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A NOTE ON VERTEX COLORINGS OF PLANE GRAPHS[1]

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Abstract

Given an integer valued weighting of all elements of a 2-connected plane graph G with vertex set V, let $c(v)$ denote the sum of the weight of $v \in V$ and of the weights of all edges and all faces incident with v. This vertex coloring of G is *proper* provided that $c(u) \neq c(v)$ for any two adjacent vertices u and v of G. We show that for every 2-connected plane graph there is such a proper vertex coloring with weights in $\{1, 2, 3\}$. In a special case, the value 3 is improved to 2.

Keywords: plane graph, vertex coloring.

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1. Introduction

We consider a simple, finite, and undirected graph G with vertex set V and edge set E. If G is plane, then F denotes the set of faces of G. The set $V \cup E$ and the set $V \cup E \cup F$ is the set of *elements* of G. For further notation and terminology, we refer to [7] and [10].

Colorings of a graph defined by weightings (labellings) of elements of that graph are popular topics in research. Here we will consider *vertex colorings* of G, this is a mapping c of V into the set of positive integers ([13]).

For each vertex $v \in V$, let $S(v)$ be a nonempty subset of the set of elements of G and $\mathcal{S} = \{S(v) \mid v \in V\} = \{S(v)\}$. For a positive integer k we consider a *weighting* of $\bigcup_{v\in V} S(v)$, this is a mapping w from $\bigcup_{v\in V} S(v)$ into the set of integers i with $1 \le i \le k$.

Furthermore, we define the corresponding vertex coloring c by $c(v)$ and $c(v) = \sum_{x\in S(v)} w(x)$ for $v \in V$. The vertex coloring c is called *irregular* if $c(u) \ne c(v)$ for any two vertices u and v of G, and *proper*, if $c(u) \ne c(v)$ for any two adjacent vertices u and v of G, unless $S(u) = S(v)$.

Moreover, for fixed $\mathcal{S}$, let $k_i(\mathcal{S})$ and $k_p(\mathcal{S})$ be the minimum k such that there exists a corresponding irregular coloring and a corresponding proper coloring, respectively. If $S = \bigcup_{v\in V} S(v)$ is ordered and the k-th member of S gets the weight 2^k, then $k_p(\mathcal{S}) \le k_i(\mathcal{S}) < 2^{|S|}$.

Note that $k_i(\{\{v\}\}) = |V|$ and $k_p(\{\{v\}\}) = \chi(G)$, where $\chi(G)$ is the chromatic number of G ([13]).

Modifying the sets $S(v)$, next we will survey several coloring concepts considered so far. The case $\mathcal{S} = \{N_V(v)\}$, where $N_V(v)$ denotes the set of vertices adjacent to $v \in V$, was recently considered in [6] and [9]. The following result of Norin can be found there.

Theorem 1 [6]**.** *Let G be a graph with chromatic number $\chi(G) = r$ and coloring number* $\operatorname{col}(G) = k$. *Let $n_1, \ldots, n_r$ be pairwise coprime integers with $n_i \ge k$ for $i = 1, \ldots, r$. Then $k_p(\{N_V(v)\}) \le n_1 n_2 \cdots n_r$.*

By taking $n_1 = 7$, $n_2 = 8$, $n_3 = 9$, and $n_4 = 11$, it follows from Theorem 1 that $k_p(\{N_V(v)\}) \le 5544$ for a planar graph G. In [6], this bound is improved to 468. Moreover, it is shown there that $k_p(\{N_V(v)\}) \le 36$ for a 3-colorable planar graph, that $k_p(\{N_V(v)\}) \le 4$ for a planar graph of girth ≥ 13, and that $k_p(\{N_V(v)\}) \le 2$ if G is a tree.

Recently [4], it was proved that $k_p(\{N_V[v]\}) \le \Delta^2 - \Delta + 1$ for a graph with maximum degree Δ, where $N_V[v] = \{v\} \cup N_V(v)$ for $v \in V$, $k_p(\{N_V[v]\}) \le \Delta - 1$ if G is bipartite, and $k_p(\{N_V[v]\}) \le 2$ if G is a tree.

Let $N_E(v)$ denote the set of edges incident with $v \in V$. Karoński, Łuczak, and Thomason posed the following conjecture for graphs having no component K_2.

Conjecture 2 [16]**.** $k_p(\{N_E(v)\}) \leq 3$.

We remark that Conjecture 2 is true for 3-colorable graphs [16] and $k_p(\{N_E(v)\}) \leq 30$ is shown in [1]. This bound is reduced to 16 in [2] and to 13 in [18]. The best known result is $k_p(\{N_E(v)\}) \leq 5$ by Kalkowski, Karoński, and Pfender [15].

Note that $k_i(\{N_E(v)\})$ is called the *irregularity strength* of G [8, 11]. The latest results and a survey about this topic can be found in [9].

The case $\mathcal{S} = \{\{v\} \cup N_E(v)\}$ was firstly introduced by Bača, Jendrol', Miller, and Ryan in [5]. Here, $k_i(\{\{v\} \cup N_E(v)\})$ is called the *total vertex irregularity strength*. Motivated by [5] and [15], Przybyło and Woźniak posed the following conjecture.

Conjecture 3 [17]**.** $k_p(\{\{v\} \cup N_E(v)\}) \leq 2$.

In addition, Przybyło and Woźniak showed

Theorem 4 [17]**.** $k_p(\{\{v\} \cup N_E(v)\}) \leq \min\{11, 1 + \lfloor \chi(G)/2 \rfloor\}$.

It follows from Theorem 4 that Conjecture 3 is true for 3-colorable graphs. The breakthrough is done by Kalkowski [6] showing that $k_p(\{\{v\} \cup N_E(v)\}) \leq 3$ by using the weights for the vertices in $\{1, 2\}$ and the weights for the edges in $\{1, 2, 3\}$.

Motivated by the above mentioned conjectures and results and by the paper of Wang and Zhu [19], Jendrol' and Šugerek [12] introduced a concept for a 2-connected plane graph G by considering $k_p(\{\{v\} \cup N_E(v) \cup N_F(v)\})$, where $N_F(v)$ denotes the set of faces of G incident with v. In [4], $k_i(\{\{v\} \cup N_E(v) \cup N_F(v)\})$ is called the *entire vertex irregularity strength*.

Jendrol' and Šugerek formulated the following conjecture

Conjecture 5 [12]**.** *If G is a 2-connected plane graph, then* $k_p(\{\{v\} \cup N_E(v) \cup N_F(v)\}) \leq 2$.

In Section 2, we will show that $k_p(\{\{v\} \cup N_E(v) \cup N_F(v)\}) \leq 3$ for each 2-connected plane graph G and that Conjecture 5 is true, if the subgraph of G spanned by the vertices of degree at least 4 is bipartite.

2. Results

Jendrol' and Šugerek proved

Theorem 6 [12]**.** *If G is a 2-connected plane graph, then* $k_p(\{\{v\} \cup N_E(v) \cup N_F(v)\}) \leq \chi(G)$.

We will show

Theorem 7. *If G is a 2-connected plane graph, then $k_p(\{\{v\} \cup N_E(v) \cup N_F(v)\}) \leq 3$.*

Proof. From the Four Color Theorem [3], we know that $\chi(G) \leq 4$. If $\chi(G) \leq 3$, then we are done by Theorem 6.

Suppose $\chi(G) = 4$ and let $f(v) \in \{1, 2, 3, 4\}$ for $v \in V$ be a proper vertex coloring of G. Now we associate the following weights to the members of $S = V \cup E \cup F$: put $w(v) = f(v)$ for $v \in V(G)$, $w(e) = 2$ for $e \in E$, and $w(\alpha) = 2$ for $\alpha \in F$. Clearly, $c(v) \equiv f(v) \pmod 4$ for $v \in V$, hence, $c(u) \neq c(v)$ if u and v are adjacent vertices of G.

Next we gradually relabel vertices weighted with weight 4. Therefore, let u and v be two adjacent vertices of G connected by the edge e with $w(u) = 4$, $w(v) \leq 3$ and $w(e) = 2$. We relabel u, v, and e as follows.

If $w(v) = 2$ or 3, then the new labels are $w^*(u) = 3$, $w^*(v) = w(v) - 1$, and $w^*(e) = 3$. If $w(v) = 1$, then $w^*(u) = 1$, $w^*(v) = 2$ and $w^*(e) = 1$.

Note that $c(v) \equiv f(v) \pmod 4$ for each $v \in V$ after this relabeling and that each edge incident with a remaining vertex of weight 4 still has weight 2 (i.e. the relabelling can proceed). ■

Conjecture 5 is true for every 2-connected bipartite plane graph, see Theorem 6. We prove the next theorem supporting Conjecture 5, too.

Theorem 8. *Let G be a 2-connected plane graph and H be the subgraph of G induced by all vertices of degree at least 4. If H is empty or bipartite, then $k_p(\{\{v\} \cup N_E(v) \cup N_F(v)\}) \leq 2$ and there is a corresponding vertex coloring c such that the weights of all faces of G equal 2.*

Proof. *Case* 1: H is the empty graph. If G is isomorphic to K_4, then the assertion is easily checked.

Hence, we may assume that $\chi(G) \leq 3$. Using Theorem 4, we may assume that there is a coloring c' realizing $k_p(\{\{v\} \cup N_E(v)\}) \leq 2$. We extend c' to a coloring c realizing $k_p(\{\{v\} \cup N_E(v) \cup N_F(v)\}) \leq 2$ by the additional weights $w(\alpha) = 2$ for every face $\alpha \in F$. Note that all vertices of G have degree 2 or 3 and that $c(v) = c'(v) + 2d$ for a vertex $v \in V$ of degree d. Hence, $c(u) \neq c(v)$ for any two adjacent vertices $u, v \in V$ of the same degree.

