Discussiones Mathematicae
Graph Theory 35 (2015) 215–227
doi:10.7151/dmgt.1793

ON •-LINE SIGNED GRAPHS $L_\bullet(S)$

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Abstract

A *signed graph* (or *sigraph* for short) is an ordered pair $S = (S^u, \sigma)$, where S^u is a graph, $G = (V, E)$, called the underlying graph of S and $\sigma : E \to \{+, -\}$ is a function from the edge set E of S^u into the set $\{+, -\}$. For a sigraph S its •*-line sigraph*, $L_\bullet(S)$ is the sigraph in which the edges of S are represented as vertices, two of these vertices are defined adjacent whenever the corresponding edges in S have a vertex in common, any such L-edge ee' has the sign given by the product of the signs of the edges incident with the vertex in $e \cap e'$. In this paper we establish a structural characterization of •-line sigraphs, extending a well known characterization of line graphs due to Harary. Further we study several standard properties of •-line sigraphs, such as the balanced •-line sigraphs, sign-compatible •-line sigraphs and $\mathcal{C}$-sign-compatible •-line sigraphs.

Keywords: sigraph, line graph, •-line sigraph, balance, sign-compatibility, $\mathcal{C}$-sign-compatibility.

2010 Mathematics Subject Classification: 05C22, 05C75.

1. Introduction

For standard terminology and notation in graph theory we refer the reader to Harary [8] and West [19], and Zaslavsky [21, 22] for sigraphs. Throughout the paper, we consider finite, undirected graphs with no loops or multiple edges.

A *signed graph* (or *sigraph* for short; see [7]) is an ordered pair $S = (S^u, \sigma)$, where S^u is a graph $G = (V, E)$, called the *underlying graph* of S and $\sigma : E \to \{+, -\}$ is a function from the edge set E of S^u into the set $\{+, -\}$, called the *signature* of S. The edges of S with positive and negative signs are called *positive edges* and *negative edges*, respectively. In a pictorial representation of a sigraph S, when S is small enough, its positive edges are shown as bold oriented line segments and negative edges as broken line segments. The *positive* (*negative*) *degree* of a vertex $v \in V(S)$ denoted by $d^+(v)(d^-(v))$ is the number of positive (negative) edges incident with the vertex v and $d(v) = d^+(v) + d^-(v)$. The edge degree $d_e(e_j)$ of an edge e_j in a sigraph S is the total number of edges adjacent to e_j in S. If the end vertices of the edge e_j are u and v, then *edge-degree* of e_j is defined as the number $d_e(e_j) = d(u) + d(v) - 2$. A vertex is called *pendent* if its degree is one.

A sigraph is *all-positive* (*all-negative*) if all its edges are positive (negative); further, it is said to be *homogeneous* if it is either all-positive or all-negative and *heterogeneous* otherwise.

A *marked sigraph* is an ordered pair $S_\mu = (S, \mu)$ where $S = (S^u, \sigma)$ is a sigraph and $\mu : V(S) \to \{+, -\}$ is a function from the vertex set $V(S)$ of S into the set $\{+, -\}$, called a *marking* of S. In particular, a sigraph $S = (S^u, \sigma)$ has a *canonical marking* or *$\mathcal{C}$-marking*, μ_σ, defined for each vertex $v \in V(S)$ by $\mu_\sigma(v) = \prod_{e \in E_v} \sigma(e)$.

The *line graph* $L(G)$ of a graph G is that graph whose vertex set can be put in one-to-one correspondence with the edge set of G, such that two L-vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. The edges of the line graph $L(G)$ are called L-edges. The line graphs were first studied by Whitney [20] and the first characterization of line graphs in terms of complete subgraphs was obtained by Krausz [11]. In the literature, we find that different authors gave different name to line graphs; particularly, line graphs are termed as *derivative* (see [14]), *interchange graph* ([13]), *adjoint* ([12]), *derived graph* ([4]) and *covering graph* (see [10]). Harary and Norman [9] finally fixed the terminology by calling it a 'line graph'. A forbidden subgraph characterization of line graphs was established by Beineke [5]. The following theorem is the well known characterization of a *line graph* given in most of the standard text-books on graph theory (e.g., see Harary [8], Ch. 8, p. 74), originally due to Beineke [4].

