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CHARACTERIZATION OF SUPER-RADIAL GRAPHS

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Abstract

In a graph G, the distance $d(u,v)$ between a pair of vertices u and v is the length of a shortest path joining them. The eccentricity $e(u)$ of a vertex u is the distance to a vertex farthest from u. The minimum eccentricity is called the radius, $r(G)$, of the graph and the maximum eccentricity is called the diameter, $d(G)$, of the graph. The super-radial graph $R^*(G)$ based on G has the vertex set as in G and two vertices u and v are adjacent in $R^*(G)$ if the distance between them in G is greater than or equal to $d(G) - r(G) + 1$ in G. If G is disconnected, then two vertices are adjacent in $R^*(G)$ if they belong to different components. A graph G is said to be a super-radial graph if it is a super-radial graph $R^*(H)$ of some graph H. The main objective of this paper is to solve the graph equation $R^*(H) = G$ for a given graph G.

Keywords: radius, diameter, super-radial graph.

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1. Introduction

The graphs considered are simple, non-trivial, undirected and finite. $G = (V, E)$ is a graph with vertex set $V(G)$ and edge set $E(G)$. In a graph G, the *distance* $d(u, v)$ between a pair of vertices u and v is the length of a shortest path joining them. The *eccentricity* $e(u)$ of a vertex u is the distance to a vertex farthest from u. The *radius* $r(G)$ of G is defined by $r(G) = \min\{e(u) : u \in V(G)\}$ and the *diameter* $d(G)$ of G is defined by $d(G) = \max\{e(u) : u \in V(G)\}$. A graph G for which $r(G) = d(G)$ is called a *self-centered* graph of radius $r(G)$. A vertex v is called an *eccentric vertex of a vertex* u if $d(u, v) = e(u)$. A vertex v of G is called an *eccentric vertex of* G if it is an eccentric vertex of some vertex of G. The concept of antipodal graph was initially introduced by Singleton [21] and was further expanded by Aravamudhan and Rajendran [2, 3]. The *antipodal graph* of a graph G, denoted by $A(G)$, is the graph on the same set of vertices as of G, two vertices being adjacent if the distance between them is equal to the diameter of G while G is connected and if G is disconnected, then two vertices are adjacent in $A(G)$ if they belong to different components of G. A graph G is said to be *antipodal* if it is the antipodal graph of some graph H.

Aravamudhan and Rajendran [2, 3] have proved the following theorem. A graph G is an antipodal graph if and only if it is the antipodal graph of its complement $\overline{G}$. In [4] the same authors observed that if H is a connected graph with $diam(H) > 2$, then $A(H) = A(H')$, where H' is the graph on the same vertex set such that two vertices are adjacent in H' if the distance between them in H is less than $diam(H)$. This observation is still true when $diam(H) = 2$ (for then $H' = H$) and when H is disconnected. In this case, the components of H and H' consists of the same vertices and the edges of $A(H)$ and $A(H')$ are exactly the edges joining vertices in different components. This extension leads to another proof of the characterization of antipodal graphs which involves showing that $A(H') = \overline{H'}$ by Johns [9].

Kathiresan and Marimuthu [14] introduced the *radial graph* $R(G)$ of a graph G on the same vertex set as G and two vertices u and v are adjacent in $R(G)$ if and only if the distance between them is equal to the radius. If G is disconnected, then two vertices are adjacent in $R(G)$ if they belong to different components of G. A graph G is called a *radial graph* if $R(H) = G$ for some graph H. Kathiresan and Marimuthu [15] characterized graphs G with specified radius for its radial graph.

In paper [20], the author defines a metric operator $X_{\mathcal{P}}$ which unifies every known digraph operator related to a distance property $\mathcal{P}$. In Theorem 1 [20] the author characterizes those digraphs G such that $X_{\mathcal{P}}(G) = H$ for some digraph G when $\mathcal{P}$ is both unitary and vertex free distance property. In particular, the characterization of both antipodal and radial graphs arises from it.

Kathiresan *et al.* [16] defined a graph G to be *periodic* if $R^m(G) = G$ for some m. If p is the least positive integer with this property, then G is called a *periodic graph with iso-period* p. A graph G is said to be an *eventually periodic graph* if there exist positive integers m and $k > 0$, such that $R^{m+i}(G) = R^i(G)$, for all $i \geq k$. They proved that every graph is either periodic or eventually periodic. In their paper they characterized all periodic graphs.

Akiyama *et al.* [1] defined the *eccentric graph* G_e of G on the same set of vertices, by joining two vertices if and only if one of the two vertices has the maximum possible distance from the other, that is $d(u,v) = \min\{e(u), e(v)\}$. Iqbalunnisa *et al.* [10] defined the *super-eccentric graph* $J(G)$ of a graph G on the same set of vertices of G and the adjacency relation between vertices is defined by $d(u,v) \geq rad(G)$ while G is connected and when G is disconnected, two vertices are adjacent in $J(G)$ if they belong to different components of G. Kathiresan *et al.* [18] have given a characterization of super-eccentric graphs.

For a digraph D, the *antipodal digraph* $A(D)$ of D is the digraph which $V(A(D)) = V(D)$ and $E(A(D)) = \{(u,v) : u, v \in V(D) \text{ and } d_D(u,v) = d(D)\}$. Johns and Sleno [8] obtained a characterization of antipodal digraphs. A digraph D is *self-antipodal* if $A(D)$ is isomorphic to D.

Kathiresan and Sumathi [17] extended the definition of radial graph to a digraph D where the arc (u,v) is included in $R(G)$ if $d(u,v)$ is the radius of D. According to them a digraph D is called a *radial digraph* if $R(H) = D$ for some digraph H.

Buckley [6] defined the *eccentric digraph* $ED(G)$ of graph G to be the digraph that has the same vertex set as G such that there is an arc from v to u provided that u is an eccentric vertex of v. He examined eccentric digraphs of graphs.

Gimbert *et al.* [12] considered the behaviour of an iterated sequence of eccentric graphs or digraphs of a graph or a digraph. They concluded with several open problems. Boland *et al.* [11] defined the eccentric digraph of a digraph. They examined eccentric digraphs of digraphs for various families of digraphs and they considered the behaviour of an iterated sequence of eccentric digraphs of a digraph.

