

## EIGENVALUE CONDITIONS FOR INDUCED SUBGRAPHS

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### Abstract

Necessary conditions for an undirected graph  $G$  to contain a graph  $H$  as induced subgraph involving the smallest ordinary or the largest normalized Laplacian eigenvalue of  $G$  are presented.

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### 1. INTRODUCTION

We consider two fixed finite, undirected, and simple graphs: Let  $G = (V, E)$  be a graph without isolated vertices, where  $V = \{1, \dots, n\}$  and  $E$  (with  $|E| = m$ ) denote the vertex set and the edge set of  $G$ , respectively. Let  $\delta \geq 1$  denote the minimum degree of  $G$ . Furthermore, let  $d_H = \frac{2e}{h}$  be the average degree of a graph  $H = (V(H), E(H))$ , where  $|V(H)| = h$  and  $|E(H)| = e$ .

The eigenvalues  $\lambda_1 \leq \dots \leq \lambda_n$  of the adjacency matrix  $A$  of  $G$  are the *ordinary eigenvalues* (or shortly the eigenvalues) of  $G$ . Note that  $-r \leq \lambda \leq \lambda_n = r$  for all eigenvalues  $\lambda$  of an  $r$ -regular graph  $G$ , and if  $G$  is connected, then  $\lambda_1 = -\lambda_n$  if and only if  $G$  is bipartite [4, 7].

Let  $D$  be the *degree matrix* of  $G$ , that is an  $(n \times n)$  diagonal matrix, where the degree  $d_i$  of vertex  $i \in V$  is the  $i$ -th entry at the main diagonal. Moreover, let  $0 = \eta_1 \leq \dots \leq \eta_n$  be the eigenvalues of the *Laplacian*  $L = D - A$  of  $G$  [1, 13]. If  $G$  is  $r$ -regular, then  $\eta$  is an eigenvalue of the Laplacian if and only if  $r - \eta$  is an eigenvalue of  $A$ .

For  $G$  without isolated vertices, the *normalized Laplacian* is the  $(n \times n)$  matrix  $\mathcal{L} = (l_{ij})$  with  $l_{ij} = 1$  if  $i = j$ ,  $l_{ij} = -\frac{1}{\sqrt{d_i d_j}}$  if  $ij \in E$ , and  $l_{ij} = 0$  otherwise. The eigenvalues  $0 = \sigma_1 \leq \dots \leq \sigma_n$  of  $\mathcal{L}$  are the *normalized Laplacian eigenvalues* of  $G$  [5, 6, 13]. It is known that  $1 < \sigma_n \leq 2$  and that  $G$  is bipartite if and only if  $\sigma_n = 2$  [10, 12, 13]. For an  $r$ -regular graph  $G$ ,  $\sigma$  is a normalized Laplacian eigenvalue if and only if  $r(1 - \sigma)$  is an eigenvalue of  $A$ .

For further notation and terminology we refer to [8].

In the present paper, we are interested in necessary conditions in terms of eigenvalues for the fact that  $G$  contains a copy of  $H$  as an induced subgraph. If all eigenvalues of  $G$  and all eigenvalues  $\phi_1 \leq \dots \leq \phi_h$  of the adjacency matrix  $A_H$  of  $H$  are taken into consideration, then Theorem 1 is a typical result of this kind.

**Theorem 1** (Cauchy's Inequalities, Interlacing Theorem [4, 7]). *If  $H$  is an induced subgraph of  $G$  with eigenvalues  $\phi_1 \leq \dots \leq \phi_h$ , then  $\lambda_i \leq \phi_i \leq \lambda_{n-h+i}$  for  $i = 1, \dots, h$ .*

In general, it is difficult to determine the spectra of large graphs  $G$  and  $H$ , however, the largest and the smallest eigenvalues of the matrices  $A$ ,  $L$ , and  $\mathcal{L}$  of a graph are well investigated ([1, 4, 5, 6]). Hence, we focus on simpler necessary conditions for  $H$  being an induced subgraph of  $G$  just involving smallest or largest eigenvalues. The inequalities (1) obtained from Theorem 1 are possible results of this type.

$$(1) \quad \lambda_1 \leq \phi_1 \quad \text{and} \quad \lambda_n \geq \phi_h.$$

If the largest Laplacian eigenvalue  $\eta_n$  of  $G$  and the degrees of the vertices of  $H$  in  $G$  are taken into account, then the assertion of Theorem 2 holds.