It remains to consider adjacent vertices $u, v \in V$ of degree 2 and 3, respectively. Let e be the edge connecting u and v. Since $w(\alpha) = 2$ for every face $\alpha \in F$, $c(u) \leq w(e) + 8$ and $c(v) \geq w(e) + 9$ and we are done in Case 1.

Case 2: H is a non-empty graph. Let $V(H)$ and $E(H)$ denote the vertex set and the edge set of H, respectively. Let the graph G' be obtained from G by simultaneously replacing each vertex $v \in V(H)$ of degree d as follows. Since G is embedded into the plane, let $e_1, \ldots, e_d \in E$ be the edges of E incident with v in clockwise order. Delete v, add the cycle on $\{v_1, \ldots, v_d\}$ with edge set $\{v_1v_2, v_2v_3, \ldots, v_{d-1}v_d, v_dv_1\}$, and let e_i be incident with v_i for $i = 1, \ldots, d$. Although v is replaced by v_i, the edge e_i is considered to be an edge of G and an edge of G' as well ($i = 1, \ldots, d$), thus, $E \subset E(G')$. A vertex in $V \setminus V(H)$ is also considered to be a vertex of G', hence, $V \setminus V(H) \subset V(G')$. Obviously, $G' = (V(G'), E(G'), F(G'))$ is a plane 2-connected graph of maximum degree 3.

By Case 1, G' admits a weighting w' with $S'(v) = \{v\} \cup N_{E(G')}(v) \cup N_{F(G')}(v)$ for $v \in V(G')$ and $k_p(\{\{v\} \cup N_{E(G')}(v) \cup N_{F(G')}(v) \mid v \in V(G')\}) \leq 2$ for the corresponding vertex coloring c' and $w'(\alpha) = 2$ for every face $\alpha \in F(G')$. We will define step by step a weight $w(x) \in \{1, 2\}$ for all $x \in S = V \cup E \cup F$ as follows.

For each face $\alpha \in F$ we put $w(\alpha) = 2$. If $v \in V \setminus V(H)$ and $e \in E \setminus E(H)$, then let $w(v) = w'(v)$ and $w(e) = w'(e)$, respectively. Note that the weight $w(x)$ is already defined for all $x \in S(v) = \{v\} \cup N_E(v) \cup N_F(v)$, if $v \in V \setminus V(H)$, hence, $c(u) \neq c(v)$ for two adjacent vertices of $V \setminus V(H)$.

Furthermore, let $w(e) = 2$ for all $e \in E(H)$. It remains to define $w(v)$ for $v \in V(H)$ and, finally, to show that $c(u) \neq c(v)$ for two adjacent vertices $u \in V \setminus V(H)$ and $v \in V(H)$. Therefore, consider an arbitrary component (a bipartite graph) K of H and let v_0 be a fixed vertex of K. If $v \in V(K)$, then let $\mathrm{dist}(v)$ be the distance of v to v_0 in K. Note that $\mathrm{dist}(v_0) = 0$ and that $\mathrm{dist}(u) \neq \mathrm{dist}(v)$ for any two adjacent vertices $u, v \in V(K)$, otherwise we have an odd cycle in K.

We put $w(v_0) = 2$ and determine $c(v_0)$. Consider $u \in V(K)$ with $\mathrm{dist}(u) > 0$ and let $w(v)$ and, hence, also $c(v)$ be already defined for all $v \in V(K)$ with $\mathrm{dist}(v) < \mathrm{dist}(u)$.

Since $w(x)$ is defined for $x \in S(u) \setminus \{u\}$, let $t \in \{1, 2\}$ be chosen such that $t + \sum_{x \in S(u) \setminus \{u\}} w(x) \not\equiv (c(v_0) + \mathrm{dist}(u)) \pmod 2$ and put $w(u) = t$. Note that the colors $c(x)$ of all vertices x of K having the same value of $\mathrm{dist}(x)$ are of the same parity. Thus, we may assume now that $w(v)$ is defined for all $v \in V(H)$ and that $c(u) \neq c(v)$ for any two adjacent vertices $u, v \in V(H)$.

Eventually, let $u \in V \setminus V(H)$ and $v \in V(H)$ be connected by the edge e and it remains to show that $c(u) \neq c(v)$. Since the degree of u is at most 3, $c(u) = \sum_{x \in S(u)} w(x) \leq w(e) + 12$. Let v have degree $d \geq 4$ in G. If $v = v_0$ then $w(v_0) = 2$. If $v \neq v_0$, then at least one edge of H is incident with v and such an edge has weight 2. In both cases, it follows $c(v) \geq 2d + (d-1) + 2 + w(e) = 3d + 1 + w(e) \geq w(e) + 13$, since $w(\alpha) = 2$ for each face $\alpha \in F$. ■

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