Huilgol *et al.* [19] considered an open problem, which is found in [11]. They characterized graphs with specified maximum degree such that $ED(G) = G$.

Gimbert *et al.* [13] presented a characterization of eccentric digraphs, which in the undirected case says that a graph G is eccentric if and only if its complement graph $\overline{G}$ is either self-centered of radius two or it is the union of complete graphs.

In [5], the k^{th} power G^k of the graph G has the same vertex set as G and vertices u and v are adjacent in G^k if the distance between them in G is at most k.

Motivated by these works, we introduce a new concept called *super-radial graph* $R^*(G)$ of a graph G on the same vertex set of G and two vertices u and v

are adjacent in $R^*(G)$ if and only if the distance between them is greater than or equal to $d(G) - r(G) + 1$. If G is disconnected, then two vertices are adjacent in $R^*(G)$ if they belong to different components of G. A graph G is said to be a *super-radial graph* if there exists a graph H such that $R^*(H) = G$. In this paper, we have given a characterization for a graph to be a super-radial graph.

The following notation can be found in [14].

Let $F_{11}, F_{12}, F_{22}, F_{23}, F_{24}, F_3$ denote the set of all connected graphs G for which $r(G) = d(G) = 1, r(G) = 1$ and $d(G) = 2$, $r(G) = d(G) = 2, r(G) = 2$ and $d(G) = 3, r(G) = 2$ and $d(G) = 4, r(G) \geq 3$, respectively. F_4 denote the set of all disconnected graphs. For graph theoretic terminology we follow [5], which is devoted entirely to the area of distance in graphs.

The following results will be used throughout this article.

Theorem A [5]. *If G is a simple graph with diameter at least 3, then $\overline{G}$ has diameter at most 3.*

Theorem B [5]. *If G is a simple graph with diameter at least 4, then $\overline{G}$ has diameter at most 2.*

Theorem C [5]. *If G is a simple graph with radius at least 3, then $\overline{G}$ has radius at most 2.*

Theorem D [23]. *If G is a selfcentred graph with radius at least 3, then $\overline{G}$ is a self centered graph of radius 2.*

From the above theorems, we have the following.

If $G \in F_{11}$, then $\overline{G}$ is a totaly disconnected graph and if $G \in F_{12}$, then $\overline{G}$ has at least one isolated vertex. If $G \in F_{22}$, then $\overline{G}$ is a member of $F_{22} \cup F_{23} \cup F_{24} \cup F_3 \cup F_4$. If $G \in F_{23}$, then $\overline{G}$ is a member of $F_{22} \cup F_{23}$. If $G \in F_{24}$, then $\overline{G}$ is a member of F_{22}. If $G \in F_3$, then $\overline{G} \in F_{22}$. If every component of G is non-trivial, then $\overline{G} \in F_{22}$. If G has at least one isolated vertex, then $\overline{G}$ is a member of F_{12}.

Lemma E [23]. *Let u, v be two vertices of a graph G. Then $d_{G^k}(u,v) = \left\lceil \frac{d_G(u,v)}{k} \right\rceil$.*

2. The Relation Between the Super-radial Operator and the Complement Operator

In this section we find a graph G for which $R^*(G) = H$ for a given graph H.

Proposition 1. *For any graph G on p vertices, $R^*(G) = K_p$ if and only if either G is self-centered or $G = \overline{K_p}$.*

Proof. If either G is self-centered or $G = \overline{K_p}$, then the result follows from the definition of $R^*(G)$. Suppose that G is connected and $r(G) \neq d(G)$. This shows that $d(G) - r(G) + 1 \geq 2$. Therefore $R^*(G) \subseteq \overline{G}$. This is a contradiction to the fact that $R^*(G) = K_p$. If G is a disconnected graph in which $|V(G_i)| = 2$, for some i^{th} component G_i of G, then $uv \notin E(R^*(G))$ whenever u and v belong to $V(G_i)$. This implies that $R^*(G) \neq K_p$. ∎

Proposition 2. *For any graph G with $p \geq 3$ vertices, $R^*(G) = K_{1,p-1}$ if and only if G is disconnected with exactly two components out of which one is an isolated vertex.*

Proof. If G is disconnected with exactly two components out of which one is an isolated vertex, then by the definition of $R^*(G), R^*(G) = K_{1,p-1}$.

Let v_1 be the vertex of degree $p-1$ and $v_2, v_3, \ldots, v_p$ be the pendant vertices of $R^*(G)$. If G is connected, then $d_G(v_1, v_i) \geq d(G) - r(G) + 1$ for all $i \neq 1$ and hence $d_G(v_1, v_i) \geq 2$. This is a contradiction to the fact that $R^*(G) = K_{1,p-1}$. If G is disconnected with more than two nontrivial components, then we arrive at a contradiction to the fact that $R^*(G) = K_{1,p-1}$. If G has exactly two nontrivial components, then $R^*(G)$ is a complete bipartite graph.

Therefore the above argument forces us to conclude that G is a disconnected graph with exactly two components out of which one is an isolated vertex. ∎

Proposition 3. *If G is a graph with $d(G) \geq r(G) + 1$, then $R^*(G) \subseteq \overline{G}$.*

Proof. By the definition of $R^*(G)$ and $\overline{G}$, we have $V(R^*(G)) = V(\overline{G}) = V(G)$. $d(G) \geq r(G) + 1$ implies that $d(G) - r(G) + 1 \geq 2$. This shows that $R^*(G) \subseteq \overline{G}$. ∎

Lemma 4. *Let G be a graph of order p. Then $R^*(G) = \overline{G}$ if and only if G is a graph with $d(G) = r(G) + 1$ or G is disconnected in which each component is complete.*

Proof. If $d(G) = r(G)+1$, then $d(G)-r(G)+1 = 2$. Therefore $R^*(G) \subseteq \overline{G}$. Also, any two adjacent vertices in G are not adjacent in $R^*(G)$. Therefore $\overline{G} \subseteq R^*(G)$. Thus $R^*(G) = \overline{G}$.

If G is disconnected with each component complete, then by the definition, $R^*(G) = \overline{G}$.

If $d(G) < r(G) + 1$, then G is self-centred and by Proposition 1, $R^*(G) = \overline{G} = K_p$. As a consequence $G = \overline{K_p}$, which is a contradiction to the fact that G is connected. This implies that $R^*(G)$ is a complete graph.