**Theorem 2** (Bollobás, Nikiforov [3]). *If  $H$  is an induced subgraph of  $G$ , then  $(\sum_{i \in V(H)} d_i - 2e) n \leq \eta_n h(n - h)$ .*

In general, it is not easy to determine the value  $\sum_{i \in V(H)} d_i$  exactly. If the degrees of  $G$  do not differ too much, then the inequality  $\sum_{i \in V(H)} d_i \geq \delta h$  is reasonable and it follows

**Corollary 3.** *If  $H$  is an induced subgraph of  $G$ , then  $\eta_n h \leq (d_H + \eta_n - \delta)n$ .*

Note that Corollary 3 only makes sense if  $\delta > d_H$ . If  $G$  is  $r$ -regular, then  $\delta = r$ ,  $\eta_n = r - \lambda_1$ , and  $\sum_{i \in V(H)} d_i = rh$ , hence, Theorem 2, Corollary 3, and the following Corollary 4, proved by Haemers already in [9], coincide in this case.

**Corollary 4** (Haemers [9]). *If  $H$  is an induced subgraph of the  $r$ -regular graph  $G$ , then  $(r - \lambda_1)h \leq (d_H - \lambda_1)n$ .*

The *identity matrix* is the  $(n \times n)$  square matrix with ones on the main diagonal and zeros elsewhere. It is denoted simply by  $I$  if the size is immaterial or can be trivially determined by the context. In the sequel,  $\underline{x}$  denotes a vector, where  $\underline{1} = (1, 1, \dots, 1)^T$  and  $\underline{0} = (0, 0, \dots, 0)^T$ , and we write  $\underline{x} \geq \underline{0}$  if  $x_i \geq 0$  for each entry  $x_i$  of  $\underline{x}$ .

Our first result is Theorem 5 concerning the case that  $G$  is regular and involving the smallest eigenvalue  $\lambda_1$  of  $G$ .

**Theorem 5.** *Let  $G$  be  $r$ -regular. If  $H$  is an induced subgraph of  $G$ , then  $(A_H - \lambda_1 I)\underline{x} = \underline{1}$  is solvable, and, for any solution  $\underline{x}$  of this equation,*

$$\frac{r - \lambda_1}{n} \leq \min \left\{ \underline{z}^T (A_H - \lambda_1 I) \underline{z} \mid \underline{z} \in R^{|V(H)|}, \underline{1}^T \underline{z} = 1 \right\} = \frac{1}{\underline{1}^T \underline{x}}.$$

Moreover, if  $\lambda_1 < \phi_1$ , then  $A_H - \lambda_1 I$  is regular and  $\underline{1}^T \underline{x}$  equals the sum of all entries of  $(A_H - \lambda_1 I)^{-1}$ .

If  $\underline{z} = (\frac{1}{h}, \dots, \frac{1}{h})^T \in R^h$ , then  $\underline{1}^T \underline{z} = 1$  and  $\underline{z}^T (A_H - \lambda_1 I) \underline{z} = \frac{2e - \lambda_1 h}{h^2}$ . Thus, Theorem 5 is an extension of Corollary 4. If in Theorem 5, additionally,  $H$  is assumed to be  $\rho$ -regular, then  $\underline{x} = \left(\frac{1}{\rho - \lambda_1}, \dots, \frac{1}{\rho - \lambda_1}\right)^T$  is a solution of  $(A_H - \lambda_1 I)\underline{x} = \underline{1}$ , thus,  $\frac{1}{\underline{1}^T \underline{x}} = \frac{(\rho - \lambda_1)}{h} = \frac{(d_H - \lambda_1)}{h}$ , hence, Corollary 4 and Theorem 5 coincide in this case.

Now consider the following example, where the assertion of Theorem 5 is stronger than that one of Corollary 4 and inequalities (1) only lead to trivial statements. We ask for a necessary condition that the  $r$ -regular graph  $G$  contains  $k \geq 1$  disjoint and independent copies of the path  $P_3$  on 3 vertices, that is,  $H$  consists of  $k$  components each of them is isomorphic to  $P_3$ . The eigenvalues of  $P_3$  are  $-\sqrt{2}, 0, \sqrt{2}$  ([4]), hence, with Theorem 1 we may assume  $\lambda_1 \leq -\sqrt{2} < -\frac{4}{3}$ . With  $h = 3k$  and  $d_H = \frac{4}{3}$ , Corollary 4 leads to  $k \leq \frac{4 - 3\lambda_1}{9(r - \lambda_1)}n$ .