Theorem 1 [8]**.** *The following statements are equivalent:*
(a) $G = (V, E)$ *is a line graph.*

(b) *The edges of G can be partitioned into some of its complete subgraphs in such a way that no vertex lies in more than two of the subgraphs.*

(c) *G does not have $K_{1,3}$ as an induced subgraph, and if two odd triangles have a common edge then the subgraph induced by their vertices is K_4.*

(d) *None of the nine subgraphs shown in Figure* 1 *is an induced subgraph of G.*

A triangle is said to be *odd* if there is a vertex in the graph adjacent to an odd number of vertices of the triangle.

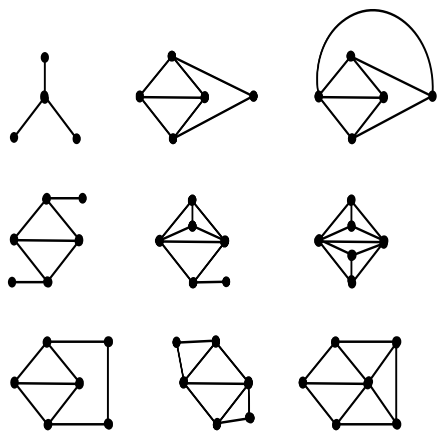

Figure 1. Beineke's nine forbidden subgraphs for a line graph.

The •-line sigraph $L_\bullet(S)$ of a sigraph S is the line graph of $S = (S^u, \sigma)$, with each L-edge ee' ($e, e' \in E(S)$) signed with $\mu_\sigma(e \cap e')$. There are two more notions of a '*signed line graph*' of a given sigraph $S = (S^u, \sigma)$ in the literature, viz., $L(S)$ and $L_\times(S)$, both have $L(S^u)$ as their underlying graph; only the rule to assign signs to the edges of $L(S^u)$ differ. An L-edge ee' in $L(S)$ is negative if and only if both the edges e and e' in S are negative [3] and an L-edge ee' in $L_\times(S)$ has the product $\sigma(e)\sigma(e')$ as its sign [6].

2. BALANCE IN •-LINE SIGRAPHS

The *sign* of a cycle Z in a sigraph S is the product of the signs of all its edges and is denoted by $\theta(Z)$. A cycle in a sigraph S is said to be *positive* (*negative*) if its sign is positive (negative). A sigraph S is said to be *balanced* if and only if all its cycles are positive.

Harary [7] derived the following structural criterion called *partition criterion* for balance in sigraphs.

Theorem 2 [7]**.** *A sigraph S is balanced if and only if its vertex set $V(S)$ can be partitioned into two subsets V_1 and V_2, one of them possibly empty, such that every positive edge joins two vertices in the same subset and every negative edge joins two vertices from different subsets.*

The following important lemma on balanced sigraphs is given by Zaslavsky.

Lemma 3 [23]**.** *A sigraph in which every chordless cycle is positive, is balanced.*

Now, the following theorem gives us the solution for $L_\bullet(S)$ to be balanced.

Theorem 4. *For a sigraph S, $L_\bullet(S)$ is balanced if and only if the following conditions hold:*

(i) *for every cycle Z in S, Z has even number of negatively marked vertices and*

(ii) *for $v \in V(S)$, if $d(v) > 2$, then $d^-(v) \equiv 0 \pmod 2$.*

Proof. *Necessity*: Suppose $L_\bullet(S)$ is balanced. Then, by definition of $L_\bullet(S)$, every L-cycle Z' in $L_\bullet(S)$ contains an even number of negative edges. Suppose there is any cycle in S that has odd number of negatively marked vertices. Then, by the definition of $L_\bullet(S)$, there are odd number of negative L-edges in $L_\bullet(S)$, a contradiction to the hypothesis. Thus, (i) follows. Now, by the definition of $L_\bullet(S)$, any three edges of S incident with v are the L-vertices of an all-negative L-triangle in $L_\bullet(S)$, contrary to the hypothesis. Thus condition (ii) is necessary. Hence, both conditions are necessary.

Sufficiency: Suppose conditions (i) and (ii) hold for a given sigraph S. We shall show that $L_\bullet(S)$ is balanced. If S is all-positive then, by definition, $L_\bullet(S)$ is also all-positive and hence, it is trivially balanced. Now, suppose that $L_\bullet(S)$ is not balanced. Then, we may assume the negative L-cycle Z'_k is of least possible length with this property. Let it be $(e_1, e_2, \ldots, e_k, e_1)$, where each e_i is an L-vertex. Suppose it has a chord e_ie_j, then one of the L-cycles $(e_i, e_{i+1}, \ldots, e_j, e_i)$ or $(e_j, e_{j+1}, \ldots, e_k, e_1, \ldots, e_i, e_j)$ is negative, contrary to the least length assumption. So we may assume Z'_k chordless. Now for any graph G, a chordless L-cycle of $L(G)$ must consist either of three L-vertices corresponding to edges of G incident with a single vertex, or of an L-cycle whose L-vertices are the edges of a chordless cycle of G. The result follows. ■

3. Sign-Compatibility and Canonical Sign-Compatibility of $L_\bullet(S)$

A sigraph $S = (S^u, \sigma)$ is *sign-compatible* [16] if it has a vertex marking μ such that each edge $e = vw$ has $\sigma(e) = -$ if and only if $\mu(v) = \mu(w) = -$. If the canonical marking μ_σ has this property, then S is said to be *canonically sign-compatible* (or $\mathcal{C}$*-sign-compatible.*)

The conditions for a sigraph S to have these properties are known (Theorems 5, 6 and 8 below). In this section we establish the conditions for a •-line sigraph $L_\bullet(S)$ to have each of these properties.

Theorem 5 [17]**.** *A sigraph S is sign-compatible if and only if there is a subset W of $V(S)$ whose induced subsigraph has for its edge set exactly the negative edges of S.*

Theorem 6 [17]**.** *A sigraph S is sign-compatible if and only if S does not contain a subsigraph isomorphic to either of the two sigraphs, S_1 formed by taking the path $P_4 = (x, u, v, y)$ with both the edges xu and vy negative and the edge uv positive and S_2 formed by taking S_1 and identifying the vertices x and y Figure 2.*

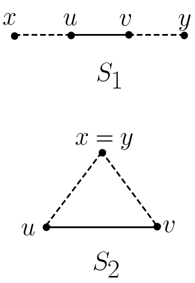

Figure 2. Acharya and Sinha forbidden subsigraphs for a sign-compatible sigraph.

Now, we present the condition for a •-line sigraph to be sign-compatible.

Theorem 7. *Let $S = (S^u, \sigma)$ be a canonically-marked sigraph. Then $L_\bullet(S)$ is sign-compatible if and only if S has the following property: Let $e_i, e_j, e_k, e_l \in E(S)$ such that there are vertices v, w, x (not necessarily distinct) with $e_i \cap e_j = v$, $e_k \cap e_i = w$ and $e_l \cap e_j = x$ and $\mu_\sigma(v) = +$. Then, either $\mu_\sigma(w) = +$ or $\mu_\sigma(x) = +$.*

Proof. *Necessity*: Suppose $L_\bullet(S)$ is sign-compatible. Then, by Theorem 6, $L_\bullet(S)$ does not contain a subsigraph isomorphic to S_1 or S_2 in Figure 2.

Let e_i and e_j be two adjacent edges in S and v be the common vertex between them and $\mu_\sigma(v) = +$. Now, suppose that e_k is adjacent with e_i and e_l is adjacent with e_j and w and x are common vertices between them respectively. If possible, suppose the condition is false. Then, $\mu_\sigma(v) = +$ and $\mu_\sigma(w) = \mu_\sigma(x) = -$. If $e_k = e_l$ then, by the definition of $L_\bullet(S)$, we have an L-triangle with two negative and one positive L-edges in $L_\bullet(S)$. Thus, we have a subsigraph isomorphic to S_2 in Figure 2 in $L_\bullet(S)$, a contradiction to the hypothesis. Now, suppose $e_k \neq e_l$, then we have an L-path $P_4' = (e_k, e_i, e_j, e_l)$ in $L_\bullet(S)$ such that e_ie_j is a positive L-edge while e_ke_i and e_je_l are negative L-edges. Thus, we have a subsigraph isomorphic to S_1 in Figure 2 in $L_\bullet(S)$, a contradiction to the hypothesis. Thus, in both the conditions we have a contradiction. Hence, the conditions are necessary.

Sufficiency: Suppose the condition in the statement of the theorem holds for a sigraph S. We want to show that $L_\bullet(S)$ is sign-compatible. Suppose to the contrary that $L_\bullet(S)$ contains a subsigraph isomorphic to S_1 or S_2.

Case I: Suppose $L_\bullet(S)$ contains a subsigraph, say ${P_4}'$, isomorphic to S_1. Let ${P_4}' = (e_i, e_j, e_k, e_l)$ be such that e_ie_j and e_ke_l are negative L-edges and e_je_k is a positive L-edge in $L_\bullet(S)$. Then, by the definition of $L_\bullet(S)$, there exists a vertex between e_j and e_k in $L_\bullet(S)$ such that it is positively marked in S and the common vertices between e_i, e_j and e_k, e_l are marked negatively in S, a contradiction to the hypothesis. Thus, $L_\bullet(S)$ does not contain a subsigraph isomorphic to S_1.

Case II: Now, let $L_\bullet(S)$ contain a subsigraph isomorphic to S_2. Then this L-triangle is either due to the edges of a triangle or due to a vertex $v \in V(S)$ in S with $d(v) \geq 3$. Let Z' be an L-triangle in $L_\bullet(S)$ which is isomorphic to S_2.

Case II(a): If all the vertices of Z' are due to the adjacent edges of a single triangle Z in S, then by the definition of $L_\bullet(S)$, we have a triangle Z in S such that its two vertices are marked negatively while one vertex is marked positively, a contradiction to the hypothesis.

Case II(b): Now, since this triangle is not due to any triangle of S, therefore Z' must contain an L-vertex, say e_p, which corresponds to an edge e_p incident with a vertex v with $d(v) \geq 3$ in S. Then either $\mu_\sigma(v) = +$ or $\mu_\sigma(v) = -$. Then, by the definition of $L_\bullet(S)$, such L-triangle is either all-positive or all-negative, a contradiction to the hypothesis. Hence, in all the conditions $L_\bullet(S)$ does not contain a subsigraph isomorphic to S_2. Hence, by Theorem 6, $L_\bullet(S)$ is sign-compatible. ∎

The following characterization of the $\mathcal{C}$-sign-compatible sigraph is given by the authors in [18]. This theorem is useful for our further investigation of $\mathcal{C}$-sign-compatible $L_\bullet(S)$.

Theorem 8 [18]**.** *A sigraph $S = (S^u, \sigma)$ is $\mathcal{C}$-sign-compatible if and only if the following conditions hold in S:*

(i) *for every vertex $v \in V(S)$ either $d^-(v) = 0$ or $d^-(v) \equiv 1 \pmod 2$, and*

(ii) *for every positive edge $e_k = v_iv_j$ in S, $d^-(v_i) = 0$ or $d^-(v_j) = 0$.*

Now, the following theorem determines the condition for $L_\bullet(S)$ to be $\mathcal{C}$-sign-compatible.

Theorem 9. *For a given sigraph $S = (S^u, \sigma)$, $L_\bullet(S)$ is $\mathcal{C}$-sign-compatible if and only if the following conditions hold in S:*

(a) *for each edge e of S, the number of edges that are adjacent with e and incident with a negative vertex that is also adjacent with e, is zero or odd, and*

(b) *for every positively marked vertex in S, say v_i, if there are two negatively marked vertices, say v_j and v_k, adjacent with v_i, then there is no other vertex adjacent with v_j or there is no other vertex adjacent with v_k.*

Proof. *Necessity*: Let $\bullet$-line sigraph $L_\bullet(S)$ be $\mathcal{C}$-sign-compatible. Then, conditions (i) and (ii) of Theorem 8 hold for $L_\bullet(S)$.

Let θ denote the signature of $L_\bullet(S)$. Thus, for any edge $e = vw$ of S, the L-vertex e of $L_\bullet(S)$ has $\mu_\theta(e)$ equal to the number of edges of S that are adjacent with e and incident with v (if v is negative) or w (if w is negative). If this number is even and positive, then $L_\bullet(S)$ is not $\mathcal{C}$-sign-compatible. Hence condition (a) is necessary.

Now, suppose there is a positively marked vertex v_i in S and two negatively marked vertices v_j and v_k such that v_j and v_k are adjacent with v_i. Suppose v_j and v_k are adjacent with some vertices. Then, one possibility is that v_j and v_k are adjacent with each other. In this case, we have an L-triangle $(v_iv_j, v_jv_k, v_kv_i, v_iv_j)$ with one positive L-edge $v_iv_jv_kv_i$ and two negative L-edges in $L_\bullet(S)$. So there exists a positive L-edge in $L_\bullet(S)$ such that it does not satisfy condition (ii) of Theorem 8. Thus, by Theorem 8, $L_\bullet(S)$ is not $\mathcal{C}$-sign-compatible, a contradiction to the hypothesis. Hence, v_j and v_k are not adjacent with each other in $L_\bullet(S)$.