If $d(G) > r(G) + 1$, then $d(G) - r(G) + 1 \geq 3$ and hence $R^*(G) \subset \overline{G}$. Thus $d(G) = r(G) + 1$.

Suppose that G has a non-complete component, say G_1. Then G_1 has two non-adjacent vertices u and v. It follows from the definitions that $uv \in E(\overline{G})$ and $uv \notin E(R^*(G))$. ∎

Corollary 5. *If* $G \in F_{12}$*, then* $R^*(G) = \overline{G}$.

Proof. Since $G \in F_{12}, d(G) = r(G) + 1$, by Lemma 4, $R^*(G) = \overline{G}$. ■

Lemma 6. *If* $G \in F_3$ *with* $r(G) + 2 \leq d(G) \leq 2r(G) - 1$*, then* $R^*(G) \in F_{22} \cup F_{23}$ *and* $\overline{R^*(G)} \in F_{tt+1}$ *for some* $t \geq 2$.

Proof. Suppose $R^*(G) \in F_{11}$. Then by Proposition 1, either G is self-centered or G is totally disconnected. This is a contradiction to $G \in F_3$ with $r(G) + 2 \leq d(G) \leq 2r(G) - 1$. Suppose $R^*(G) \in F_{12}$. Then $R^*(G)$ has at least one vertex u of eccentricity one. Then $d(u, v) \geq d(G) - r(G) + 1 \geq 3$ in G for all $u \in V(G) - \{u\}$. Since G is connected, u has at least one adjacent vertex w in G. Therefore $d(u, w) = 1$ in G. Then u is not adjacent to w in $R^*(G)$. Which is a contradiction to $R^*(G) \in F_{12}$. Therefore $R^*(G) \notin F_{12}$. Now we claim that $R^*(G)$ has at least one vertex of eccentricity two. Let u be any peripheral vertex. Then there exists a vertex v in G such that $d(u, v) = d(G)$ in G. Therefore u and v are adjacent in $R^*(G)$.

Consider the set $\overline{N}(u) = \{w : d(u, w) \leq d(G) - r(G)\}$ in G. Clearly in $R^*(G), u$ is not adjacent to any vertex of $\overline{N}(u)$.

Let $w \in \overline{N}(u)$. Then $d(u, w) \leq d(G) - r(G)$ for all $w \in \overline{N}(u)$. Now $d(u, v) \leq d(u, w) + d(w, v)$ in G. Therefore $d(G) \leq d(G) - r(G) + d(w, v)$ in G. Hence

$$d(w, v) \geq r(G) \text{ in } G. \tag{1}$$

Futher $r(G) + 2 \leq d(G) \leq 2r(G) - 1$, which implies,

$$d(G) - r(G) + 1 \leq r(G) \text{ in } G. \tag{2}$$

From (1) and (2),

$$d(w, v) \geq r(G) \geq d(G) - r(G) + 1 \text{ in } G.$$

Hence by the definition, v is adjacent to all the vertices of $\overline{N}(u)$ in $R^*(G)$. Let d be the distance in $R^*(G)$. Therefore, $d(u, w) = d(u, v) + d(v, w) = 1 + 1 = 2$ for all $w \in \overline{N}(u)$. Thus, $R^*(G)$ has a vertex of eccentricity two. Hence $R^*(G) \in F_{22} \cup F_{23} \cup F_{24}$. Let $S = \{w : e(w) = d(G) \text{ in } G\}$. Clearly, $e(w) = 2$ for all $w \in S$ in $R^*(G)$. Let $x \in V(G) - S$. Let $\overline{N}(x) = \{y : d(x, y) \leq d(G) - r(G)$ in $G\}$. Clearly, x is not adjacent to any vertex of $\overline{N}(x)$ in $R^*(G)$. Since $d(x, u) \geq d(G) - r(G) + 1, d(x, u) = 1$ in $R^*(G)$ for all $u \notin \overline{N}(x)$. That is $xu \in E(R^*(G))$.

Let $v' \in S$. Then there exists a vertex $v'' \in S$ such that

$$d(v', v'') = d(G) \text{ in } G. \tag{3}$$

Clearly, $v'v'' \in E(R^*(G))$. Suppose both v' and v'' are in $\overline{N}(x)$ in G. Since $r(G)+2 \le d(G) \le 2r(G)-1$,

$$\begin{aligned} d(v',v'') &\le d(v',x)+d(x,v'') \\ &\le d(G)-r(G)+d(G)-r(G), \\ d(v',v'') &\le 2(d(G)-r(G)) < d(G) \text{ since } d < 2n. \end{aligned}$$

Therefore, $d(v',v'') < d(G)$ in G which is a contradiction to (3).

Hence among v' and v'' at most one vertex can be in $\overline{N}(x)$ in G. Without loss of generality, $v' \notin \overline{N}(x)$ in G. $xv' \in E(R^*(G))$. Let $w \in \overline{N}(x)$ in G. In $R^*(G), d(x,w) \le d(x,v')+d(v',w) \le 1+2$ (because $e(v')=2$). That is $d(x,w) \le 3$ for all $w \in \overline{N}(x)$.

Suppose both $v', v'' \notin \overline{N}(x)$. Then $d(x,w) \le 3$ in $R^*(G)$ for all $w \in \overline{N}(x)$ in G. This is true for all $x \in V(G)-S$. Therefore $2 \le e(u) \le 3$ in $R^*(G)$ for all $u \in V(R^*(G))$. That is $R^*(G) \notin F_{24}$ and $R^*(G) \in F_{22} \cup F_{23}$.

Claim. $\overline{R^*(G)} \in F_{tt+1}$ *where* $t \ge 2$.

By the definition of the k^{th} power of a graph G, we have $d_{G^k}(u,v) = \left\lceil \frac{d_G(u,v)}{k} \right\rceil$. Hence $G^k = \overline{R^*(G)}$ where $k = d(G)-r(G)$. $r(G) \le e(u) \le d(G)$ for all u in G implies $\left\lceil \frac{r(G)}{d(G)-r(G)} \right\rceil \le e_{\overline{R^*(G)}}(u) \le \left\lceil \frac{d(G)}{d(G)-r(G)} \right\rceil$ for all $u \in V(\overline{R^*(G)})$. Since $\frac{d(G)}{d(G)-r(G)} = 1 + \frac{r(G)}{d(G)-r(G)}, \left\lceil \frac{d(G)}{d(G)-r(G)} \right\rceil = 1 + \left\lceil \frac{r(G)}{d(G)-r(G)} \right\rceil$.