If we consider the system  $(A_H - \lambda_1 I)\underline{x} = \underline{1}$ , then, by Theorem 5, it is solvable and it follows  $\underline{1}^T \underline{x} = k \underline{1}^T \underline{y}$ , where  $\underline{y}$  is a solution of  $(A_{P_3} - \lambda_1 I)\underline{y} = \underline{1}$ . It is easy to see that  $\underline{1}^T \underline{y} = \frac{4 + 3\lambda_1}{2 - \lambda_1^2}$ , thus, again by Theorem 5,  $k \leq \frac{2 - \lambda_1^2}{(4 + 3\lambda_1)(r - \lambda_1)}n$ , which is stronger than  $k \leq \frac{4 - 3\lambda_1}{9(r - \lambda_1)}n$ .

If, additionally,  $G$  is assumed to be bipartite, then  $\lambda_1 = -r$  and  $\lambda_n = r$ . The inequalities (1) just imply  $\sqrt{2} \leq r$  in this case.

Next we consider again the case that  $G$  is not necessarily regular and try to establish a result similar to Theorem 5. Therefore, let  $M(G, H)$  be the set of non-empty induced subgraphs  $H^*$  of  $H$  such that  $B\underline{y} = \underline{1}$  has a solution  $\underline{y} = (y_1, \dots, y_t)^T$  with  $y_s > 0$  for  $s = 1, \dots, t = |V(H^*)|$ , where  $A_{H^*}$  denotes the adjacency matrix of  $H^*$  and  $B = A_{H^*} + (\sigma_n - 1)\delta I$ . In this case  $\underline{y}$  is called a *positive solution* of  $B\underline{y} = \underline{1}$ . With  $H^* = K_1$  and  $y_1 = \frac{1}{(\sigma_n - 1)\delta} > 0$  (note that  $\sigma_n > 1$ ), it follows  $K_1 \in M(G, H) \neq \emptyset$ .

If  $H^* \in M(G, H)$  and  $\underline{y}_1$  and  $\underline{y}_2$  are positive solutions of  $B\underline{y} = \underline{1}$ , then, since  $B$  is symmetric,  $\underline{1}^T \underline{y}_1 = \underline{y}_2^T B \underline{y}_1 = \underline{y}_2^T \underline{1} = \underline{1}^T \underline{y}_2$ , hence, the value  $\underline{1}^T \underline{y}$  is independent on the choice of the positive solution  $\underline{y}$ . We define  $g(G, H^*) = \underline{1}^T \underline{y}$ , where  $\underline{y}$  is an arbitrary positive solution of  $B\underline{y} = \underline{1}$ .

If the induced subgraph  $H^*$  of  $H$  is  $\rho$ -regular, then it is easy to see that  $(A_{H^*} + (\sigma_n - 1)\delta I)\underline{y} = \underline{1}$  has a positive solution  $\underline{y} = \left(\frac{1}{\rho + (\sigma_n - 1)\delta}, \dots, \frac{1}{\rho + (\sigma_n - 1)\delta}\right)^T$ , hence,  $H^* \in M(G, H)$ .

If  $H_1^*$  and  $H_2^*$  are independent induced subgraphs of  $H$  and  $H_1^*, H_2^* \in M(G, H)$ , then the disjoint union  $H_1^* \cup H_2^*$  of  $H_1^*$  and  $H_2^*$  also belongs to  $M(G, H)$  and  $g(G, H_1^* \cup H_2^*) = g(G, H_1^*) + g(G, H_2^*)$ .

Eventually, let  $f(G, H) = \min_{H^* \in M(G, H)} \frac{1}{g(G, H^*)}$ . Our second result is Theorem 6 involving the largest normalized Laplacian eigenvalue  $\sigma_n$  of  $G$ .

