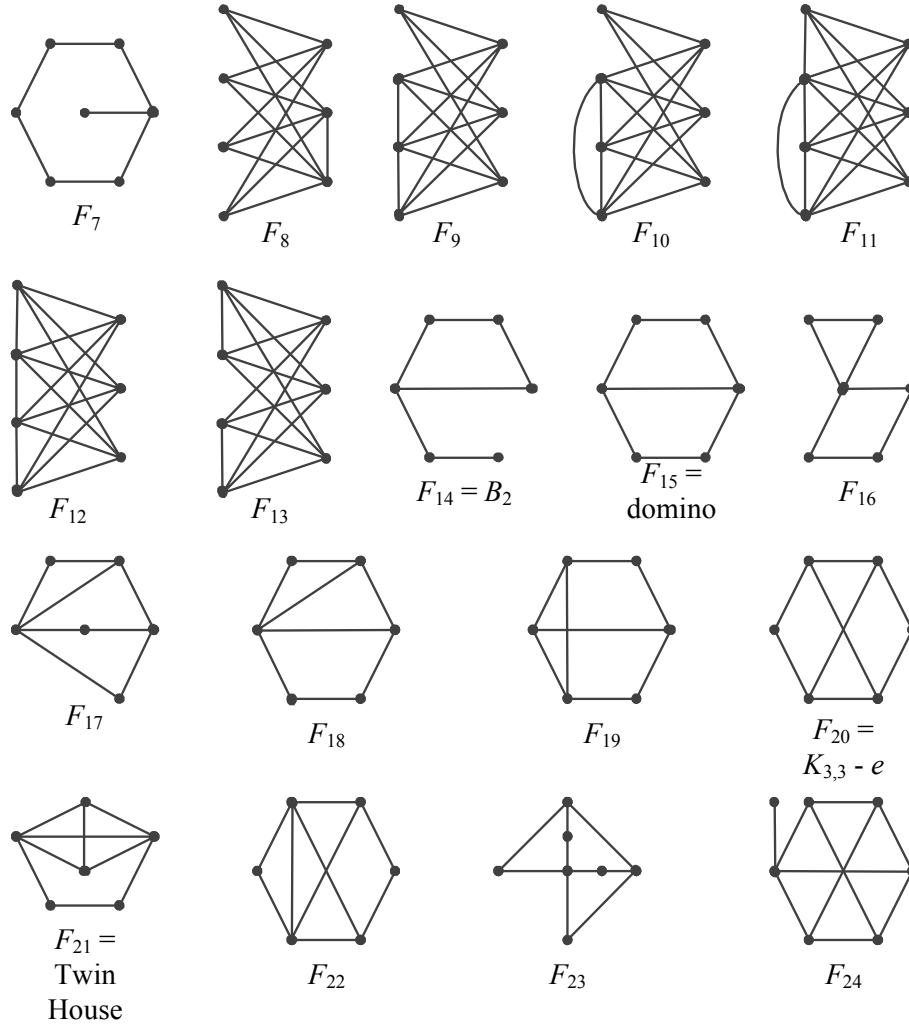


Figure 2. Some forbidden subgraphs for Algorithm MIN.

Zverovich [35] obtained another set, say  $\mathcal{F}_3$ , of forbidden induced subgraphs for MIN, where  $\mathcal{F}_3 = \{F_1, F_4, F_5, F_6, F_7, F_{14}, F_{15}, F_{16}, F_{17}, F_{18}, F_{19}, F_{20}, F_{21}, F_{22}, F_{23}, F_{24}\}$ .

Following this direction, we will describe a set of forbidden induced subgraphs for Algorithm MAX in Section 2.

Given a graph  $G$ , Algorithm MAX chooses a maximum degree vertex, deletes it from the graph together with all incident edges. The process is repeated until there are no edges remaining. Then the remaining vertices form a maximal independent set of  $G$ . This idea is based on an assumption that a maximum degree vertex has very small possibility to belong to some maximum independent set and hence can be deleted from the graph without (or with very small possibility

of) changing the independence number. Obviously, it will be better if we can find and delete only a vertex not belonging to some maximum independent set. Such vertices are called  $\alpha$ -redundant vertices [5].

**Definition.** [5] Given a graph  $G$ , a vertex  $v$  of  $G$  will be called  $\alpha$ -redundant if  $\alpha(G - v) = \alpha(G)$ .

Here, for a graph  $G$ , and a vertex  $v$  of  $G$ , we denote by  $G - v$  the graph obtained from  $G$  by deleting  $v$  and all incident edges. Unfortunately, the problem of finding such vertices, is polynomially equivalent with the MIS problem and hence also NP-hard in general. However, in some cases,  $\alpha$ -redundant vertices can be recognized efficiently. In the following, we present some conditions to recognize  $\alpha$ -redundant vertices. First, we have the following obvious fact.

**Proposition 1.** Given a graph  $G$ , a vertex  $v$  of  $G$  is  $\alpha$ -redundant if and only if there exists some maximum independent set  $S$  such that  $v \notin S$ .

**Lemma 2** [13]. *Let  $G$  be a graph containing an induced  $K_{1,m}$ ,  $u(v_1, v_2, \dots, v_m)$ , where  $u$  is the center-vertex. Then either  $u$  is  $\alpha$ -redundant or there exists some  $u_1, u_2, \dots, u_m$  such that  $\{u, u_1, u_2, \dots, u_m\}$  is independent and there is a perfect matching between  $\{u_i\}$  and  $\{v_i\}$ .*

The  $\alpha$ -redundant technique was used successfully in [5, 6, 12, 13, 36, 20] to extend some polynomial results of the MIS problem in some subclasses of  $P_5$ -free graphs. Note that, in Lemma 2, for the case  $m = 1$ , we have a so-called *neighborhood reduction* technique [14]: let  $a$  and  $b$  be adjacent vertices in a graph  $G$ . If every neighbor vertex of  $b$  is also adjacent to  $a$ , then for any independent set  $S$  containing  $a$ , the set  $(S \setminus \{a\}) \cup \{b\}$  is also independent, i.e.,  $a$  is  $\alpha$ -redundant. And for the case  $m = 2$ , we have a so-called *vertex deletion* technique [4]: let three vertices  $a, b$ , and  $c$  induce a  $P_3$  with edges  $ab$  and  $bc$ . If  $(N(a) \cup N(c)) \setminus N[b]$  is a clique, then the removal of  $b$  does not change the independence number of the graph, i.e.,  $b$  is  $\alpha$ -redundant.

In Section 2, we will describe conditions under which a maximum degree vertex is not  $\alpha$ -redundant in order to describe forbidden subgraphs for Algorithm MAX. In Section 3, we will describe some techniques to recognize some  $\alpha$ -redundant vertex in the neighborhood of a maximum degree vertex  $u$ . In case we succeed, we will delete such a vertex instead of  $u$ . It leads us to a new hybrid algorithm for solving the MIS problem. Our motivation is finding an algorithm better than MAX and VO in the sense of forbidden induced subgraph set. In Section 4, we will combine Algorithm MAX with  $K_{1,m}$ -reduction to use some polynomial time solution of the MIS problem in some subclass of  $K_{1,m}$ -free graphs. Section 5 is devoted to comparison of the new algorithms with classical heuristic methods. In Section 6, we will give some discussion around the topic and related results.