Let $t = \left\lceil \frac{r(G)}{d(G)-r(G)} \right\rceil$, since $r(G) \ge 3$ and $r(G)+2 \le d(G) \le 2r(G)-1, t \ge 2$. Therefore $t \le e_{\overline{R^*(G)}}(u) \le 1+t$ for all $u \in V(\overline{R^*(G)})$. Suppose u and v are antipodal vertices of G. Then $d(u,v) = d(G)$.

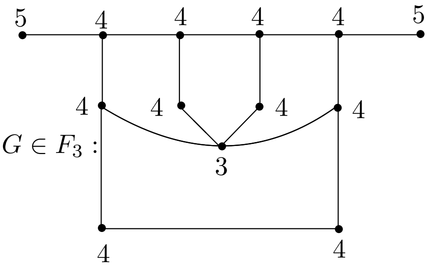
$d_{G^k}(u,v) = \left\lceil \frac{d_G(u,v)}{k} \right\rceil = \left\lceil \frac{d(G)}{d(G)-r(G)} \right\rceil = 1 + \left\lceil \frac{r(G)}{d(G)-r(G)} \right\rceil = 1+t, t \ge 2.$

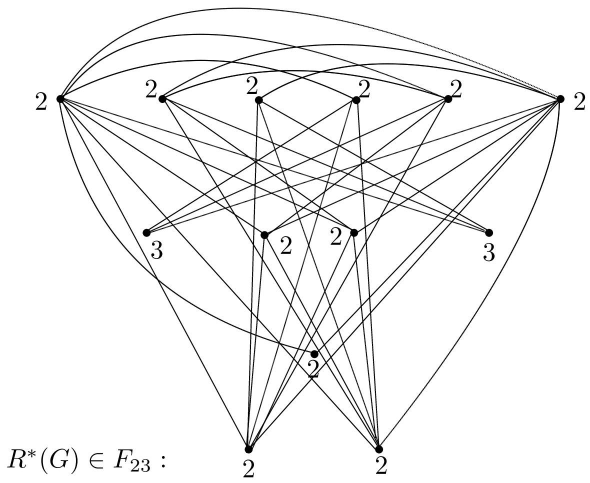
That is $d_{G^k}(u,v) = 1+t, t \ge 2$. Suppose $e(u) = 1+t$, $t \ge 2$. w is any central vertex of G. Then $d(w,u) = r(G) = d(w,v)$

$d_{G^k}(w,u) = \left\lceil \frac{d_G(w,u)}{d(G)-r(G)} \right\rceil = \left\lceil \frac{r(G)}{d(G)-r(G)} \right\rceil = t.$

That is $\overline{R^*(G)} \in F_{tt+1}$ where $t \ge 2$. Hence the proof. ■

$G \in F_3$:





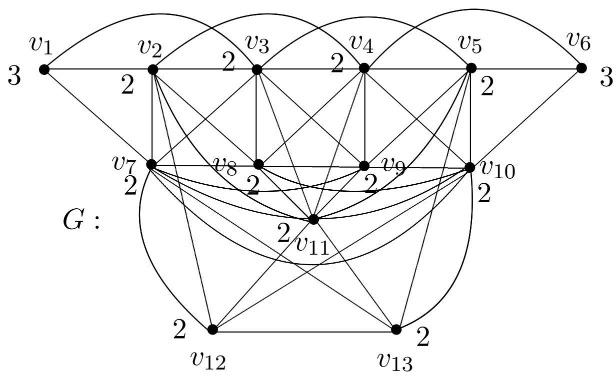


Figure 1. A graph G, its super-radial graph $R^*(G)$ and its complement $\overline{R^*(G)}$ with eccentricities.

Note that there is no characterization of G for which $R(G) = G$. But we have the following.

3. Characterization of Super-radial Graphs

The concept of super-radial graph does not fall into any one of the cases in the metric operator X_p defined by [20]. The property defined by the super-radial graph operator is vertex free but no unitary, so it does not fall into Theorem 1 in [20]. This motivate us to characterize all super-radial graphs.

Proposition 7. *For any graph $G, R^*(G) = G$ if and only if either $G \in F_{11}$ or $G \in F_{23}$ with $G = \overline{G}$.*

Proof. If $G \in F_{11}$, then $R^*(G) = G$. If $G \in F_{23}$ with $G = \overline{G}$, then by Lemma 4, $R^*(G) = \overline{G}$. $G = \overline{G}$ implies that $R^*(G) = G$. Suppose $R^*(G) = G$. If $G \in F_{23}$ with $G \neq \overline{G}$, then by Lemma 4, $R^*(G) = \overline{G}$, but by our assumption $R^*(G) = G$ implies $G = \overline{G}$, which is a contradiction to $G \neq \overline{G}$.

Now let $G \in \mathcal{A} = F_{12} \cup F_{22} \cup F_{24} \cup F_3 \cup F_4$. If $G \in F_{12} \cup F_{22} \cup F_{24}$, then by Proposition 1, Proposition 3 and Corollary 5, $R^*(G) = \overline{G}$ or $R^*(G) = K_p$ or $R^*(G) \in F_4$. Since by assumption $R^*(G) = G$, either $G = K_p$ or $G \in F_4$, which is a contradiction to $G \in F_{12} \cup F_{22} \cup F_{24}$. If $G \in F_3$ with G being a self-centered graph, then $R^*(G) = K_p$. That is $G = K_p$, which is a contradiction to $G \in F_3$. If $G \in F_3$ with $d(G) = r(G) + 1$, then by Lemma 4, $R^*(G) = \overline{G}$. But by our assumption $R^*(G) = G, G = \overline{G}$. Since $G \in F_3, d(\overline{G}) \leq 2$, which is contradiction to $G = \overline{G}$,

Suppose $G \in F_3$ with $r(G) + 2 \leq d(G) \leq 2r(G) - 1$, then by Lemma 6, $R^*(G) \in F_{22} \cup F_{23}$. Since by our assumption $R^*(G) = G, G \in F_{22} \cup F_{23}$, which is a contradiction to $G \in F_3$. Suppose $G \in F_3$ with $d(G) = 2r(G)$. Then by definition the center vertex in G is isolated in $R^*(G)$. Therefore $R^*(G) \in F_4$. By our assumption $R^*(G) = G, G \in F_4$, which is a contradiction to $G \in F_3$. Suppose $G \in F_4$. Then $R^*(G) \in F_{11} \cup F_{12} \cup F_{22}$. By our assumption $R^*(G) = G, G \in F_{11} \cup F_{12} \cup F_{22}$, which is a contradiction to $G \in F_4$. Therefore if $R^*(G) = G$ then either $G \in F_{11}$ or $G \in F_{23}$ with $G = \overline{G}$. ■

Motivated by the above proposition we state the following open problem.