**Theorem 6.** *If  $H$  is an induced subgraph of  $G$ , then*

$$\frac{\sigma_n \delta^2}{2m} \leq \min \left\{ \underline{z}^T (A_H + (\sigma_n - 1)\delta I) \underline{z} \mid \underline{z} \in \mathbb{R}^{|V(H)|}, \underline{1}^T \underline{z} = 1, \underline{z} \geq \underline{0} \right\} = f(G, H).$$

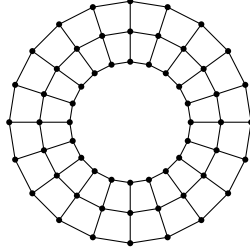
If  $G$  is  $r$ -regular, then the assertion of Theorem 6 is weaker than that one of Theorem 5 because  $\lambda_1 = r(1 - \sigma_n)$ ,  $\frac{2m}{\sigma_n \delta^2} = \frac{n}{r - \lambda_1}$ , and  $\min \left\{ \underline{z}^T (A_H - \lambda_1 I) \underline{z} \mid \underline{z} \in \mathbb{R}^{|V(H)|}, \underline{1}^T \underline{z} = 1 \right\} \leq \min \left\{ \underline{z}^T (A_H + (\sigma_n - 1)\delta I) \underline{z} \mid \underline{z} \in \mathbb{R}^{|V(H)|}, \underline{1}^T \underline{z} = 1, \underline{z} \geq \underline{0} \right\}$  in this case.

In general, it is not easy to calculate  $\min \left\{ \underline{z}^T (A_H + (\sigma_n - 1)\delta I) \underline{z} \mid \underline{1}^T \underline{z} = 1, \underline{z} \geq \underline{0} \right\}$ , however, in special cases it can be done efficiently.

Therefore, we consider an example, where the graph  $G$  is non-regular (i.e., Corollary 4 and Theorem 5 are not applicable),  $f(G, H)$  can be determined easily, and the necessary condition of Theorem 6 for the graph  $H$  to be an induced subgraph of  $G$  is stronger than that one of Theorem 2.

For positive integers  $p$  and  $q$ , where  $p$  is even, let  $G = C_p \square P_3$  be the Cartesian product<sup>1</sup> of the cycle  $C_p$  and the path  $P_3$  on 3 vertices (for  $p = 20$ ,  $G$  is shown in the figure) and let  $H$  consist of  $q$  copies of  $K_{1,4}$ .

<sup>1</sup>Given graphs  $G_1$  and  $G_2$  with vertex set  $V_1$  and  $V_2$ , respectively, their Cartesian product  $G_1 \square G_2$  is the graph with vertex set  $V_1 \times V_2$ , where  $(v_1, v_2)(w_1, w_2) \in E(G_1 \square G_2)$  when either  $v_1 = w_1$  and  $v_2 w_2 \in E(G_2)$  or  $v_2 = w_2$  and  $v_1 w_1 \in E(G_1)$ .



We have  $n = 3p$ ,  $m = 5p$ ,  $\delta = 3$ , and, since  $G$  is bipartite,  $\sigma_n = 2$ . The Laplacian eigenvalues of  $C_p$  and of  $P_3$  are  $2 - 2\cos(\frac{2\pi j}{p})$  for  $j = 0, \dots, p - 1$  and  $0, 1, 3$ , respectively ([4]). Moreover, if  $\eta'$  and  $\eta''$  are Laplacian eigenvalues of  $G'$  and  $G''$ , respectively, then  $\eta' + \eta''$  is a Laplacian eigenvalue of  $G' \square G''$  ([4]). Because  $p$  is even, it follows  $\eta_n = 2 - 2\cos(\pi) + 3 = 7$ .

It is easy to see that  $\sum_{i \in V(H)} d_i - 2e = 10q$  and, using  $h = 5q$ , Theorem 2 implies  $q \leq \frac{3}{7}p$  in this case.

If  $H^*$  is an induced subgraph of  $K_{1,4}$ , then  $H^* = K_{1,s}$  or  $H^* = \overline{K}_s$  (the edgeless graph on  $s$  vertices) for suitable  $s \in \{1, 2, 3, 4\}$ .

Let  $H^* = K_{1,s}$  and consider the system  $(A_{H^*} + (\sigma_n - 1)\delta I)\underline{y} = (A_{H^*} + 3I)\underline{y} = \underline{1}$ . It is easy to see that  $K_{1,4}, K_{1,3} \notin M(G, H)$ ,  $K_{1,2}, K_{1,1} \in M(G, H)$ ,  $g(G, K_{1,2}) = \frac{5}{7}$ , and  $g(G, K_{1,1}) = \frac{1}{2}$ .