Now, to some other notions which will be used throughout the paper. In this paper, we consider only finite undirected, simple graphs  $G = (V(G), E(G))$ . For  $u, v \in V(G)$ , we write  $u \sim v$  if  $uv \in E(G)$  and  $u \not\sim v$  if  $uv \notin E(G)$ . Denote by  $N(x)$  the *neighborhood* of a vertex  $x \in V(G)$ , by  $d(x) = |N_G(x)|$  the *degree* of  $x$  in  $G$ . For a set  $M \subseteq V(G)$ , denote by  $G[M]$  the *induced subgraph* of  $G$  on  $M$ . For a vertex  $x$ , we also denote  $N_M(x) = N(x) \cap M$  and  $d_M(x) = |N_M(x)|$ . For an induced subgraph  $H$  of  $G$ , denote  $N_H(x) = N_{V(H)}(x)$  and  $d_H(x) = |N_H(x)|$ . Let  $\Delta(G) = \max\{d(u) : u \in V(G)\}$ . As usual, we also denote by  $K_p$  the complete graph having  $p$  vertices and by  $K_{m,n}$  the complete bipartite graph with cardinalities  $m$  and  $n$  of the two parts. For  $K_{1,m}$ , the vertex of degree  $m$  is called the center-vertex. Denote by  $P_n$  the induced path having  $n$  vertices.

2. FORBIDDEN SUBGRAPHS FOR ALGORITHM MAX

First, we recall the well-known greedy Algorithm MAX (see Algorithm 1) for finding a maximal independent set in a graph, which was introduced in [15].

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**Algorithm 1** Algorithm MAX (Maximum degree).

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**Input:**  $G = (V, E)$ .

**Output:**  $S$ , a maximal independent set of  $G$ .

- 1:  $H_n := G; i := n;$
  - 2: **while**  $E(H_i) \neq \emptyset$  **do**
  - 3:     Choose a vertex  $v_i \in V(H_i)$  such that  $d_{H_i}(v_i) = \Delta(H_i);$
  - 4:      $H_{i-1} := H_i - v_i; i := i - 1;$
  - 5: **end while**
  - 6:  $S := V(H_i);$
  - 7: **return**  $S$
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Obviously, the set  $S$ , generated by Algorithm MAX, is a maximal (but not necessarily maximum) independent set in  $G$ , and hence  $\alpha(G) \geq |S|$ . In this section, we describe some set of forbidden induced subgraphs, under which  $\alpha(G) = |S|$ . Let  $G$  be a graph such that the algorithm fails for  $G$ , i.e., the algorithm chooses and deletes some vertex of maximum degree  $u$  of  $G$  but the deletion of  $u$  reduces the independence number of  $G$ . It implies that  $u$  belongs to every maximum independent set of  $G$  or, in other words,  $u$  is not  $\alpha$ -redundant by Proposition 1. Now, we consider some conditions, under which  $u$  is not  $\alpha$ -redundant.

**Lemma 3.** *Let  $G$  be a graph,  $u$  be a vertex,  $v \in N(u)$ , and  $S$  be a maximum independent set such that  $u \in S$ . Then either  $u$  is  $\alpha$ -redundant or  $d_S(v) \geq 2$ .*

**Proof.** If  $d_S(v) = 1$ , then  $(S \setminus \{u\}) \cup \{v\}$  is a maximum independent set not containing  $u$ , i.e.,  $u$  is  $\alpha$ -redundant by Proposition 1. ■

**Lemma 4.** *Let  $G$  be a graph,  $E(G) \neq \emptyset$ ,  $u$  be a maximum degree vertex and  $S$  be a maximum independent set such that  $u \in S$ . Then either  $u$  is  $\alpha$ -redundant or there exist some  $v_1, v_2 \in N(u)$  such that  $v_1 \approx v_2$ .*

**Proof.** Let  $|N(u)| = p$ . Since  $u$  is of maximum degree and  $E(G) \neq \emptyset$ ,  $p > 0$ . Assume that  $N(u)$  is a clique. Then for every  $v \in N(u)$ ,  $d(v) \geq p - 1 + 1 = p = d(u) \geq d(v)$ . Hence, for every  $v \in N(u)$ ,  $d(v) = p$ , i.e.,  $N[u]$  is a clique and a connected component of  $G$ . Now,  $u$  is  $\alpha$ -redundant because there exists some maximum independent set  $S$  containing a neighbor of  $u$  and not containing  $u$ . ■

**Lemma 5.** *Let  $G$  be a graph and  $(v_1, u, v_2)$  be an induced  $P_3$  ( $u$  is the mid-vertex). Then either  $u$  is  $\alpha$ -redundant or there exist  $u_1, u_2$  such that  $\{u_1, v_1, u, v_2, u_2\}$  induces a  $P_5$  ( $u$  is the mid-vertex) or a banner ( $F_3$ ) ( $v_1$  or  $v_2$  is of degree three) or a  $K_{2,3}$  ( $v_1, v_2$  are of degree three).*

**Proof.** Assume that  $u$  is not  $\alpha$ -redundant. Then by Lemma 2, there exist  $u_1, u_2$  such that  $\{u, u_1, u_2\}$  is independent and there exists a perfect matching between  $\{u_1, u_2\}$  and  $\{v_1, v_2\}$ , i.e.,  $\{u_1, v_1, u, v_2, u_2\}$  induces a  $P_5$ , a banner, or a  $K_{2,3}$  depending on the adjacency between  $\{u_1, u_2\}$  and  $\{v_1, v_2\}$ . ■

Given a graph  $G$ , denote by  $k_{MAX}(G)$  the minimum cardinality of a (maximal) independent set given by Algorithm MAX. The following result describes a set of forbidden induced subgraphs, under which Algorithm MAX gives a maximum independent set.

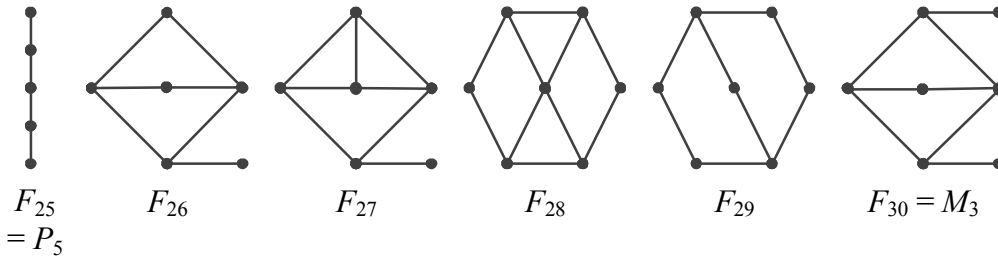


Figure 3. Some forbidden subgraphs for Algorithm MAX and Algorithm MMAX.

**Theorem 6** (Forbidden subgraphs for MAX Algorithm). *Let  $G$  be an  $\mathcal{F}_4$ -free graph of order  $n \geq 7$ , where  $\mathcal{F}_4 = \{F_4, F_{15}, F_{19}, F_{20}, F_{21}, F_{24}, F_{25}, F_{26}, F_{27}\}$ . Then  $k_{MAX}(G) = \alpha(G)$ .*

**Proof.** Assume that the statement in the theorem is not correct. That means there exists a (without loss of generality) connected graph  $G$  satisfying the assumption of the theorem and some vertex  $u \in V(G)$  such that

1.  $u$  is of maximum degree in  $G$  and

2.  $u$  belongs to every maximum independent set of  $G$ ,

i.e., Algorithm MAX will fail.

Let  $S$  be a maximum independent set of  $G$  (hence  $u \in S$ ) and  $T = V(G) \setminus S$ . Since  $G$  is  $P_5$ -free and  $u$  is not  $\alpha$ -redundant, we obtain the following observation from Lemma 5.

**Claim 7.** *Let  $v_1, v_2 \in N(u)$  be two non-adjacent vertices. Then there exists some  $u' \in S$  such that  $v_1, v_2 \in N(u) \cap N(u')$ .*

Also by Lemma 5, for two non-adjacent neighbors  $v_1, v_2$  of  $u$ , there exist some  $u_1, u_2$  such that  $\{u, u_1, u_2, v_1, v_2\}$  induces a  $P_5$ , or a  $K_{2,3}$ , or an  $F_3$ . The first case cannot happen because  $G$  is  $P_5$ -free. We consider the two other cases by Claims 8 and 9.