Problem 8. Discuss the behaviour of the iterated sequence $G, R^*(G), R^*(R^*(G)), \ldots$.

Corollary 9. *A self-centered graph G is self super-radial if and only if $G \in F_{11}$.*

Proof. Let G be a self-centered graph. Suppose $G \in F_{11}$. Then $R^*(G) = K_p = G$. Therefore G is self super-radial graph. Conversely, suppose G is self super-radial graph. Then there exists a graph G such that $R^*(G) = G$. Now we claim that $G \in F_{11}$. Suppose $G \in F_{ii}$ where $i \geq 2$. Then by definition, $R^*(G) = K_p$, also by assumption $R^*(G) = G, G = K_p$, which is a contradiction to $G \in F_{ii}, i \geq 2$. Hence $G \in F_{11}$. ■

Lemma 10. *If G is a disconnected graph, then each component of $\overline{R^*(G)}$ is complete.*

Proof. Since G is a disconnected graph, by definition $R^*(G)$ is connected. Suppose u and v are two vertices of a component G_i of G. If $uv \in E(G_i)$, then $uv \notin E(R^*(G))$ and $uv \in E(\overline{R^*(G)})$.

Also, if $uv \notin E(G_i)$, then $uv \notin E(R^*(G))$ and $uv \in E(\overline{R^*(G)})$.

Therefore for any two vertices in a component G_i of G that are either adjacent or nonadjacent in G, that vertices are not adjacent in $R^*(G)$. But in $\overline{R^*(G)}$, the above two vertices are adjacent. This is true for any pair of vertices in the component G_i of G. Hence G_i is complete in $\overline{R^*(G)}$. ∎

Lemma 11. *Let* $G \in F_{12}$.

(i) *If each component of* $\overline{G}$ *is complete, then* G *is super-radial.*

(ii) *If at least one component of* $\overline{G}$ *is not complete, then* G *is not super-radial.*

Proof. (i) Since each component of $\overline{G}$ is complete, by Lemma 4, $R^*(G) = \overline{\overline{G}} = G$. That is $R^*(\overline{G}) = G$. Therefore G is super-radial.

(ii) Since $G \in F_{12}$ by Corollary 5, $R^*(G) = \overline{G}$, $\overline{G}$ is disconnected. Suppose $\overline{G}$ has at least one component which is not complete. Then by definition of super-radial $R^*(\overline{G}) \subset G$. Therefore neither $R^*(G) = G$ nor $R^*(\overline{G}) = G$. Let H be a graph such that $R^*(H) = G$, which is not isomorphic to G and $\overline{G}$.

Suppose H is a self-centered graph, then by Proposition 1, $R^*(H) = K_p, G = K_p$, which is a contradiction to $G \in F_{12}$. Suppose $H \in F_{23} \cup F_{24}$. Then $R^*(H) \in F_{22} \cup F_{23} \cup F_4$. By our assumption $R^*(H) = G, G \in F_{22} \cup F_{23} \cup F_4$, which is a contradiction to $G \in F_{12}$.

Suppose $H \in F_3$ with $d(H) = r(H) + 1$, then by Lemma 4, $R^*(H) = \overline{H}$. By our assumption $R^*(H) = G$, $G = \overline{H}, \overline{G} = H$. Since $G \in F_{12}, \overline{G}$ is disconnected, H is disconnected which is a contradiction to $H \in F_3$. Suppose $H \in F_3$ with $d(H) = 2r(H)$, then by definition $R^*(H) \in F_4$. By our assumption $R^*(H) = G, G \in F_4$ which is a contradiction to $G \in F_{12}$.

Suppose $H \in F_3$ with $r(H) + 2 \leq d(H) \leq 2r(H) - 1$, by Lemma 6, $R^*(H) \in F_{22} \cup F_{23}$. By our assumption $R^*(H) = G$, $G \in F_{22} \cup F_{23}$, which is a contradiction to $G \in F_{12}$.

Suppose $H \in F_4$. Then $R^*(H) \in F_{11} \cup F_{12} \cup F_{22}$. If $R^*(H) \in F_{11} \cup F_{22}$, by our assumption $R^*(H) = G, G \in F_{11} \cup F_{22}$, which is a contradiction to $G \in F_{12}$. If $R^*(H) \in F_{12}$, then by Lemma 10, each component of $\overline{R^*(H)}$ is complete. By our assumption $R^*(H) = G, \overline{R^*(H)} = \overline{G}$, each component of $\overline{G}$ is complete, which is a contradiction to our hypothesis $\overline{G}$ has at least one non complete component. Therefore $R^*(H) \notin F_{12}$. By all the above arguments there is no graph H such that $R^*(H) = G$.

Hence G is not super-radial graph. ∎

Lemma 12. *Let* $G \in F_{22}$.

(i) *If* $\overline{G} \in F_{22}$, *then* G *is not a super-radial graph.*

(ii) *If* $\overline{G} \in F_{23}$, *then* G *is a super-radial graph.*

(iii) *If* $\overline{G} \in F_{24}$, *then* G *is not a super-radial graph.*

(iv) *If* $\overline{G} \in F_3$, *then* G *is a super-radial graph if and only if* $d(\overline{G}) = r(\overline{G}) + 1$.