If  $H^* = \overline{K}_s$ , then  $H^* \in M(G, H)$  and  $(A_{H^*} + 3I)\underline{y} = \underline{1}$  lead to  $g(G, H^*) = \frac{8}{3}$ , hence,  $f(G, H) = \frac{3}{4q}$ . By Theorem 6, it follows  $q \leq \frac{5}{12}p < \frac{3}{7}p$ .

If  $H^*$  with  $|V(H^*)| \geq 1$  is an arbitrary induced subgraph of  $H$  and  $\underline{z} = (z_1, \dots, z_h)^T$  with  $z_i = \frac{1}{|V(H^*)|}$  if  $i \in V(H^*)$  and  $z_i = 0$  otherwise, then  $\underline{1}^T \underline{z} = 1$  and  $\underline{z}^T (A_H + (\sigma_n - 1)\delta I)\underline{z} = \frac{d_{H^*} + (\sigma_n - 1)\delta}{|V(H^*)|}$ , where  $d_{H^*}$  denotes the average degree of  $H^*$ . Thus, Corollary 7 is a consequence of Theorem 6.

**Corollary 7.** *If  $H$  is an induced subgraph of  $G$ , then  $\frac{\sigma_n \delta^2}{2m} \leq \frac{d_{H^*} + (\sigma_n - 1)\delta}{|V(H^*)|}$ , where  $H^*$  is an arbitrary induced subgraph of  $H$  with  $|V(H^*)| \geq 1$ .*

Obviously, Corollary 7 is an extension of Corollary 4 if  $G$  is regular. We conclude with an example, where Corollary 3 is weaker than Corollary 7 for not necessarily regular  $G$ . Therefore, let  $V(H)$  be an independent set of  $G$ , i.e.  $d_H = 0$ . By Corollary 3 and Corollary 7, it follows that  $h \leq \frac{\eta_n - \delta}{\eta_n} n$  and  $h \leq \frac{2(\sigma_n - 1)}{\sigma_n \delta} m$  if  $G$  contains  $h$  independent vertices, respectively. In [11], it is shown that there are infinitely many graphs  $G$  such that  $\frac{2(\sigma_n - 1)}{\sigma_n \delta} m < \frac{\eta_n - \delta}{\eta_n} n$ .

2. PROOFS

In [11], the following Lemma 8 is proved. For completeness we give a proof here.

**Lemma 8.** *If  $x_1, \dots, x_n$  are real numbers, then*

$$(2) \quad \sigma_n \left( \sum_{i=1}^n d_i x_i \right)^2 - 2(\sigma_n - 1)m \sum_{i=1}^n d_i x_i^2 \leq 4m \sum_{ij \in E} x_i x_j.$$

**Proof.** It is easy to see that  $\sigma$  is an eigenvalue of  $\mathcal{L}$  if and only if  $\mu = 1 - \sigma$  fulfills  $\det(A - \mu D) = 0$ , see [10, 12, 14]. Let  $\mu_i = 1 - \sigma_{n-i+1}$  for  $i = 1, \dots, n$ .

Note that  $D$  is positive definite since  $\delta \geq 1$ . Define  $\underline{x}^T D \underline{y}$  as the inner product for vectors  $\underline{x}, \underline{y} \in R^n$  and let  $\underline{x}$  and  $\underline{y}$  be called *D-orthogonal* if  $\underline{x}^T D \underline{y} = 0$ . If  $\underline{x}^T D \underline{x} = 1$  then  $\underline{x}$  is called *D-normal*. A set of *D-normal* vectors being pairwise *D-orthogonal* is a *D-orthonormal set*.

We consider the generalized eigenvalue problem  $A \underline{x} = \mu D \underline{x}$  for  $\mu \in R$  and  $\underline{x} \in R^n$  with  $\underline{x} \neq \underline{0}$ . If the pair  $(\mu, \underline{x})$  is a solution of this equation, then  $\underline{x}$  is a *D-eigenvector of G* and  $\mu$  is the corresponding *D-eigenvalue of G*.

We use the well known fact (e.g. see [14]) that there is a *D-orthonormal basis* of  $R^n$  consisting of *D-eigenvectors of G*. Next we will show the following assertion. If  $\{u_1, \dots, u_n\}$  is a *D-orthonormal basis* of  $R^n$  such that  $u_i$  is a *D-eigenvector* with corresponding *D-eigenvalue*  $\mu_i$  for  $i = 1, \dots, n$ , then, for any vector  $\underline{x} \in R^n$ ,

$$(3) \quad (\mu_2 - \mu_1)(\underline{x}^T D \underline{u}_2)^2 + \dots + (\mu_n - \mu_1)(\underline{x}^T D \underline{u}_n)^2 + \mu_1 \underline{x}^T D \underline{x} = \underline{x}^T A \underline{x}.$$

To see this, let  $\underline{x}$  be given. There are real numbers  $a_1, \dots, a_n$  such that  $\underline{x} = a_1 u_1 + \dots + a_n u_n$ .

Then  $\underline{x}^T A \underline{x} = \mu_1 a_1^2 + \dots + \mu_n a_n^2$ ,  $\underline{x}^T D \underline{x} = a_1^2 + \dots + a_n^2$ , and  $\underline{x}^T D \underline{u}_i = a_i$  for  $i = 1, \dots, n$ . The desired equality (3) is equivalent to  $(\mu_2 - \mu_1)a_2^2 + \dots + (\mu_n - \mu_1)a_n^2 + \mu_1(a_1^2 + \dots + a_n^2) = \mu_1 a_1^2 + \dots + \mu_n a_n^2$ .

As a consequence,

$$(4) \quad (\mu_n - \mu_1)(\underline{x}^T D \underline{u}_n)^2 + \mu_1 \underline{x}^T D \underline{x} \leq \underline{x}^T A \underline{x}.$$

The vector  $\frac{1}{\sqrt{2m}} \underline{1}$  is a *D-normal D-eigenvector of G* with corresponding *D-eigenvalue*  $\mu_n = 1$ , thus, inequality (4) and  $\sigma_n = 1 - \mu_1$  imply the lemma. ■

**Proof of Theorem 5.** Inequality (2) and  $\lambda_1 = r(1 - \sigma_n)$ , if  $G$  is  $r$ -regular, imply the fact that if  $G$  is  $r$ -regular and  $x_1, \dots, x_n$  are real numbers, then

$$(5) \quad (r - \lambda_1) \left( \sum_{i=1}^n x_i \right)^2 + \lambda_1 n \sum_{i=1}^n x_i^2 \leq 2n \sum_{ij \in E} x_i x_j.$$

Let  $U$  be an induced subgraph of  $G$  isomorphic to  $H$  and  $\phi : V(H) \rightarrow V(U)$  be a graph isomorphism from  $H$  to  $U$ .

For real numbers  $z_1, \dots, z_h$  with  $\sum_{q=1}^h z_q = 1$ , let  $x_1, \dots, x_n$  be defined as follows: If  $i \in V(U)$ , then there is a suitable  $q \in \{1, \dots, h\}$  such that  $i = \phi(v_q)$ . Set  $x_i = z_q$  in this case. If  $i \in V \setminus V(U)$ , then let  $x_i = 0$ .

With  $\underline{z} = (z_1, \dots, z_h)^T$ , we obtain  $\sum_{i \in V} x_i = \sum_{q=1}^h z_q = 1$ ,  $\sum_{i \in V} x_i^2 = \sum_{q=1}^h z_q^2$ , and  $2 \sum_{ij \in E} x_i x_j = 2 \sum_{v_q v_{q'} \in E(H)} z_q z_{q'} = \underline{z}^T A_H \underline{z}$ .

Inequality (5) implies  $(r - \lambda_1) + \lambda_1 (\sum_{q=1}^h z_q^2) n \leq \underline{z}^T A_H \underline{z} n$ , hence, with  $B = (A_H - \lambda_1 I)$ ,  $1 \leq \frac{n}{(r-\lambda_1)} \min \underline{z}^T B \underline{z} = \frac{n}{(r-\lambda_1)} MIN$ , where the minimum is taken over all vectors  $\underline{z} = (z_1, \dots, z_h)^T$  with  $\sum_{q=1}^h z_q = 1$ .

Note that this minimum exists, because  $\lambda_1 \leq \phi_1$  follows from Theorem 1, hence, all eigenvalues  $\phi_1 - \lambda_1, \phi_2 - \lambda_1, \dots, \phi_h - \lambda_1$  of  $B$  are non-negative. It follows that  $B$  is positive semidefinite.

To investigate this value  $MIN$ , we consider the Lagrange function  $L(\underline{z}, \kappa) = \underline{z}^T B \underline{z} - 2\kappa (\sum_{q=1}^h z_q - 1)$  with Lagrange multiplier  $2\kappa$  and the necessary optimality conditions  $L_{z_q} = 0$  for  $q = 1, \dots, h$  (for more details an Lagrange Theory see [2]).

We obtain that the equations  $B \underline{z} = \kappa \underline{1}$  and  $\underline{1}^T \underline{z} = 1$  are simultaneously solvable.

Next we will show that  $\kappa$  is unique. If  $B \underline{z}_1 = \kappa_1 \underline{1}$ ,  $\underline{1}^T \underline{z}_1 = 1$ ,  $B \underline{z}_2 = \kappa_2 \underline{1}$ , and  $\underline{1}^T \underline{z}_2 = 1$ , then  $\kappa_1 = \kappa_1 \underline{1}^T \underline{z}_2 = \underline{z}_1^T B \underline{z}_2 = \kappa_2 \underline{z}_1^T \underline{1} = \kappa_2$ .

With  $1 \leq \frac{n}{(r-\lambda_1)} MIN$ , it follows  $MIN = \underline{z}^T B \underline{z} = \kappa > 0$ .

If  $\underline{x} = \frac{1}{\kappa} \underline{z}$ , then  $B \underline{x} = \underline{1}$  and  $\underline{1}^T \underline{x} = \frac{1}{\kappa}$ .

If  $\lambda_1 < \phi_1$ , then  $B$  is regular and  $1 = \underline{1}^T \underline{z} = \kappa \underline{1}^T B^{-1} \underline{1}$ , hence,  $\underline{1}^T \underline{x} = \underline{1}^T B^{-1} \underline{1}$ . ■

**Proof of Theorem 6.** The proof of Theorem 6 is similar to that one of Theorem 5.

Let  $x_i \geq 0$  for  $i = 1, \dots, n$  and, since  $\sigma_n > 1$ , inequality (2) implies

$$\sigma_n (\sum_{i=1}^n d_i x_i)^2 - \frac{2(\sigma_n - 1)m}{\delta} \sum_{i=1}^n (d_i x_i)^2 \leq \frac{4m}{\delta^2} \sum_{ij \in E} (d_i x_i)(d_j x_j).$$

Substituting  $w_i = d_i x_i$  for  $i = 1, \dots, n$ , it follows

$$(6) \quad \sigma_n \delta^2 - 2(\sigma_n - 1)m\delta \sum_{i=1}^n w_i^2 \leq 4m \sum_{ij \in E} w_i w_j,$$

for arbitrary  $w_i \geq 0$  for  $i = 1, \dots, n$  with  $\sum_{i=1}^n w_i = 1$ .

Again, let  $U$  be an induced subgraph of  $G$  isomorphic to  $H$  and  $\phi : V(H) \rightarrow V(U)$  be a graph isomorphism from  $H$  to  $U$ , and, for real numbers  $z_1, \dots, z_h \geq 0$  with  $\sum_{q=1}^h z_q = 1$ , let  $w_1, \dots, w_n$  be defined as follows: If  $i \in V(U)$ , then there is a suitable  $q \in \{1, \dots, h\}$  such that  $i = \phi(v_q)$ . Set  $w_i = z_q$  in this case. If  $i \in V \setminus V(U)$ , then let  $w_i = 0$ .

Inequality (6) implies  $\frac{\sigma_n \delta^2}{2m} \leq \min(\underline{z}^T A_H \underline{z} + (\sigma_n - 1)\delta \underline{z}^T \underline{z}) = MIN$ , where the minimum is taken over  $\mathcal{S}_h = \{\underline{z} = (z_1, \dots, z_h)^T \mid z_q \geq 0 \text{ for } q = 1, \dots, h, \sum_{q=1}^h z_q = 1\}$ . Note that this minimum exists because  $\underline{z}^T A_H \underline{z} + (\sigma_n - 1)\delta \underline{z}^T \underline{z}$  is a continuous function and  $\mathcal{S}_h$  is a compact set.

Let  $\underline{z} = (z_1, \dots, z_h)^T \in \mathcal{S}_h$  with  $\underline{z}^T A_H \underline{z} + (\sigma_n - 1)\delta \underline{z}^T \underline{z} = MIN$ . Furthermore, let  $H'$  be the induced subgraph of  $H$  with vertex set  $V(H') = \{q \in V(H) \mid z_q > 0\} \neq \emptyset$ .

If  $t = |V(H')| = 1$ , then  $H' = K_1 \in M(G, H)$  with  $V(H') = \{q\}$ ,  $z_q = 1$ , and  $MIN = (\sigma_n - 1)\delta > 0$ . Hence,  $\underline{y} = (\frac{1}{(\sigma_n - 1)\delta})$  is a positive solution of  $(A_{H'} + (\sigma_n - 1)\delta I)\underline{y} = \underline{1}$  and it follows  $g(G, H') = \underline{1}^T \underline{y} = \frac{1}{(\sigma_n - 1)\delta} = \frac{1}{MIN}$  and  $1 \leq \frac{2m}{\sigma_n \delta^2 g(G, H')}$ .

If  $t \geq 2$ , then  $0 < z_q < 1$  for all  $q \in V(H')$ . Thus,  $MIN = \min(\underline{u}^T A_{H'} \underline{u} + (\sigma_n - 1)\delta \underline{u}^T \underline{u})$ , where the minimum is taken over the relative interior  $rint(\mathcal{S}_t) = \{\underline{u} = (u_1, \dots, u_t)^T \mid u_s > 0 \text{ for } s = 1, \dots, t, \sum_{s=1}^t u_s = 1\}$  of  $\mathcal{S}_t$ , consequently, this minimum is a local minimum at the hyperplane  $\mathcal{H}_t = \{\underline{u} = (u_1, \dots, u_t)^T \mid \sum_{s=1}^t u_s = 1\}$ .

To investigate this value  $MIN$ , we consider the Lagrange function  $L(\underline{u}, \kappa) = \underline{u}^T A_{H'} \underline{u} + (\sigma_n - 1)\delta \underline{u}^T \underline{u} - 2\kappa(\sum_{s=1}^t u_s - 1)$  with Lagrange multiplier  $2\kappa$  and the necessary optimality conditions  $L_{u_s} = 0$  for  $s = 1, \dots, t$ .

With  $B = A_{H'} + (\sigma_n - 1)\delta I$ , we obtain that the system  $B\underline{u} = \kappa \underline{1}$ ,  $\underline{1}^T \underline{u} = 1$  has a positive solution  $\underline{u}$ .

Next we will show that  $\kappa$  is unique. If  $Bu_1 = \kappa_1 \underline{1}$ ,  $\underline{1}^T u_1 = 1$ ,  $Bu_2 = \kappa_2 \underline{1}$ , and  $\underline{1}^T u_2 = 1$ , then  $\kappa_1 = \kappa_1 \underline{1}^T u_2 = u_1^T B u_2 = \kappa_2 u_1^T \underline{1} = \kappa_2$ .

With  $1 \leq \frac{2m}{\sigma_n \delta^2} MIN$ , it follows  $MIN = \underline{u}^T B \underline{u} = \kappa > 0$ .

If  $\underline{y} = \frac{1}{\kappa} \underline{u}$ , then  $B\underline{y} = \underline{1}$  has a positive solution  $\underline{y}$ , consequently,  $H' \in M(G, H)$ . Moreover,  $g(G, H') = \underline{1}^T \underline{y} = \frac{1}{\kappa} = \frac{1}{MIN}$  and we obtain  $1 \leq \frac{2m}{\sigma_n \delta^2 g(G, H')}$ .

To see that  $f(G, H) = \frac{1}{g(G, H')}$ , assume there is  $H'' \in M(G, H)$  with  $g(G, H'') > g(G, H')$ . Then there exists  $\underline{x} \in rint(\mathcal{S}_t)$  with  $t = |V(H'')|$  such that  $\underline{x}^T A_{H''} \underline{x} + (\sigma_n - 1)\delta \underline{x}^T \underline{x} < MIN$ .

Let  $x_i = u_i$  if  $i \in V(H'')$  and  $x_i = 0$  for  $i \in V(H) \setminus V(H'')$ .

It follows  $\underline{x} = (x_1, \dots, x_h)^T \in \mathcal{S}_{|V(H)|}$  and  $\underline{x}^T A_H \underline{x} + (\sigma_n - 1)\delta \underline{x}^T \underline{x} < MIN$ , contradicting the definition of  $MIN$ . ■

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