**Claim 8.** *There are no vertices  $v_1, v_2 \in T$  and  $u_1, u_2 \in S$  such that  $\{v_1, v_2, u, u_1, u_2\}$  induces a  $K_{2,3}$ .*

**Proof.** Suppose that  $\{v_1, v_2, u, u_1, u_2\}$  induces a  $K_{2,3}$ , for some  $v_1, v_2 \in T$  and  $u_1, u_2 \in S$ . Let  $H$  be a maximal (by inclusion) induced complete bipartite subgraph of  $G$  with parts  $A$  and  $B$  such that  $\{u, u_1, u_2\} \subseteq A \subseteq S$  and  $\{v_1, v_2\} \subseteq B \subseteq T$ .

*Case 1.*  $|B| < |A|$ . Since  $d(u) \geq d(v_2)$ , there exists  $v_3 \in T \setminus V(H)$  such that  $v_3 \sim u$  and  $v_3 \approx v_2$ .

If  $N_A(v_3) = A$ , then  $v_3 \sim v$  for some  $v \in B$  (otherwise,  $H$  is not maximal). Without loss of generality, let  $v = v_1$ . Then  $\{u, u_1, u_2, v_1, v_2, v_3\}$  induces an  $F_4$ , a contradiction.

Hence, there exists some  $v_3 \in T \setminus V(H)$  such that  $v_3 \sim u$ ,  $v_3 \approx v_1$ , and  $v_3 \approx u'$  for some  $u' \in A$ , say  $v_3 \approx u_1$ .

If  $v_3 \approx u' \forall u' \in A \setminus \{u\}$ , then, by Claim 7, there exists some  $u_3 \in S \setminus A$  such that  $u_3$  is adjacent to  $v_1$  and  $v_3$ . Moreover,  $v \approx u_3$  for some  $v \in B$  (otherwise we have a contradiction with the maximality of  $H$ ). Assume that  $v_2 \approx u_3$ . Then  $v_3 \sim v_2$ , otherwise  $\{u_2, v_2, u, v_3, u_3\}$  induces a  $P_5$ , a contradiction. Now,  $\{u, u_1, u_3, v_1, v_2, v_3\}$  induces an  $F_{28}$ , a contradiction.

Hence,  $u_3 \sim u'$  for some  $u' \in A$ , say  $u' = u_2$ . But then  $\{u, u_1, u_2, v_1, v_2, v_3\}$  induces an  $F_{20}$  or an  $F_{21}$ , depending on  $v_3 \sim v_2$  or not, a contradiction.

*Case 2.*  $|B| \geq |A|$ , i.e, there exists  $v_3 \in B \setminus \{v_1, v_2\}$ . The set  $S' = (S \setminus A) \cup B$  cannot be an independent set of  $G$ , otherwise, since  $|S'| \geq |S|$  and  $u \notin S'$ ,  $u$  is  $\alpha$ -redundant. Hence, there exists some  $u_3 \in S \setminus A$  such that  $u_3 \sim v$  for some  $v \in B$ , say  $v = v_1$ . Moreover, the maximality of  $H$  implies that  $u_3$  cannot be adjacent to every vertex of  $B$ . Without loss of generality, assume that  $u_3 \approx v_2$ . Then  $\{u, u_1, u_2, u_3, v_1, v_2, v_3\}$  induces an  $F_{24}$  or an  $F_{20}$ , depending on  $v_3 \sim u_3$  or not, a contradiction. □

**Claim 9.** *There are no vertices  $v_1, v_2 \in T$  and  $u_1, u_2 \in S$  such that  $\{v_1, v_2, u, u_1, u_2\}$  induces an  $F_3$ .*

**Proof.** For contradiction, let  $v_1, v_2 \in T$  and  $u_1, u_2 \in S$  be vertices such that  $\{u, u_1, u_2, v_1, v_2\}$  induces an  $F_3$  (without loss of generality, assume that  $u_2$  is the vertex of degree one in the  $F_3$ ).

Let  $H = G[\{u, u_1, u_2, v_1, v_2\}]$ . Since  $d_H(v_1) > d_H(u)$ , there exists some  $v_3 \notin V(H)$  such that  $u \sim v_3$  and  $v_3 \approx v_1$ . By Claim 8,  $v_3 \approx u_1$  or  $v_3 \approx u_2$ .

If  $v_3 \sim u_1$  and  $v_3 \approx u_2$ , then  $\{u, u_1, u_2, v_1, v_2, v_3\}$  induces an  $F_{20}$  or an  $F_{26}$ , depending on  $v_2 \sim v_3$  or not, a contradiction.

If  $v_3 \sim u_2$  and  $v_3 \approx u_1$ , then  $\{u, u_1, u_2, v_1, v_2, v_3\}$  induces an  $F_{15}$  or an  $F_{19}$ , depending on  $v_2 \sim v_3$  or not, a contradiction.

If  $v_3 \approx u_1$  and  $v_3 \approx u_2$ , then by Claim 7, there exists some  $u_3 \in S$  such that  $u_3 \sim v_3, u_3 \sim v_1$ . Moreover, by Claim 8,  $v_2 \approx u_3$ . Now,  $\{u, u_1, u_2, v_1, v_2, v_3\}$  induces an  $F_{15}$  or an  $F_{27}$ , depending on  $v_2 \sim v_3$  or not, a contradiction.  $\square$

Lemmas 4, 5 and Claims 7, 8, 9 finish the proof of the theorem.  $\blacksquare$

### 3. A MODIFICATION OF ALGORITHM MAX

Algorithms MAX, MIN, VO are heuristic methods, i.e., for a graph  $G$ , they give a maximal (and not necessarily maximum) independent set in polynomial time. Some other useful techniques for solving the problem are transformation methods. Transformation methods include graph transformation (changing some part of the graph) with controlled changing of the independence number. Some examples of graph transformation methods are STRUCTION [11], Magnet [16], BAT [18], vertex folding [8], and vertex splitting [2, 32]. Other transformation methods are deletion (or insertion) of an edge [7] or deletion of a vertex (along with deletion of all incident edges) [4], simplicial reduction [33], neighborhood reduction [14], and twin reduction [9]. We can repeat the deletion of a vertex or an edge until we obtain a simple (enough) graph, i.e., a graph for which we already have some efficient algorithm for solving the problem.

In this section, we describe a modification of Algorithm MAX using  $\alpha$ -redundant technique. The idea here is to pick a maximum degree vertex  $u$  of  $G$ , but before deleting  $u$  from  $G$ , we check if some neighbor of  $u$  is  $\alpha$ -redundant (in the sense of Lemma 2) and delete it instead of  $u$  in the case it is. Algorithm 2 is the pseudo-code of our new algorithm.

Consider an arbitrary simple graph  $G$ , let  $n = |V(G)|$ . Clearly, MMAX gives a maximal independent set. The algorithm repeatedly checks if the remaining graph still contains edges and chooses a maximum degree vertex  $u$  (Step 3). In



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**Algorithm 2** Algorithm MMAX (Modification of MAX).

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**Input:**  $G = (V, E)$ .

**Output:**  $S$ , a maximal independent set of  $G$ .

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1:  $H_n := G; i := n; S := \emptyset$ 
2: while  $E(H_i) \neq \emptyset$  do
3:   Choose a vertex  $u \in V(H_i)$  such that  $d_{H_i}(u) = \Delta(H_i)$ ;
4:   for  $v \in N_{H_i}(u)$  do
5:     if there exists some  $u_1 \in N_{H_i}(v) \setminus N_{H_i}[u]$  such that there exists no  $v_1 \in$ 
        $N_{H_i}(u_1) \setminus N_{H_i}[v]$  then
6:        $H_{i-1} := H_i - v; i := i - 1$ ; break;
7:     end if
8:     if there exist some  $u_1, u_2 \in N_{H_i}(v) \setminus N[u]$  and  $u_1 \approx u_2$  such that there
       exist no  $v_1, v_2, v_3 \in N_{H_i}(u) \setminus N_{H_i}[v]$  and  $v_1 \sim u, v_2 \sim u_1, v_3 \sim u_2$  then
9:        $H_{i-1} := H_i - v; i := i - 1$ ; break;
10:    end if
11:  end for
12:   $H_{i-1} := H_i - u; i := i - 1$ ;
13: end while
14:  $S := V(H_i)$ ;
15: return  $S$ 

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Steps 5 and 8, the algorithm checks and removes a vertex  $v \in N(u)$  if  $v$  is  $\alpha$ -redundant by applying Lemma 2 for the case  $m = 1$  and  $m = 3$ , respectively. If no vertex in  $N(u)$  is  $\alpha$ -redundant in this sense, then Step 12 removes  $u$  with the assumption that  $u$  is  $\alpha$ -redundant. In the case that there is no remaining edge, the remaining vertices form a maximal independent set (Step 14).

We can find a maximum degree vertex  $u$  of  $G$  in time  $O(n^2)$ . Then  $|N(u)|$  is at most  $n - 1$ . For  $v \in N(u)$ , we can check if  $v$  is  $\alpha$ -redundant in time  $O(n^2)$  in Step 5, and in time  $O(n^5)$  in Step 8. The removal of vertices will be performed at most  $n$  times. Therefore, we obtain the following result.

**Theorem 10.** *For an arbitrary graph  $G$ , Algorithm MMAX finds a maximal independent set in time  $O(n^7)$ .*

Given a graph  $G$ , denote by  $k_{MMAX}(G)$  the minimum cardinality of a maximal independent set found by Algorithm MMAX. The following theorem provides a set of forbidden induced subgraphs for Algorithm MMAX.

**Theorem 11.** *Let  $G$  be an  $\mathcal{F}_5$ -free graph of order  $n \geq 7$ , where  $\mathcal{F}_5 = \{F_1, F_5, F_7, F_8, F_{14}, F_{15}, F_{18}, F_{20}, F_{21}, F_{24}, F_{28}, F_{29}, F_{30}\}$ . Then  $MMAX(G) = \alpha(G)$ .*

**Proof.** We basically follow the idea of the proof of Theorem 6. Assume that the

statement in the theorem is not correct, that means there exists some graph  $G$  satisfying the assumption of Theorem 11 and a vertex  $u_0 \in V(G)$  such that:

1.  $u_0$  is of maximum degree in  $G$ ,
2. for every vertex  $v \in N(u_0)$ ,  $v$  is not  $\alpha$ -redundant (as described in Steps 4.1 and 4.2), and
3.  $u_0$  belongs to every maximum independent set of  $G$ ,

i.e., the MMAX algorithm will fail.

Let  $S$  be a maximum independent set of  $G$ , i.e.,  $u_0 \in S$ . Denote  $T = V(G) \setminus S$ , i.e.,  $N(u_0) \subset T$ . We have the following observations.

**Claim 12.** *There exists  $v_0 \in N(u_0)$  and  $u_1, u_2 \in S$  such that  $\{v_0, u_0, u_1, u_2\}$  induces  $K_{1,3}$ , where  $v_0$  is the center-vertex.*

**Proof.** By Lemma 4, there exist  $v_1, v_2 \in N(u_0)$  such that  $v_1 \approx v_2$ . Since  $u_0$  is not  $\alpha$ -redundant, by Lemma 5, the only case we have to consider is that there exists  $u_1, u_2 \in S$  such that  $\{u_1, v_1, u_0, v_2, u_2\}$  induces a  $P_5$ , because for the remaining cases (say  $K_{2,3}$  and banner), we have a desired  $K_{1,3}$ . Since  $v_1, v_2$  are not  $\alpha$ -redundant, by applying Lemma 2 for the case  $m = 1$ ,  $N(u_1) \setminus N[v_1]$  and  $N(u_2) \setminus N[v_2]$  are not empty. We proceed by considering the following cases.

*Case 1.* There exists some  $v_3 \in (N(u_1) \setminus N[v_1]) \cap (N(u_2) \setminus N[v_2])$ . If  $v_3 \sim u_0$ , then  $v_3$  is such a vertex  $v_0$  of the conclusion of the claim. Hence, we assume that  $v_3 \not\sim u_0$ . Since  $(S \setminus \{u_0, u_1, u_2\}) \cup \{v_1, v_2, v_3\}$  is not independent, there exists some  $u_3 \in S \setminus \{u_0, u_1, u_2\}$  such that  $u_3$  is adjacent to at least one vertex among  $v_1, v_2, v_3$ . Now,  $u_3$  is adjacent to at least two vertices among  $\{v_1, v_2, v_3\}$ , otherwise  $\{u_0, u_1, u_2, u_3, v_1, v_2\}$  induces an  $F_7$ , a contradiction. Hence,  $v_1$  or  $v_2$  is such a vertex  $v_0$  of the conclusion of the claim.

*Case 2.* There exists some  $v_3 \in N(u_1) \setminus (N[v_1] \cup N(u_2))$  and  $v_4 \in N(u_2) \setminus (N[v_2] \cup N(u_1))$ .

*Subcase 2.1.*  $v_3 \sim v_4$ .

*Subcase 2.1.1.*  $v_3 \sim u_0$  (similar for the case  $v_4 \sim u_0$ ). Then  $v_3 \sim v_2$ , otherwise  $\{u_2, v_2, u_0, v_1, u_1, v_3\}$  induces an  $F_{14}$ , a contradiction. Moreover,  $v_4 \not\sim v_1$ , otherwise  $\{u_1, u_2, v_1, v_2, v_3, v_4\}$  induces an  $F_{15}$ , a contradiction. Hence,  $v_4 \sim u_0$ , otherwise  $\{u_2, v_4, v_3, u_1, v_1, u_0\}$  induces an  $F_{14}$ , a contradiction. But now,  $\{u_1, v_1, u_0, v_2, u_2, v_4\}$  induces an  $F_{14}$ , a contradiction.

*Subcase 2.1.2.*  $u_0$  is adjacent to neither  $v_3$  nor  $v_4$ . If  $v_3 \sim v_2$  (similar for the case  $v_4 \sim v_1$ ), then  $v_1 \sim v_4$ , otherwise  $\{v_1, u_0, v_2, u_2, v_4, v_3\}$  induces an  $F_{14}$ , a contradiction. Now,  $\{u_1, u_2, v_1, v_2, v_3, v_4\}$  induces an  $F_{15}$ , a contradiction.

Hence, we assume that  $v_1 \approx v_4$  and  $v_2 \approx v_3$ . Since  $(S \setminus \{u_1, u_0, u_2\}) \cup \{v_1, v_2, v_3\}$  (and  $(S \setminus \{u_1, u_0, u_2\}) \cup \{v_1, v_2, v_4\}$  either) is not independent, there exists some  $u_3 \in S \setminus \{u_0, u_1, u_2\}$  such that  $u_3$  is adjacent to at least one vertex among  $\{v_1, v_2, v_3, v_4\}$ . Hence,  $u_3$  is adjacent to  $v_1$  or  $v_2$ , otherwise  $\{u_0, u_1, u_2, u_3, v_1, v_2, v_3, v_4\}$  induces an  $F_1$ , a contradiction. Now,  $v_1$  or  $v_2$  is such a vertex  $v_0$  of the conclusion of the claim.

*Subcase 2.2.  $v_3 \approx v_4$ .*

*Subcase 2.2.1.  $v_3 \sim u_0$*  (similar for the case  $v_4 \sim u_0$ ). Then  $v_3 \sim v_2$ , otherwise  $\{u_2, v_2, u_0, v_1, u_1, v_3\}$  induces an  $F_{14}$ , a contradiction. Now,  $v_1 \sim v_4$  if and only if  $v_4 \sim u_0$ , otherwise  $\{u_2, v_4, u_0, v_1, u_1, v_3\}$  induces an  $F_{14}$ , a contradiction. Hence, we have the two following subcases.

(i)  $v_4$  is adjacent to  $v_1, u_0$ . Then  $\{u_0, u_1, u_2, v_1, v_2, v_3, v_4\}$  induces an  $F_{28}$ , a contradiction.

(ii)  $v_4$  is not adjacent to  $u_0$  and  $v_1$ . Since  $(S \setminus \{u_0, u_1, u_2\}) \cup \{v_1, v_2, v_4\}$  is not independent, there exists some  $u_3 \in S \setminus \{u_0, u_1, u_2\}$  such that  $u_3$  is adjacent to at least one vertex among  $v_1, v_2, v_4$ . Hence  $u_3$  is adjacent to  $v_1$  or  $v_2$ , otherwise  $u_3 \sim v_4$  and  $\{u_3, v_4, u_2, v_2, u_0, v_1, u_1\}$  induces an  $F_1$ , a contradiction. Now,  $v_1$  or  $v_2$  is such a vertex  $v_0$  of the conclusion of the lemma.

*Subcase 2.2.2.  $u_0$  is not adjacent to  $v_3, v_4$ .*

(i)  $v_3 \sim v_2$  (similar for the case  $v_4 \sim v_1$ ). Then  $v_4 \approx v_1$ , otherwise  $\{u_0, u_1, u_2, v_1, v_2, v_3, v_4\}$  induces an  $F_{29}$ , a contradiction. Since  $(S \setminus \{u_0, u_1, u_2\}) \cup \{v_1, v_2, v_4\}$  is not independent, there exists some  $u_3 \in S \setminus \{u_0, u_1, u_2\}$  such that  $u_3$  is adjacent to at least one vertex among  $v_1, v_2, v_4$ . Hence  $u_3$  is adjacent to  $v_1$  or  $v_2$ , otherwise  $u_3 \sim v_4$  and  $\{u_3, v_4, u_2, v_2, u_0, v_1, u_1\}$  induces an  $F_1$ , a contradiction. Now,  $v_1$  or  $v_2$  is such a vertex  $v_0$  of the conclusion of the claim.

(ii)  $v_3 \approx v_2$ , i.e.,  $v_3$  is not adjacent to  $u_0, v_2$ , and  $v_4$  and similarly  $v_4$  is not adjacent to  $v_1$  and  $u_0$ . Now,  $\{v_3, u_1, v_1, u_0, v_2, u_2, v_4\}$  induces an  $F_1$ , a contradiction. □

**Claim 13.** *Let  $v_0 \in N(u_0)$  and  $u_1, u_2 \in N_S(v_0) \setminus \{u_0\}$  be two non-adjacent vertices. Then there exist some  $v_1, v_2$  such that  $\{u_0, u_1, u_2, v_0, v_1, v_2\}$  induces a  $K_{3,3}$ .*

**Proof.** Since  $\{v_0, u_0, u_1, u_2\}$  induces a  $K_{1,3}$  and  $v_0$  is not  $\alpha$ -redundant, by Lemma 2 (for the case  $m = 3$ ), there exist some  $v_1, v_2, v_3 \in V(G)$  such that  $\{v_0, v_1, v_2, v_3\}$  is independent and  $u_i \sim v_{i+1}$  for  $i = 0, 1, 2$ . Let  $X = \{u_0, u_1, u_2\}$ . By the symmetry, we consider the following cases.

*Case 1.*  $|N_X(v_i)| = 3$  for at least two positive integers  $i$ . Then  $\{u_0, u_1, u_2, v_0, v_1, v_2, v_3\}$  induces a  $K_{3,3}$ .

*Case 2.*  $|N_X(v_2)| = 2$  and  $N_X(v_2) = \{u_1, u_2\}$ . Then  $\{u_0, u_1, u_2, v_0, v_1, v_2\}$  induces an  $F_{14}$  in the case  $|N_X(v_1)| = 1$ , an  $F_{15}$  in the case  $|N_X(v_1)| = 2$  or an  $F_{20}$  in the case  $|N_X(v_1)| = 3$ , a contradiction.

*Case 3.*  $|N_X(v_1)| = |N_X(v_2)| = 1$  and  $|N_X(v_3)| = 3$ . Then  $\{u_0, u_1, u_2, v_0, v_1, v_2, v_3\}$  induces an  $F_{30}$ , a contradiction.

*Case 4.*  $|N_X(v_i)| = 1$  for  $i = 1, 2, 3$ . Let  $H$  be a maximal (inclusion sense) graph consisting of  $k$  induced paths of length two of the form  $v_0u_iv_{i+1}$  where  $v_0$  is the common initial vertex. Since  $(S \setminus \{u_0, u_i\}) \cup \{v_0, v_{i+1}\}$  is not independent for every  $i$ , for each  $i$  ( $2 \leq i \leq k$ ), there exists some  $w_i \in S \setminus \{u_0, u_{i-1}\}$  such that  $w_i$  is adjacent to  $v_0$  or  $v_i$ . The rest of the proof is processed by considering the following subcases.

*Subcase 4.1.* There exists some  $i$  such that  $w_i \sim v_i$ , without loss of generality, assume that  $w_2 \sim v_2$ . By Lemma 3, there exists some  $u \in N_S(v_1) \setminus \{u_0\}$ . If  $u = w_2$ , then  $\{u_0, u_1, u_2, w_2, v_0, v_1, v_2\}$  induces an  $F_{15}$  or an  $F_7$  depending on  $w_2 \sim v_0$  or not, a contradiction. If  $u \neq w_2$ , then  $\{u, v_1, u_0, v_0, u_1, v_2, w_2\}$  induces an  $F_1$ , an  $F_7$ , an  $F_{14}$ , or an  $F_{15}$  depending on the adjacency between  $\{u, w_2\}$  and  $\{v_0, v_1, v_2\}$ , a contradiction.

*Subcase 4.2.* There exists some  $w \in S$  such that  $w$  is not adjacent to any  $v_i$  for  $i \geq 2$  and  $w \sim v_0$ . Then  $w \approx v_1$ , otherwise  $\{v_2, u_1, v_0, u_0, v_1, w\}$  induces an  $F_{15}$ , a contradiction. Since  $v_0$  is not  $\alpha$ -redundant, by Lemma 2 (for the case  $m = 1$ ), there exists some  $t \in N(w) \setminus N[v_0]$  and by the maximality of  $H$ ,  $t$  is adjacent to some vertex  $u_i$  or  $v_i$ . Moreover, by Lemma 3, there exists some  $u \in S \setminus V(H)$  such that  $v_1 \sim u$ .

*Subcase 4.2.1.*  $t$  is not adjacent to any  $v_i$ . Then  $t \sim u_i$  for some  $u_i$  and  $t$  is adjacent to all others  $u_j$ , otherwise  $\{v_{j+1}, u_j, v_0, u_i, t, w\}$  induces an  $F_{14}$ , a contradiction. Thus,  $\{u, u_0, w, v_0, v_1, t\}$  induces an  $F_{15}$  or an  $F_{14}$  depending on  $t \sim u$  or not, a contradiction.

*Subcase 4.2.2.*  $t \sim v_i$  for some  $i \geq 1$ .

(i)  $t \approx v_1$ . Then, without loss of generality, assume that  $t \sim v_2$ . Hence,  $t$  is not adjacent to  $u_0, u_1$ , and  $u$ , otherwise  $\{u_0, u_1, u, w, v_0, v_1, t\}$  induces an  $F_7$ , an  $F_{14}$ , or an  $F_{15}$ , a contradiction. Now,  $\{t, v_2, u_1, v_0, u_0, v_1, u\}$  induces an  $F_1$ , a contradiction.

(ii)  $t \sim v_1$ . We consider the three following subcases.

(a)  $t$  is adjacent to  $u_i$  for some  $i \geq 1$ . Then  $t \sim u$ , otherwise  $\{u, v_1, t, w, v_0, u_i\}$  induces an  $F_{14}$ , a contradiction. Now,  $\{t, u, v_0, v_1, u_0, u_i\}$  induces an  $F_{18}$  or an  $F_5$  depending on  $t \sim u_0$  or not, a contradiction.

(b)  $t \approx u_i$  for  $i \geq 1$  and  $t \sim u_0$ . Then  $t \sim v_{i+1}$  for every  $i \geq 1$ , otherwise  $\{v_{i+1}, u_i, v_0, u_0, t, w\}$  induces an  $F_{14}$ , a contradiction. Now,  $\{v_1, t, v_0, v_2, v_3, u_1, u_2\}$  induces an  $F_7$ , a contradiction.

(c)  $t \approx u_i$  for every  $i$ . Then  $t \approx u$ , otherwise  $\{t, w, v_0, u_0, v_1, u\}$  induces an  $F_5$ , a contradiction. Now,  $\{u, v_1, t, w, v_0, u_1, v_2\}$  induces an  $F_1$ , a contradiction.

□

**Claim 14.** *There exist no  $u_1, u_2 \in S$  and  $v_1, v_2, v_3$  such that  $\{u_0, u_1, u_2, v_1, v_2, v_3\}$  induces a  $K_{3,3}$ .*

**Proof.** Indeed, suppose there exist  $u_1, u_2 \in S$  and  $v_1, v_2, v_3$  such that  $\{u_0, u_1, u_2, v_1, v_2, v_3\}$  induces a  $K_{3,3}$ . Let  $H$  be a maximal (inclusion sense) induced complete bipartite subgraph of  $G$  with parts  $A$  and  $B$  such that  $A = \{u_0, u_1, \dots, u_p\} \subseteq S$  and  $B = \{v_1, v_2, \dots, v_q\} \subseteq T$  ( $p \geq 2$  and  $q \geq 3$ ).

*Case 1.*  $p < q$ . The set  $S' = (S \setminus A) \cup B$  is not an independent set of  $G$ , otherwise we have a maximum independent set not containing  $u$ , a contradiction. Hence, there exists some  $u \in S \setminus A$  such that  $u \sim v_i$  for some  $v_i \in B$ , say  $u \sim v_1$ . Moreover, the maximality of  $H$  implies that  $u$  is not adjacent to some vertex of  $B$ . Without loss of generality, assume that  $u \not\sim v_2$ . Then  $u$  is not adjacent to any vertex  $v_i \in B \setminus \{v_1\}$ , otherwise  $\{u, u_0, u_1, v_1, v_2, v_i\}$  induces an  $F_{20}$ , a contradiction. Now,  $\{u, u_0, u_1, u_2, v_1, v_2, v_3\}$  induces an  $F_{24}$ , a contradiction.

*Case 2.*  $3 \leq q \leq p$ . Since  $d(u_0) \geq d(v_2)$ , there exists  $v \in T \setminus V(H)$  such that  $v \sim u_0$  and  $v \not\sim v_2$ .

*Subcase 2.1.*  $N_A(v) = A$ . By the maximality of  $H$ ,  $v \sim v_i$  for some  $v_i \in B$ . Without loss of generality, assume that  $v \sim v_1$ . Now,  $\{u_0, u_1, u_2, u_3, v_1, v_2, v\}$  induces an  $F_8$ , a contradiction.

*Subcase 2.2.*  $v \not\sim u_i$  for some  $u_i$ , without loss of generality, assume that  $v \not\sim u_1$ . Then  $v$  is not adjacent to any vertex  $u_i, i \geq 2$ , otherwise  $\{u_0, u_1, u_i, v, v_1, v_2\}$  induces an  $F_{21}$  or an  $F_{20}$  depending on  $v \sim v_1$  or not, a contradiction. Now, by Lemma 3, there exists some  $u \in N_S(v) \setminus A$ .

*Subcase 2.2.1.*  $u$  is adjacent to some vertex  $v_i$  of  $B$ , without loss of generality, assume that  $u \sim v_1$ . By the maximality of  $H$ ,  $u$  is not adjacent to some vertex  $v_i$  of  $B$  different from  $v_1$ , without loss of generality, assume that  $u \not\sim v_2$ . Thus,  $\{u, u_0, u_1, u_2, v_1, v_2, v_3\}$  induces an  $F_{20}$  or an  $F_{24}$  depending on  $u \sim v_3$  or not, a contradiction.

*Subcase 2.2.2.*  $u$  is not adjacent to any vertex  $v_i$  of  $B$ . So,  $v$  is adjacent to every vertex  $v_i$  of  $B \setminus \{v_2\}$ , otherwise  $\{u, v, u_0, v_2, u_1, v_i\}$  induces an  $F_{14}$ . Now,  $\{v, u_1, u_2, v_1, v_2, v_3\}$  induces an  $F_{20}$ , a contradiction. □

Together, all above claims give us the theorem. ■

#### 4. ALGORITHM MAX AND $K_{1,m}$ REDUCTION

In the literature, there are some results about polynomial time solution for the MIS problem in subclasses of  $K_{1,m}$ -free graphs, for example  $(P_k, K_{1,m})$ -free graphs [22],  $(S_{1,2,j}, \text{banner}, K_{1,m})$ -free graphs,  $(S_{1,2,3}, B_k, K_{1,m})$ -free graphs [19] and

$(S_{2,2,2}, B_2(F_{14}), \text{domino}(F_{15}), M_3(F_{30}))$ -free graphs [20]. So one possible heuristic approach for the MIS problem is to remove all maximum degree vertices which are the centers of some  $K_{1,m}$  and then apply one polynomial solution for some subclass of  $K_{1,m}$ -free graphs. In this section, we describe the method of combining Algorithm MAX and  $K_{1,m}$  reduction. Recall that in [20] we have shown that the MIS problem is solvable in time  $O(n^{m+2})$  for  $(S_{2,2,2}, F_{14}, F_{15}, F_{30}, K_{1,m})$ -free graphs. In this section, we fix  $m \geq 3$ . Now, we combine Algorithm MAX and this result to obtain the Algorithm MMAX- $l$  (see Algorithm 3).

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**Algorithm 3** Algorithm MMAX- $l$  (Combination of MAX and  $K_{1,l}$  reduction).

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**Input:**  $G = (V, E)$ , an  $(S_{2,2,2}, F_{14}, F_{15}, F_{30})$ -free graph.

**Output:**  $S$ , an independent set of  $G$ .

- 1:  $H_n := G; i := n; S := \emptyset;$
  - 2: **while**  $H_i$  contains a  $K_{1,l}$  **do**
  - 3:   Choose a vertex  $u \in V(H_i)$  such that  $u$  is the center-vertex of some  $K_{1,l}$  and  $u$  is of maximum degree among center vertices of all induced copies of  $K_{1,l}$  in  $H_i$ ;
  - 4:    $i := i - 1; H_i := H_{i+1} - u;$
  - 5: **end while**
  - 6: Let  $S$  be the maximum independent set of  $H_i$  obtained by the technique for  $(S_{2,2,2}, F_{14}, F_{15}, F_{30}, K_{1,l})$ -free graphs described in [20];
  - 7: **return**  $S$
- 

To find and remove center vertices of all  $K_{1,l}$ 's can be done in time  $O(n^{l+2})$  and Step 6 can be performed in time  $O(n^{l+2})$  [20]. Hence, Algorithm MMAX- $l$  can be performed in time  $O(n^{l+2})$  and gives us an independent set. Given a graph  $G$ , we denote by  $k_{\text{MMAX-}l}(G)$  the minimum cardinality of an independent set given by Algorithm MMAX- $l$ . The following theorem describes a set of forbidden induced subgraphs for this algorithm.

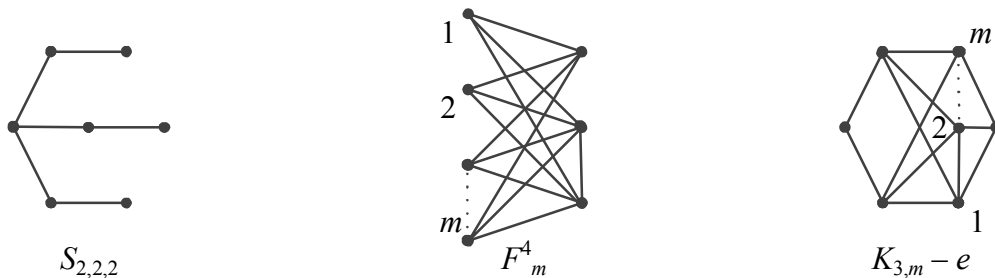


Figure 4. Some forbidden subgraphs for Algorithm MMAX- $l$  ( $l \geq 2m - 2$ ).

**Theorem 15** (Forbidden subgraphs for Algorithm MMAX- $l$ ). *Let  $G$  be an  $\mathcal{F}_6$ -free graph of order  $n \geq 7$ , where  $\mathcal{F}_6 = \{S_{2,2,2}, F_{14}, F_{15}, F_{30}, K_{3,m} - e, F_4^m\}$ . Then*

$k_{MMAX-l}(G) = \alpha(G)$  for  $l \geq 2m - 2$  and  $m \geq 4$ .

**Proof.** We will follow the proof of Theorem 6. Assume that the statement in the theorem is not correct. That means there exists a (without loss of generality) connected graph  $G$  satisfying the assumption of the theorem and some vertex  $u \in V(G)$  such that

1.  $u$  is the center-vertex of some  $K_{1,l} \{u, v_1, v_2, \dots, v_l\}$ ,
2.  $u$  is of maximum degree among such center vertices of  $K_{1,l}$ 's in  $G$ , and
3.  $u$  belongs to every maximum independent set of  $G$ ,

i.e., the MMAX- $l$  algorithm will fail.

Let  $S$  be a maximum independent set of  $G$  (hence,  $u \in S$ ) and  $T = V(G) \setminus S$ . Let  $W = \{v_1, \dots, v_l, \dots, v_p\}$  be the maximal (inclusion sense) independent set such that  $W \subset N(u) \subset T$ . By Lemma 2, there exist  $u_1, u_2, \dots, u_p \in S$  such that there exists a perfect matching between  $\{u_i\}$  and  $\{v_i\}$ . Without loss of generality, assume that  $u_i \sim v_i$  for  $1 \leq i \leq p$ . Let  $B = \{u, u_1, \dots, u_l\}$ . Then  $H = (W, B, E)$  is a bipartite graph.

**Claim 16.** *There exists some  $u_i \in B \setminus \{u\}$  such that  $u_i$  is adjacent to every vertex of  $W$ .*

**Proof.** Since  $H$  does not induce an  $S_{2,2,2}$ , there exist some  $u_i, v_j$  such that  $i \neq j$  and  $u_i \sim v_j$ . Let  $u_i$  be a vertex such that  $|N_W(u_i)|$  is maximum. Assume that  $u_i \not\sim v_k$  for some  $k$ . If  $u_k$  is not adjacent to any vertex of  $N_W(u_i)$ , then  $\{u_i, v_i, v_j, u, v_k, u_k\}$  induces an  $F_{14}$  for some  $v_j \in N_W(u_i) \setminus \{v_i\}$ , a contradiction. Now, assume that  $u_k \sim v_j$  for some  $v_j \in N_W(u_i)$ . By the choice of  $u_i$ ,  $u_k \not\sim v_l$  for some  $v_l \in N_W(u_i)$ . Now,  $\{u, v_l, u_i, v_k, u_k, v_j\}$  induces an  $F_{15}$ , a contradiction. Hence,  $u_i$  is adjacent to every vertex of  $W$ . □

Without loss of generality, assume that  $u_p$  is adjacent to every vertex of  $W$ .

**Claim 17.**  *$H$  is  $2K_2$ -free.*

**Proof.** Assume that  $H$  contains an induced  $2K_2$   $u_i v_j + u_k v_r$ , for some  $u_i, u_k \in B \setminus \{u\}$  and  $v_j, v_r \in W$ . Note that  $i, k \neq p$ . Now,  $W \cup \{u, u_p, u_i, u_k, v_j, v_r, v_p\}$  induces an  $F_{30}$  in the case  $u_i, u_k$  are not adjacent to  $v_p$ . If  $u_i \sim v_p$  (similar for the case  $u_k \sim v_p$ ), then  $\{u, u_i, u_k, v_j, v_l, v_p\}$  induces an  $F_{15}$  or an  $F_{14}$  depending on  $u_k \sim v_p$  or not, a contradiction. □

Also by an observation in [13], since  $H$  is a  $2K_2$ -free connected bipartite graph, we have the following observation.

**Claim 18** [13]. *The vertices of each part of  $H$  can be linearly ordered under inclusion of their neighborhood.*

This leads us to the following observation.

**Claim 19.**  *$H$  induces a  $K_{p,p}$ .*

**Proof.** By Claim 18, without loss of generality, assume that  $N_W(u_i) \subset N_W(u_j)$  for  $i < j$ , i.e.,  $u_j \sim v_i$  for every  $i, j$  such that  $j \geq i$ . Moreover, by the existence of a perfect matching between  $B \setminus \{u\}$  and  $W$ , we have that  $|N_W(u_i)| \geq i$  for  $i = 1, 2, \dots$

If  $u_i \approx v_j$ , for some  $i, j$  such that  $j > i \geq m - 1$ , then  $\{u, u_{2m-2}, u_i, v_j, v_1, v_2, \dots, v_{m-1}\}$  induces a  $K_{3,m} - e$ , a contradiction. Hence,  $N_W(u_i) = W$  for  $i \geq m - 1$ .

If  $u_i \approx v_j$  for some  $i, j$  such that  $j > i$  and  $m - 2 \geq i \geq 2$ , then  $\{v_1, v_i, v_j, u_i, u_{m-1}, u_m, \dots, u_{2m-3}\}$  induces a  $K_{3,m} - e$ , a contradiction. Thus,  $N_W(u_i) = W$  for  $i \geq 2$  and hence,  $\{u, u_2, \dots, u_p\} \cup W$  induces a  $K_{p,p}$ .  $\square$

Now, without loss of generality, let  $H$  be a maximal (inclusion sense) induced complete bipartite subgraph of  $G$  with parts  $A$  and  $W$  such that  $\{u, u_2, \dots, u_p\} \subseteq A \subseteq S$ . Let  $A = \{u, u_2, \dots, u_p, \dots, u_q\}$ .

**Claim 20.** *There exists some vertex  $v \in T \setminus W$  such that  $v \sim u$  and  $v \approx v_i$  for some  $v_i \in W$ .*

**Proof.** If  $p = |W| < |A|$ , then since  $d(u) \geq d(v_1)$  and  $v_1$  is also a center-vertex of some  $K_{1,l}$  (say  $v_1(u, u_1, \dots, u_{l-1})$ ), there exists  $v \in T \setminus V(H)$  such that  $v \sim u$  and  $v \approx v_1$ .

Consider the case  $p = |W| = |A|$ . Then the set  $S' = (S \setminus A) \cup W$  cannot be an independent set of  $G$ , otherwise,  $S'$  is a maximum independent set not containing  $u$ . Hence, there exists some  $u' \in S \setminus A$  such that  $u' \sim v$  for some  $v \in W$ . Assume that  $u' \sim v_1$ . Now, again, since  $d(u) \geq d(v_1)$ , there exists  $v \in T \setminus V(H)$  such that  $v \sim u$  and  $v \approx v_1$ .  $\square$

Without loss of generality, assume that  $v \approx v_2$ . Moreover, by the maximality of  $W$ ,  $v \sim v_i$  for some  $v_i \in W$ . Without loss of generality, assume that  $v \sim v_1$ . Note that  $v$  has at most  $m - 1$  neighbors in  $A$ , otherwise  $m$  neighbors of  $v$  in  $A$ , together with  $v, v_1, v_2$  induce an  $F_m^4$ , a contradiction. Moreover,  $v$  has at most  $m - 1$  neighbors in  $W$ , otherwise  $m - 1$  neighbors of  $v$  in  $W$ , together with  $v, u_i, u_j$ , and  $v_2$  induces a  $K_{3,m} - e$  for  $u_i, u_j$  are two non-neighbors of  $v$  in  $A$ . We consider the two following cases.

*Case 1.*  $v$  is not adjacent to any vertex of  $A \setminus \{u\}$ , then there exists some  $u' \in S \setminus A$  such that  $u' \sim v$ , otherwise  $(S \setminus u) \cup \{v\}$  is a maximum independent set not containing  $u$ , a contradiction. By the maximality of  $H$ , there exists some vertex  $v_i \in W$  such that  $u' \approx v_i$ . If  $u'$  is adjacent to two vertices  $v_j, v_k \in W$ , then



$\{u, u', u_2, \dots, u_{m-1}, v_i, v_j, v_k\}$  induces a  $K_{3,m} - e$ , a contradiction. Hence,  $u'$  is adjacent to at most one vertex  $v'$  of  $W$ . Now,  $\{v_j, u_2, v_k, u, v, u'\}$  induces an  $F_{14}$ , for  $v_j, v_k$  are two non-neighbors of both  $v$  and  $u'$ , a contradiction.

*Case 2.*  $v$  is adjacent to some vertex of  $A \setminus \{u\}$ , without loss of generality, assume that  $v \sim u_2$ . Then  $m - 1$  non-neighbors of  $v$  in  $W$ , together with  $v, u, u_2$ , and  $u_i$  induces a  $K_{3,m} - e$ , for some non-neighbor  $u_i$  of  $v$  in  $A$ , a contradiction. ■

### 5. COMPARISON

First, we have the following simple observations.

- Claim 21.** (1)  $F_7, F_{29}$ , and  $F_{30}$  induce  $F_2$  and  $F_{25}$ .  
 (2)  $F_{18}$  induces  $F_3$  and  $F_{25}$ .  
 (3)  $F_{14}, F_{15}, F_{19}, F_{20}, F_{21}, F_{24}$ , and  $F_{30}$  induce  $F_3$ .  
 (4)  $F_8$  induces  $F_4$ .  
 (5)  $F_{39}, S_{2,2,2}, F_{14}$ , and  $F_{15}$  induce  $F_{25}$ .  
 (6)  $F_4^m$  induces  $F_8$ .  
 (7)  $K_{3,m} - e$  induces  $F_{20}$ .

Hence, we have the following results.

- Proposition 22.** (1) *The class of  $\mathcal{F}_1$ -free graphs and the class of  $\mathcal{F}_4$ -free graphs are subclasses of the  $\mathcal{F}_5$ -free graph class.*  
 (2) *The class of  $\mathcal{F}_4$ -free graphs is a subclass of the  $\mathcal{F}_6$ -free graph class.*

Proposition 22 shows that Algorithm MMAX and Algorithm MMAX- $l$  are at least as good as VO and MAX in the sense of forbidden subgraph classes. Now, we compare the heuristic algorithms MAX, MMAX, MIN, VO in the aspect of performance for particular graphs. For the MIN algorithm and the VO algorithm, the full instances can be found in [31] and [27], respectively. For the sake of proofs, we will only recall the main ideas of the algorithms.

For each step, the MIN algorithm chooses a vertex of minimum degree in the remaining graph, puts it in the current (maximal) independent set and deletes the vertex and all its neighbours. The process will be repeated until there is no more remaining vertex.

The VO algorithm first makes a list of vertices of the graph, in which the vertices are ordered in an increasing order of degree. The algorithm sequentially puts vertices into the independent set in the list only if they are not adjacent with any (already) chosen vertex in the current independent set. The process will be repeated until we process through all the list.

We denote by  $k_{VO}(G), k_{MIN}(G)$  the minimum cardinalities of independent sets obtained when applying VO, MIN on  $G$ , respectively.

**Theorem 23.** *For every integer  $p > 2$  there is a graph  $G$  such that*

$$\begin{aligned} k_{MAX}(G) - k_{MIN}(G) &= k_{MAX}(G) - k_{VO}(G) = k_{MMAX}(G) - k_{MIN} = \\ &= k_{MMAX} - k_{VO} = p - 2. \end{aligned}$$

**Proof.** Let  $G_1$  be an induced copy of  $K_p$ , a complete graph of order  $p$ ,  $G_2$  be the graph consisting of  $p$  independent vertices, and  $G_3$  be the single vertex graph. Let  $G$  be the graph obtained by joining an edge for all pairs of vertices  $x, y$  for  $x \in V(G_i), y \in V(G_{i+1})$ , for  $i = 1, 2$ . Then, clearly,  $k_{MIN}(G) = k_{VO}(G) = 2$ , while  $k_{MMAX}(G) = k_{MAX}(G) = p$ . ■

The following result shows that the difference between  $k_{MMAX}$  on one hand, and  $k_{MAX}, k_{VO}$  and  $k_{MIN}$  on the other hand, can be arbitrarily large.

**Theorem 24.** *For every integer  $p > 2$ , there is a graph  $G$  such that*

$$k_{MMAX}(G) - k_{MAX}(G) = k_{MMAX}(G) - k_{VO}(G) = k_{MMAX}(G) - k_{MIN}(G) = p - 2.$$

**Proof.** Let  $G_1, G_3$  be induced copies of  $K_p$  and  $G_2$  be the graph consisting of  $p$  independent vertices. Let  $G$  be the graph obtained by joining an edge for all pairs of vertices  $x, y$  for  $x \in V(G_i), y \in V(G_{i+1})$ , for  $i = 1, 2$ . Then, clearly,  $k_{MAX}(G) = k_{MIN}(G) = k_{VO}(G) = 2$ , while  $k_{MMAX}(G) = p$ . ■

**Remark.** Note that the graph  $G$  constructed in the proof of Theorem 24 is  $K_{p+1}$  free. Hence, Algorithm MMAX- $l$  always finds a maximum independent set (of cardinality  $p$ ) for  $l \geq p + 1$ .

## 6. CONCLUSION

Recently, the MIS problem in  $P_5$ -free graphs has been the subject of many researchs. In [5], it is proved that the problem can be solved for  $(P_5, K_{3,3} - e (F_{20}), \text{twin-house } (F_{21}))$ -free graphs in time  $O(n^6)$ . In [25], the authors described a method solving the problem for  $(P_5, K_{3,3} - e)$ -free graph. In [21], the first approach for solving the problem for  $P_5$ -free graphs in time  $O(n^{12})$  is presented. The three results are generalizations of the forbidden subgraphs for Algorithm MAX. On the other hand, it is obvious that Algorithm MAX has the complexity at most  $O(n^2)$ .

Moreover, there are not many results about polynomial time solution for the MIS problem in some subclasses of  $P_7$ -free graphs, except for  $(P_7, \text{banner}(F_3))$ -free graphs [3] and  $(P_7, K_{1,m})$ -free graphs [22]. We have a similar situation for

$S_{2,2,2}$ -free graphs, except  $(S_{2,2,2}, \text{banner})$ -free graphs [12],  $(S_{2,2,2}, B_2, \text{domino}, M_3, K_{m,m})$ -free graphs [20], and  $S_{2,2,2}$ -free subcubic graphs [26].

The Algorithm MMAX, as well as MIN and VO, show that the problem can be solved efficiently in some subclasses of class of  $P_7$ -free graphs. Our method is a combination between a heuristic method (Algorithm MAX) and a conditional exact method ( $\alpha$ -redundant technique). The class of forbidden induced subgraphs of the new algorithm, Algorithm MMAX, is a super class of both MAX and VO. The class of forbidden induced subgraphs of Algorithm MMAX- $l$  also gives us a polynomial solution for the MIS problem in a subclass of  $S_{2,2,2}$ -free graphs. Note that the complexity of the problem for the class of  $P_7$ -free graphs or the class of  $S_{2,2,2}$ -free graphs is still an open question.

### Acknowledgement

We would like to express our special thanks to an anonymous reviewer for his/her very useful suggestions and comments.

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Received 31 January 2014

Revised 7 January 2015

Accepted 7 January 2015