(v) *If* $\overline{G} \in F_4$, *then* G *is a super-radial graph if and only if each component of* $\overline{G}$ *is complete.*

Proof. (i) Since $\overline{G} \in F_{22}$, by Proposition 1, $R^*(G) = K_p$. Let H be a graph such that $R^*(H) = G$, which is not isomorphic to $\overline{G}$. Suppose that $H \in A = F_{11} \cup F_{12} \cup F_{22} \cup F_{23} \cup F_{24} \cup F_3 \cup F_4$. If $H \in F_{11}$, then by Proposition 1, $R^*(H) = K_p$. If $H \in F_{12}$, then by Corollary 5, $R^*(H) = \overline{H}$. But $\overline{H}$ is disconnected and $R^*(H) = G, G = \overline{H}$, G is disconnected, which is a contradiction to $G \in F_{22}$. Therefore $H \notin F_{12}$.

If H is a self-centered graph, then by Proposition 1, $R^*(H) = K_p$. Let $H \in F_{23}$ with $\overline{H} \in F_{23}$. Suppose $H = \overline{H}$, by Lemma 4, $R^*(H) = \overline{H}$ which implies $R^*(H) = H$. But by assumption $R^*(H) = G, H = G, G \in F_{23}$, which is a contradiction to $G \in F_{22}$.

Suppose $H \neq \overline{H}$, by Lemma 4, $R^*(H) = \overline{H}$. Since $R^*(H) = G, \overline{H} = G$, which implies $H = \overline{G}, \overline{G} \in F_{23}$, which is a contradiction to $\overline{G} \in F_{22}$. Let $H \in F_{23}$ with $\overline{H} \in F_{22}$. Since $H \in F_{23}$, by Lemma 4, $R^*(H) = \overline{H}$. Since by assumption $R^*(H) = G, G = \overline{H}$, implies $\overline{G} = H$, which is a contradiction to our assumption $H \neq \overline{G}$. Therefore $H \notin F_{23}$.

Suppose that $H \in F_{24}$. By Proposition 3, $R^*(H) \subseteq \overline{H}$. Also any vertex v such that $e(v) = 2$ in H is not adjacent to any vertex in $R^*(H)$. Hence $R^*(H)$ is disconnected. Also by assumption $R^*(H) = G$ implies G is disconnected, which is a contradiction to $G \in F_{22}$. Therefore $H \notin F_{24}$.

Suppose that $H \in F_3$. If $H \in F_3$ with $d(H) = r(H) + 1$, then by Lemma 4, $R^*(H) = \overline{H}$. By our assumption, $R^*(H) = G$ implies $G = \overline{H}, \overline{G} = H$, which is a contradiction to our assumption $H \neq \overline{G}$. If $H \in F_3$ with $d(H) = 2r(H)$ then by Proposition 3, $R^*(H) \subseteq \overline{H}$. Also any vertex v such that $e(v) = r(H)$ in H is not adjacent to any vertex in $R^*(H)$. Hence $R^*(H)$ is disconnected. Also by assumption $R^*(H) = G$ implies G is disconnected which is a contradiction to $G \in F_{22}$.

If $H \in F_3$ with $r(H) + 2 \leq d(H) \leq 2r(H) - 1$, then by Lemma 6, $R^*(H) \in F_{22} \cup F_{23}$. Suppose $R^*(H) \in F_{23}$. Since $R^*(H) = G, G \in F_{23}$, which is a contradiction to $G \in F_{22}$. Therefore $R^*(H) \notin F_{23}$. Suppose $R^*(H) \in F_{22}$. By hypothesis we have $G \in F_{22}$. If $R^*(H) = G$, then $\overline{R*(H)} = \overline{G}$. But by Lemma 6, $\overline{R^*(H)} \notin F_{22}$, which implies $\overline{G} \notin F_{22}$, which is a contradiction to $\overline{G} \in F_{22}$. Hence by the above arguments we conclude that there is no graph $H \in F_3$ such that $R^*(H) = G$.

If $H \in F_4$, then $R^*(H) \in F_{11} \cup F_{12} \cup F_{22}$. If $R^*(H) \in F_{11} \cup F_{12}$,, then by our assumption $R^*(H) = G$, $G \in F_{11} \cup F_{12}$, which is a contradiction to $G \in F_{22}$. Therefore, $R^*(H) \notin F_{11} \cup F_{12}$. If $R^*(H) \in F_{22}$ and by our assumption $R^*(H) = G$, then $\overline{R^*(H)} = \overline{G}$. Since $H \in F_4, \overline{R^*(H)} \in F_4$. Therefore, $\overline{G} \in F_4$, which is a contradiction to $\overline{G} \in F_{22}$. Therefore $H \notin F_4$.

Hence by all the above arguments, we conclude that there is no graph H such that $R^*(H) = G$. Therefore $G \in F_{22}$ with $\overline{G} \in F_{22}$, G is not a super-radial graph.

(ii) Since $\overline{G} \in F_{23}$, by Lemma 4, $R^*(\overline{G}) = \overline{\overline{G}} = G$. That is, $R^*(\overline{G}) = G$. Hence G is a super-radial graph.

(iii) Since $G \in F_{22}$, by Proposition 1, $R^*(G) = K_p$. Since $\overline{G} \in F_{24}$, by Proposition 3, $R^*(\overline{G}) \subseteq \overline{\overline{G}} = G$. But any vertex v such that $e(v) = 2$ in G is not adjacent to any vertex in $R^*(\overline{G})$. Hence $R^*(\overline{G})$ is disconnected.

Let H be a graph such that $R^*(H) = G$ which is not isomorphic to G and $\overline{G}$. Suppose that $H \in \mathcal{A} = F_{11} \cup F_{12} \cup F_{22} \cup F_{23} \cup F_{24} \cup F_3 \cup F_4$. If H is a self-centered graph, then $R^*(H) = K_p$. By our assumption $R^*(H) = G, G = K_p$, which is a contradiction to $G \in F_{22}$. If $H \in F_{12}$, then by Corollary 5, $R^*(H) = \overline{H}$. But $\overline{H}$ is disconnected and $R^*(H) = G, G = \overline{H}$, G is disconnected, which is a contradiction to $G \in F_{22}$. Therefore $H \notin F_{12}$. Suppose $H \in F_{23}$ with $\overline{H} \in F_{23}$. If $H = \overline{H}$, by Lemma 4, $R^*(H) = \overline{H}$ implies $R^*(H) = H$. But by assumption $R^*(H) = G, H = G$. Hence, $H \in F_{23}$ implies $G \in F_{23}$ which is a contradiction to $G \in F_{22}$.

If $H \neq \overline{H}$, then by Lemma 4, $R^*(H) = \overline{H}$. Since $R^*(H) = G$, $\overline{H} = G$ which implies $H = \overline{G}$. Since $H \in F_{23}, \overline{G} \in F_{23}$, which is a contradiction to $\overline{G} \in F_{24}$. Let $H \in F_{23}$ with $\overline{H} \in F_{22}$. Since $H \in F_{23}$, by Lemma 4, $R^*(H) = \overline{H}$. Since by our assumption $R^*(H) = G, G = \overline{H}$ implies $\overline{G} = H$, which is a contradiction to our assumption $\overline{G} \neq H$. Therefore $H \notin F_{23}$.

Suppose that $H \in F_{24}$. By Proposition 3, $R^*(H) \subseteq \overline{H}$. But any vertex v such that $e(v) = 2$ in H is not adjacent to any vertex in $R^*(H)$. That is $R^*(H)$ is disconnected. By our assumption $R^*(H) = G$ implies G is disconnected, which is a contradiction to $G \in F_{22}$. Therefore $H \notin F_{24}$.

If $H \in F_3$ with $d(H) = r(H) + 1$, then by Lemma 4, $R^*(H) = \overline{H}$. By our assumption, $R^*(H) = G$ implies $G = \overline{H}$, $\overline{G} = H$, which is a contradiction to our assumption $H \neq \overline{G}$. If $H \in F_3$ with $d(H) = 2r(H)$, then by Proposition 3, $R^*(H) \subseteq \overline{H}$. But any vertex v such that $e(v) = r(H)$ in H is not adjacent to any vertex in $R^*(H)$. That is $R^*(H)$ is disconnected. By our assumption $R^*(H) = G$, implies G is disconnected, which is a contradiction to $G \in F_{22}$. If $H \in F_3$ with $r(H) + 2 \leq d(H) \leq 2r(H) - 1$, then by Lemma 6, $R^*(H) \in F_{22} \cup F_{23}$. Suppose $R^*(H) \in F_{23}$. Since $R^*(H) = G, G \in F_{23}$, which is a contradiction to $G \in F_{22}$. Therefore $R^*(H) \notin F_{23}$.

Suppose $R^*(H) \in F_{22}$. By hypothesis we have $G \in F_{22}$. If $R^*(H) = G$ then $\overline{R^*(H)} = \overline{G}$. By Lemma 6, $\overline{R^*(H)} \notin F_{22} \cup F_{24}$, which implies $\overline{G} \notin F_{22} \cup F_{24}$. In particular, $\overline{G} \notin F_{24}$, which is a contradiction to $\overline{G} \in F_{24}$. Hence we conclude that there is no graph $H \in F_3$ such that $R^*(H) = G$.

If $H \in F_4$, then $R^*(H) \in F_{11} \cup F_{12} \cup F_{22}$. If $R^*(H) \in F_{11} \cup F_{12}$, by our assumption $R^*(H) = G$, $G \in F_{11} \cup F_{12}$, which is a contradiction to $G \in F_{22}$. If $R^*(H) \in F_{22}$

and by our assumption $R^*(H) = G$, then $\overline{R^*(H)} = \overline{G}$. By Lemma 10, $\overline{R^*(H)} \in F_4$ and implies $\overline{G} \in F_4$, which is a contradiction to $\overline{G} \in F_{24}$. Therefore $H \notin F_4$.

Hence by all the above arguments, we conclude that there is no graph H such that $R^*(H) = G$. Therefore, if $G \in F_{22}$ with $\overline{G} \in F_{24}$ is not a super-radial graph.

(iv) Suppose $\overline{G} \in F_3$ with $d(\overline{G}) = r(\overline{G}) + 1$. By Lemma 4, $R^*(\overline{G}) = \overline{\overline{G}} = G$. That is $R^*(\overline{G}) = G$. Therefore G is a super-radial graph. Conversely, suppose G is a super-radial graph. Then there exists a graph H such that $R^*(H) = G$. Suppose H is self-centered graph, then $R^*(H) = K_p$. Suppose $H \in F_{12} \cup F_{23}$, then by Lemma 4, $R^*(H) = \overline{H}$.

By our assumption $R^*(H) = G$, implies $G = \overline{H}$ implies $\overline{G} = H$. Therefore $\overline{G} \in F_{12} \cup F_{23}$, which is a contradiction to $\overline{G} \in F_3$. Suppose $H \in F_{24}$, then $R^*(H) \in F_4$. By our assumption $R^*(H) = G, G \in F_4$, which is a contradiction to $G \in F_{22}$. Suppose $H \in F_3$ with $d(H) = r(H)+1$. Then by Lemma 4, $R^*(H) = \overline{H}$. By our assumption $R^*(H) = G$ implies $\overline{H} = G$ implies, $H = \overline{G}$. That is $R^*(\overline{G}) = G$ with $d(\overline{G}) = r(\overline{G}) + 1$.

Suppose $H \in F_3$ with $r(H) + 2 \leq d(H) \leq 2r(H) - 1$. By Lemma 6, $R^*(H) \in F_{22} \cup F_{23}$. By our assumption $R^*(H) = G$ implies $G \in F_{22} \cup F_{23}$. Assuming $R^*(H) \in F_{23}$ implies $G \in F_{23}$, which is a contradiction to $G \in F_{22}$. If $R^*(H) \in F_{22}$ then by Lemma 6, $\overline{R^*(H)} \in F_{tt+1}$. Since $R^*(H) = G, \overline{R^*(H)} = \overline{G}, \overline{G} \in F_{tt+1}$. That is $d(\overline{G}) = r(\overline{G}) + 1$ which is a contradiction to our assumption $r(\overline{G}) + 2 \leq d(\overline{G}) \leq 2r(\overline{G}) - 1$. Therefore $H \notin F_3$ with $r(H) + 2 \leq d(H) \leq 2r(H) - 1$.

If $H \in F_3$ with $d(H) = 2r(H)$, then $R^*(H) \in F_4$. By our assumption $R^*(H) = G$ implies $G \in F_4$, which is a contradiction to $G \in F_{22}$. Suppose $H \in F_4$ then $R^*(H) \in F_{11} \cup F_{12} \cup F_{22}$. Since $G \in F_{22}$ and $R^*(H) = G$ implies $R^*(H) \in F_{22}$. Then $\overline{R^*(H)} \in F_4$, which implies $\overline{G} \in F_4$, which is a contradiction to $\overline{G} \in F_3$. By all the above argument, there is no graph H such that $R^*(H) \in G$. Therefore, if $\overline{G} \in F_3$, then G is a super-radial graph if and only if $d(\overline{G}) = r(\overline{G})+1$.

(v) Since $G \in F_{22}$, by Proposition 1, $R^*(G) = K_p$. Suppose $\overline{G} \in F_4$ with each component of $\overline{G}$ is complete. Then by Lemma 4, $R^*(\overline{G}) = \overline{\overline{G}} = G$. That is $R^*(\overline{G}) = G$. Hence G is a super-radial graph. Conversely, suppose G is a super-radial graph. Then there exists a graph H such that $R^*(H) = G$. Since $G \in F_{22}$, by Proposition 1, $R^*(G) = K_p$. Therefore $H \neq G$.

Suppose $\overline{G} \in F_4$ with at least one non complete componen, then $R^*(\overline{G}) \subset G$. Therefore $H \neq \overline{G}$. Suppose $\overline{G} \in F_4$ with each component of $\overline{G}$ is complete. Then by Lemma 4, $R^*(\overline{G}) = \overline{\overline{G}} = G$. That is $R^(\overline{G}) = G$. By our assumption $R^*(H) = G$ implies $R^*(H) = R^*(\overline{G})$ implies $H = \overline{G}$.

Suppose H is a self-centered graph. Then by Proposition 1, $R^*(H) = K_p, G = K_p$, which is a contradiction to $G \in F_{22}$. Suppose H satisfies $d(H) = r(H) + 1$, then by Lemma 4, $R^*(H) = \overline{H}$. By our assumption $R^*(H) = G$, $G = \overline{H}$ implies $\overline{G} = H$. But $\overline{G}$ is disconnected, H is disconnected, which is a contradiction to

$d(H) = r(H) + 1$.

Suppose H satisfies $d(H) = 2r(H)$. Then $R^*(H) \in F_4$. By our assumption $G \in F_4$ which is a contradiction to $G \in F_{22}$. Suppose H with $r(H)+2 \le d(H) \le 2r(H) - 1$. Then by Lemma 6, $R^*(H) \in F_{22} \cup F_{23}$. If $R^*(H) \in F_{23}$, then by our assumption $G \in F_{23}$ which is a contradiction to $G \in F_{22}$. If $R^*(H) \in F_{22}$, then by Lemma 6, $\overline{R^*(H)} \in F_{tt+1}$. By our assumption $R^*(H) = G$ implies $\overline{R^*(H)} = \overline{G}$. Therefore $\overline{G} \in F_{tt+1}$ which is a contradiction to $\overline{G} \in F_4$.

Suppose $H \in F_4$. Then $R^*(H) \in F_{11} \cup F_{12} \cup F_{22}$. Since by our assumption, $R^*(H) = G$, $G \in F_{11} \cup F_{12} \cup F_{22}$. By hypothesis $G \in F_{22}$ which implies $G \notin F_{11} \cup F_{12}$. Suppose $R^*(H) \in F_{22}$ and $G \in F_{22}$. But $H \neq \overline{G}$ implies $\overline{H} \neq G$. That is $R^*(H) \neq \overline{H}$. By Lemma 4, $d(H) \neq r(H) + 1$ or H is disconnected in which at least one component is non complete. Therefore, if $\overline{G} \in F_4$ then each component of $\overline{G}$ is complete if and only if G is a super-radial graph. ∎

Lemma 13. *Let $G \in F_{23}$.*

(i) *If $\overline{G} \in F_{22}$, then G is not a super-radial graph.*

(ii) *If $\overline{G} \in F_{23}$, then G is a super-radial graph.*

Proof. (i) Since $G \in F_{23}$, by Lemma 4, $R^*(G) = \overline{G}$. Since $\overline{G} \in F_{22}$, by Proposition 1, $R^*(\overline{G}) = K_p$. Let H be a graph such that $R^*(H) = G$, which is not isomorphic to G and $\overline{G}$. If H is a self-centered graph then by Proposition 1, $R^*(H) = K_p, G = K_p$, which is a contradiction to $G \in F_{23}$. Suppose H with $d(H) = r(H) + 1$, then by Lemma 4, $R^*(H) = \overline{H}$. By our assumption $R^*(H) = G, G = \overline{H}$ implies $\overline{G} = H$ which is a contradiction to $H \neq \overline{G}$.

Suppose H with $d(H) = 2r(H)$. Then $R^*(H) \in F_4$. By our assumption $R^*(H) = G, G \in F_4$, which is a contradiction. Suppose H with $r(H) + 2 \le d(H) \le 2r(H) - 1$. Then by Lemma 6, $R^*(H) \in F_{22} \cup F_{23}$. By our assumption $R^*(H) = G, G \in F_{22} \cup F_{23}$. If $R^*(H) \in F_{22}$, then $G \in F_{22}$, a contradiction to $G \in F_{23}$. If $R^*(H) \in F_{23}$, then $G \in F_{23}$. Suppose $R^*(H) = G$ implies $\overline{R^*(H)} = \overline{G}$. Since by Lemma 6, $\overline{R^*(H)} \in F_{tt+1}$ implies $\overline{G} \in F_{tt+1}$, which is a contradiction to $\overline{G} \in F_{22}$. Therefore $H \notin F_3$ with $r(H) + 2 \le d(H) \le 2r(H) - 1$.

Suppose $H \in F_4$. Then $R^*(H) \in F_{11} \cup F_{12} \cup F_{22}$. Since by our assumption, $R^*(H) = G$, which implies $G \in F_{11} \cup F_{12} \cup F_{22}$, which is a contradiction to $G \in F_{23}$. Hence there is no graph H such that $R^*(H) = G$. Therefore if $G \in F_{23}$ with $\overline{G} \in F_{22}$, then G is not a super-radial graph.

(ii) Since $G \in F_{23}$, by Lemma 4, $R^*(G) = \overline{G}$. