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ON DECOMPOSING REGULAR GRAPHS INTO ISOMORPHIC DOUBLE-STARS

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Abstract

A *double-star* is a tree with exactly two vertices of degree greater than 1. If T is a double-star where the two vertices of degree greater than one have degrees k_1+1 and k_2+1, then T is denoted by S_{k_1,k_2}. In this note, we show that every double-star with n edges decomposes every $2n$-regular graph. We also show that the double-star $S_{k,k-1}$ decomposes every $2k$-regular graph that contains a perfect matching.

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1. Introduction

By a *decomposition* of a graph G we mean a sequence $H_1, H_2, \ldots, H_k$ of subgraphs whose edge sets partition the edge set of G. If each subgraph H_i is isomorphic to a fixed graph H, then the decomposition is an H-*decomposition* of G and we say H *decomposes* G. A large amount of research has been done on the topic of graph decompositions over the last five decades (see [1] and [2] for recent surveys). Much investigation has been motivated by the following conjecture of Ringel [10].

Conjecture 1. *Every tree T with n edges decomposes the complete graph K_{2n+1}.*

A broadening of Ringel's conjecture is due to Graham and Häggkvist (see [5]).

Conjecture 2. *Every tree T with n edges decomposes every $2n$-regular graph G.*

Despite persistent attacks over the last 40 years, Ringel's conjecture and variations thereof, such as the Graceful Tree Conjecture (see [4]), still stand today. Much less work has been done on the Graham and Häggkvist conjecture however.

Results confirming Conjecture 2, in certain cases, can be found in Snevily's Ph.D. thesis [11]. For example, Snevily shows that every tree T with n edges decomposes every $2n$-regular graph G provided that the girth of G is larger than the diameter of T. He also shows that every tree with n edges decomposes the cartesian product of any n cycles. Other results on decompositions of the cartesian product of graphs into trees can be found in a recent paper by Jao, Kostochka, and West [8].

The graph $K_{1,k}$ is known as a *k-star* and is often denoted by S_k. A *double-star* is a tree with exactly two vertices of degree greater than 1. The two vertices of degree greater than 1 are called the *centers* of the double-star and the edge joining them is called the *central-edge*. If T is a double-star where the two centers have degrees $k_1 + 1$ and $k_2 + 1$, then T is denoted by S_{k_1,k_2}. Note that S_{k_1,k_2} has $k_1 + k_2 + 1$ edges and is isomorphic to S_{k_2,k_1}. The double-star $S_{k,k}$ is called *symmetric*.

Conjecture 2 is simple to verify when T is a star. We will verify it when T is a double-star. We will also show that $S_{k,k-1}$ decomposes every $2k$-regular graph that contains a perfect matching.

2. Main Results

We give some additional definitions before proceeding with our main results. An *orientation* of a graph G is an assignment of directions to the edges of G. An *Eulerian orientation* of G is an orientation where the indegree at each vertex is equal to the outdegree. It is simple to see that a graph with all even degrees has an Eulerian orientation.

Theorem 3. *Every double-star with n edges decomposes every $2n$-regular graph.*

Proof. Let H be the double-star S_{k_1,k_2} with center vertices a and b, where the degree of a is $k_1 + 1$ and the degree of b is $k_2 + 1$. Let G be a $2n$-regular graph where $n = k_1 + k_2 + 1$. We will show that H decomposes G.

Orient the edges of H so that each leaf has indegree 1. Orient the edge $\{a, b\}$ from a to b. Let F be a 2-factor in G. Then F has an Eulerian orientation. Since

$G - E(F)$ is $(2n-2)$-regular, it has an Eulerian orientation. Consider any cycle C in F, and let D_C denote the digraph in G consisting of all arcs with tail in $V(C)$. Thus every vertex in D_C will have outdegree (in D_C) either $k_1 + k_2 + 1$ or 0. Because $\{E(D_C) \colon C \text{ is a cycle in } F\}$ partitions $E(G)$, the proof will be complete if we can show that each such subgraph D_C has an H-decomposition.

Let the cycle C have length p and consist of alternating vertices and arcs labeled v_0, e_1, v_1, e_2, $\ldots$, v_{p-1}, e_p, $v_p = v_0$.

For the first copy H_1 of H in the decomposition, we use e_1 as the central arc, and identify v_0 with a and v_1 with b. Choose k_2 arcs with tail at v_1; label as X the set of endvertices of these k_2 arcs. The remaining k_1 arcs with tail at v_0 in H_1 in this construction will be determined at the end.

We construct the remaining copies $H_2, H_3, \ldots, H_p$ sequentially. After H_{i-1} is determined we construct H_i as follows. The central arc of H_i is e_i, with v_{i-1} identified with a from H, and v_i identified with b. The remaining arcs with tail at v_{i-1} are all such arcs of $D_C - C$ that were not chosen to be in H_{i-1}. From the remaining $k_1 + k_2$ arcs with tail at v_i, we choose k_2 arcs so that:

i) no arc is chosen that is adjacent with an arc chosen at this step to have tail v_{i-1} (avoid an immediate triangle), and

ii) we include in the pool all arcs with head a vertex in X.

The selection process above can always be implemented because in H_{i-1} we chose all possible arcs with tail at v_{i-1} and head at a vertex in X, so no such arc appears in H_i.

It remains only to complete the construction of H_1. After H_p has been constructed, k_1 arcs with tail at v_0 have yet to be assigned; we include these arcs in H_1. Because of the pattern noted above, none of these arcs has as a head a vertex in X. Thus H_1 also has no triangles and is therefore isomorphic to H. ■

In [5], Häggkvist states that he has proven (but has not published) a result showing that every tree with n edges and diameter d decomposes every $2n$-regular graph of girth at least d. Since the girth of a graph with no multiple edges is at least 3, Häggkvist's unpublished result would cover the result in Theorem 3.

We turn our focus to decompositions of n-regular graphs into trees with n edges. If G is n-regular and H is a tree with n edges, then H may or may not decompose G. In fact, if n is even and G has odd order, then $|E(G)|$ would not be divisible by n and thus H could not decompose G. It is also easy to see that S_n decomposes an n-regular graph G if and only if G is bipartite. Graham and Häggkvist do in fact conjecture that every tree T with n edges decomposes every n-regular bipartite graph G (see [5]). This conjecture was verified by Jacobson, Truszczyński, and Tuza [6] for T for the cases when T is a double-star and for when $T = P_5$.

In [9], Kotzig conjectured that the symmetric double-star $S_{k,k}$ decomposes a $(2k+1)$-regular graph G if and only if G contains a perfect matching. Kotzig's conjecture was proved by Jaeger, Payan, and Kouider in [7].

Theorem 4. *For $k \geq 1$, let G be a $(2k+1)$-regular graph. Then $S_{k,k}$ decomposes G if and only if G contains a perfect matching.*

It is simple to see why G must contain a perfect matching if $S_{k,k}$ decomposes it. If G has order $2m$, then the number of $S_{k,k}$'s in the decomposition is m. Since no two central edges in the decomposition can be adjacent, the central edges must form a perfect matching.

Let G be a graph that contains a perfect matching M. A *tent* in G is a pair $\{\{v,x\},\{v,y\}\}$ of adjacent edges such that $\{x,y\}$ is an edge of M. The common vertex v is called the *top* of the tent. Jaeger *et al.* [7] showed that if G is $(2k+1)$-regular, then $G - M$ has an Eulerian orientation so that every tent is a directed path.

We use a slight variation of the approach of Jaeger *et al.* to show that if G is a $2k$-regular simple graph of even order and with a perfect matching, then $S_{k,k-1}$ decomposes G.

Lemma 5. *If G is an Eulerian graph that contains a perfect matching M, then G has an Eulerian orientation such that every tent is oriented into a directed path.*

Proof. We obtain the desired Eulerian orientation as follows. Begin a walk at any vertex w, and start with any edge incident with w. At each step where there is a choice of edges to continue the walk, if we are at vertex v which is incident with tent edges $\{\{v,x\},\{v,y\}\}$, we choose one of these edges if and only if the other edge was the most recent edge in the walk. This process can only end at start vertex w. Orient the edges of the walk according to the direction in which they were traversed. Remove those edges from G, and iterate if any edges remain in G. It is easy to see this process gives the desired orientation. ■

Theorem 6. *For $k \geq 2$, let G be a $2k$-regular graph that contains a perfect matching M. Then $S_{k,k-1}$ decomposes G.*

Proof. By Lemma 5, G has an Eulerian orientation such that every tent is a directed path. For $x \in V(G)$, let $I_x = \{e_1, e_2, \ldots, e_k\}$ be the k arcs with terminal vertex x in the orientation of G and let $V_x = \{x_1, x_2, \ldots, x_k\}$ be the set of initial vertices of these arcs.

If $e = \{x,y\} \in M$, where e is oriented from x to y, then $x \in V_y$, $e \in I_y$, and $V_x \cap V_y = \emptyset$ because each tent is oriented into a directed path. It follows that the graph

$$L_e = (V_x \cup V_y \cup \{y\}, I_x \cup I_y)$$

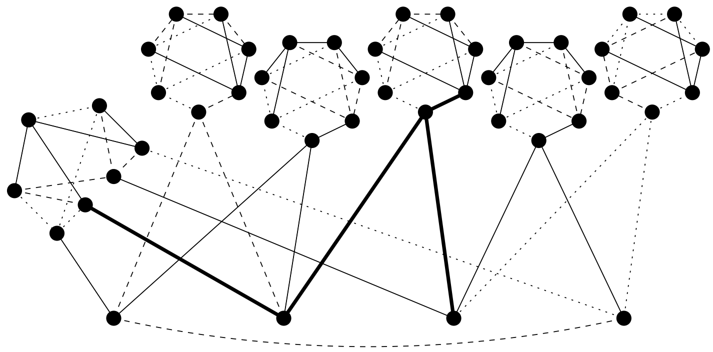

Figure 1. A 4-regular graph without a perfect matching that is $S_{2,1}$-decomposable.

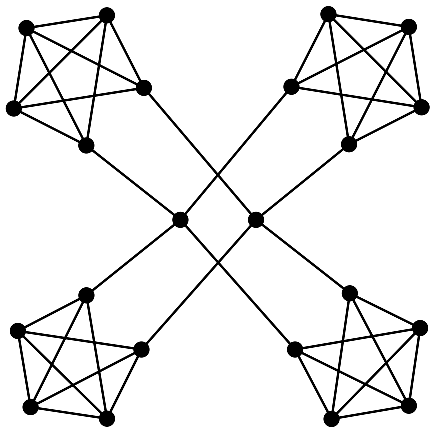

Figure 2. A 4-regular graph without a perfect matching that is not $S_{2,1}$-decomposable.

is isomorphic to $S_{k,k-1}$. Moreover, since each edge of G has exactly one terminal vertex, which is on exactly one edge of M, $\{L_e \colon e \in M\}$ forms an $S_{k,k-1}$-decomposition of G. This completes the proof. ■

If a $2k$-regular graph does not contain a perfect matching, then it may or may not be $S_{k,k-1}$-decomposable. In Figure 1, we show a 4-regular graph that does not contain a perfect matching but is $S_{2,1}$-decomposable. Figure 2 shows a 4-regular graph G that does not contain a perfect matching and is not $S_{2,1}$-decomposable. This graph consists of four vertex-disjoint copies of $K_5 - e$ with each of the degree 3 vertices in these copies joined to one of two additional vertices. Let J denote one of the four copies of $K_5 - e$ in G. Since J contains 9 edges, three edges from the complement of J are needed to get all the edges of J in an $S_{2,1}$-decomposition of G. Since a tree containing edges from more than one $K_5 - e$ in G must have diameter at least 4 and there are only 8 edges in G that are not in a $K_5 - e$, there is no $S_{2,1}$-decomposition of G.

For a graph G, let ${}^2G$ denote the multigraph obtained from G by replacing every edge in G with two parallel edges. In [3], we show that every double-star with n edges decomposes ${}^2G$ for every n-regular graph G. We also investigate decompositions of $2n$-regular multigraphs with edge multiplicity at most 2 into double-stars with n edges.

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