Now, suppose v_l is a vertex adjacent with v_j and v_m is a vertex adjacent with v_k respectively in S. Then, by the definition of $L_\bullet(S)$, we have an L-path $p_4 = (v_lv_j, v_jv_i, v_iv_k, v_kv_m)$ in $L_\bullet(S)$ such that L-edge $v_jv_iv_iv_k$ is a positive edge while L-edges $v_lv_jv_jv_i$ and $v_iv_kv_kv_m$ are negative edges. Thus, again there exists a positive L-edge in $L_\bullet(S)$ such that it does not satisfy condition (ii) of Theorem 8. So by the same argument as above, we get a contradiction to the hypothesis. Hence, condition (b) is necessary.

Sufficiency: Suppose conditions (a) and (b) hold for a given sigraph S. We shall show that $L_\bullet(S)$ is $\mathcal{C}$-sign-compatible. Suppose on contrary that $L_\bullet(S)$ is not $\mathcal{C}$-sign-compatible. Then, by Theorem 8 condition (i) or condition (ii) is not satisfied for $L_\bullet(S)$.

Suppose condition (i) of Theorem 8 is not satisfied for $L_\bullet(S)$ i.e., there is an L-vertex $e \in V(L_\bullet(S))$ such that neither $d^-(e) = 0$ nor $d^-(e) \equiv 1 \pmod 2$. This shows that there are even number of edges adjacent with e in S such that these edges are incident with vertices with negative marking, a contradiction to condition (a).

Now, suppose condition (ii) of Theorem 8 is not satisfied i.e., for any positive edge $k = ee'$ there are negative L-edges on e and e' in $L_\bullet(S)$. Suppose this positive L-edge lies on an L-path $P'_4 = (e_i, e_j, e_k, e_l)$ such that e_je_k is a positive L-edge while e_ie_j and e_ke_l are negative L-edges in $L_\bullet(S)$. Then, by the definition of $L_\bullet(S)$, we have a path $p_5 = (u, v, w, x, y)$ in S such that $e_i = uv$, $e_j = vw$, $e_k = wx$ and $e_l = xy$. Clearly, e_je_k is a positive L-edge in $L_\bullet(S)$ so, by the definition of $L_\bullet(S)$, common vertex w between e_j and e_k is surely positively marked in S. Similarly, e_ie_j and e_ke_l are negative L-edges in $L_\bullet(S)$, so their common vertices v and x, respectively, receive negative mark in S. Thus, for a positively marked vertex w in S, two negative vertices v and x are adjacent with w in S. Also, u and y are adjacent with v and x in S, a contradiction to (a).

This positive L-edge can be in a triangle also. Let Z' be such L-triangle in $L_\bullet(S)$. Now, this L-triangle is either due to the edges of a triangle of S or due to a triangle $v \in V(S)$ in S with $d(v) \geq 3$.

Case I: If all the vertices of Z' are due to the adjacent edges of a single triangle Z in S, then by the definition of $L_\bullet(S)$, we have triangle Z in S with one and two negatively marked vertices in S. Thus, again we have a contradiction to (a).

Case II: Now, since this L-triangle is not due to any triangle of S, therefore Z' must contain an L-vertex, say e_p, such that it is incident with a vertex $v \in V(S)$ with $d(v) \geq 3$ in S. Then, by the definition of $L_\bullet(S)$, either such L-triangle is all-positive or all-negative. So there is no such positive L-edge in $L_\bullet(S)$. Hence, by Theorem 8, $L_\bullet(S)$ is $\mathcal{C}$-sign-compatible. This completes the proof. ∎

4. Existential Characterization of $L_\bullet(S)$

In this section we establish the characterization of the •-line sigraph. While the characterization problem has been solved for 'line sigraph' (i.e., sigraph S for which there exists a sigraph H such that $L(H) \cong S$) as in [2], the same remains to be solved for the ×-line sigraph' (i.e., sigraph S for which there exists a sigraph H such that $L_\times(H) \cong S$) as well as for the '•-line sigraph' (i.e., sigraph S for which there exists a sigraph H such that $L_\bullet(H) \cong S$).

Hence, for any given isolate-free sigraph S, consider the sigraph equation

$$L_\bullet(H) \cong S \tag{1}$$

where any sigraph H satisfying (1) (i.e., a 'solution' of (1)) will be called an $L_\bullet$-root of S [1]. By the definition of •-line sigraph it is clear that

$$L_\bullet(S^u) \cong L(S^u),$$

so Theorem 1 is a characterization of •-line graphs also. We have the following important observation by Sampatkumar, which is useful in the upcoming theorem.

Remark 10 [15]. *Every canonically marked sigraph contains an even number of negative vertices.*

Now, we give two following lemmas which are essential for the characterization of •-line graphs.

Lemma 11. *Let S be a connected graph and let U be any even subset of $V(S)$. Then there is a signature σ for S such that $\mu_\sigma(v) = -$ if and only if $v \in U$.*

Proof. Let $|U| = 2k$. If $k = 0$ then we have the all-positive signature; now inductively suppose the statement true for even subsets of size 2, 4, ..., $2(k-1)$. Let u, v be any two elements of U. By the inductive hypothesis there is a signature σ such that $\mu_\sigma(w) = -$ if and only if $w \in U - \{u, v\}$ (or w could be anything other than u or v). There is a path from u to v; change the sign of every edge on the path, giving a signature τ. It is clear that $\mu_\tau(w) = -$ if and only if $w \in U$. ■

Lemma 12. *Let $S = (S^u, \sigma)$ be the •-line graph of sigraph $T = (T^u, \tau)$ and extend T^u to a graph $\hat{T}^u$ by adding a new vertex t and an edge $e = st$, where $s \in V(T^u)$ (so that t is pendent). Now extend σ to a signature $\hat{\sigma}$ on $L(S^u)$ as follows. The new L-edges are $e_1e, e_2e, \ldots, e_de$ where the e_i are the edges of T^u incident with s; give all these the same sign (either $+$ or $-$). Then $\hat{S} = (\hat{S}^u, \hat{\sigma})$ is a •-line graph.*

Proof. Extend τ to a signature on $\hat{T}^u$, by appropriately signing st so that the canonical marking $\hat{\mu}(s)$ of $V(T)$ agrees with the $\hat{\sigma}(e_ie)$. ■

Now, the following theorem gives us the solution of (1).

Theorem 13. *A given connected sigraph $S = (S^u, \sigma)$ is a •-line graph if and only if following conditions hold in S:*

(1) *S^u is a line graph and the edges of S can be partitioned into complete subsigraphs in such a way that no vertex lies in more than two of the subsigraphs and each such complete subsigraph is homogeneous.*

(2) *if each vertex of S belongs to exactly two of these subsigraphs, then the number of all-negative complete sigraphs is even.*

Proof. *Necessity*: We are given a sigraph S. Suppose S is a •-line sigraph. Then there exists a sigraph T, such that $S^u \cong L_\bullet(T^u)$, so that S^u is a line graph. Thus, by Theorem 1, the edges of S^u can be partitioned into complete subsigraphs such that no vertex lies in more than two of these. The vertices of any such subsigraph $\mathcal{Q}$ are the L-vertices of $L_\bullet(T^u)$ corresponding to the edges of T^u incident with some vertex v of T^u, and therefore the edges of $\mathcal{Q}$ are L-edges of $L_\bullet(T^u)$ signed with $\mu_\sigma(v)$. Thus these complete subsigraphs are homogeneous, and condition (1) is satisfied.

Next, suppose all the vertices in S lie in exactly two such subsigraphs and number of all-negative subsigraphs is odd. By the definition of $\bullet$-line sigraph, it is clear that there are odd number of negatively marked vertices in the $L_\bullet$-root of S, a contradiction to the Remark 10. Hence, by contradiction, (2) holds.

Now, suppose there are some vertices in S such that these are not in two such subsigraphs. While making the $L_\bullet$-root of S by the help of these subsets, it is clear that there are some pendent vertices in the $L_\bullet$-root of S. Since the vertices are pendent in $L_\bullet$-root of S, so there are not any other edges incident with these vertices in the $L_\bullet$-root of S. By the definition of $L_\bullet(S)$, these vertices are not giving any contribution to the signing of any complete subsigraphs in the $\bullet$-line sigraph. Thus, by including or excluding these vertices the number of negatively marked vertices, in $L_\bullet$-root of S, is even. So the number of all-negative homogeneous complete subsigraphs is either even or odd according to the number of pendent vertices and their canonical marking in the $L_\bullet$-root of S.

Sufficiency: Suppose S is a sigraph satisfying the conditions. We shall show that S is the $\bullet$-line sigraph, that is, there exists a sigraph T such that $S \cong L_\bullet(T)$.

Let T^u be the graph such that $S^u = L(T^u)$. We wish to find a signature τ on T^u such that μ_τ is the canonical marking of $V(T^u)$ giving the required signature of S^u. Assume, first that there is an even number of all-negative complete subsigraphs in the partitioning of $E(S)$. Let U be the corresponding set of vertices of T^u. By Lemma 11, there is a signature τ on T^u such that μ_τ is negative exactly on U. Finally, if T^u has a pendent vertex t, then by Lemma 12 there is a complete sigraph whose sign may be chosen independently. This completes the proof. ■

Note: If a sigraph S is disconnected then S is a $\bullet$-line sigraph if and only if its every component satisfies the conditions of Theorem 13 separately.

Corollary 14. *Every homogeneous complete sigraph K_n is a $\bullet$-line sigraph.*

Proof. Suppose we are given a homogeneous complete sigraph. It is easy to see that $L_\bullet$-root for K_n^u is star $K_{1,n}^u$. Thus, K_n^u is a line graph. Thus, the conditions of Theorem 13 are satisfied. Hence, every homogeneous complete sigraph K_n is a $\bullet$-line sigraph. ■

Corollary 15. *Every path sigraph P_n is a $\bullet$-line sigraph.*

Proof. Suppose we are given a path sigraph P_n. It is easy to see that $L_\bullet$-root for P_n^u is P_{n+1}^u. Thus, P_n^u is a line graph. We can partition P_n into $(n-1)$ K_2 complete subsigraphs in such a way that no vertex lies in more than two of the subsigraphs and each such K_2 is homogeneous. Since there are pendent vertices in P_n, number of such complete subsigraph may be even or odd. So both the conditions of Theorem 13 are satisfied. Hence, every path sigraph P_n is a •-line sigraph. ■

Corollary 16. *A cycle C_n is a •-line sigraph if it contains even number of negative edges.*

Proof. Suppose we are given a cycle C_n. It is easy to see that the $L_\bullet$-root for C_n^u is C_n^u. Thus, C_n^u is a line graph. We can partitioned C_n into n, K_2 complete subsigraphs in such a way that no vertex lies in more than two of the subsigraphs and each such K_2 is homogeneous. Since there are no pendent vertices in C_n and there is an even number of negative edges in C_n, the number of such complete subsigraphs K_2 is even. So both the conditions of Theorem 13 are satisfied. Hence, every cycle sigraph C_n is a •-line sigraph. ■

Corollary 17. *Every balanced cycle C_n is a •-line sigraph.*

Proof. The result is trivial by Lemma 3 and Corollary 16. ■

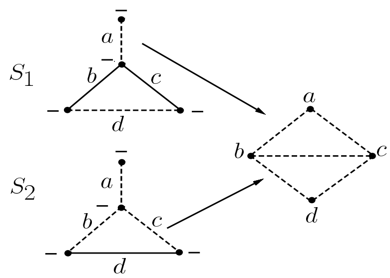

Figure 3. Showing a •-line sigraph and its two $L_\bullet$-root sigraphs.

Note: We know that if there are odd (even) number of negative edges incident with any vertex, then due to its canonical marking $\mu_\sigma(v) = -$ $(\mu_\sigma(v) = +)$. Therefore, for a given •-line sigraph its $L_\bullet$-root sigraphs is not unique. The pictorial presentation of this is shown in Figure 3. Hence, the following problem is open.

Problem 18. Characterize •-line sigraphs having exactly one $L_\bullet$-root sigraph up to isomorphism.

Acknowledgement

The authors express gratitude to Dr. B.D. Acharya, who attracted the authors to the characterization problem and to find other related properties of •-line sigraphs. Also, his paper *Signed intersection graphs* [1] gave a way to solve the problems.

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Received 25 September 2013
Revised 7 April 2014
Accepted 9 April 2014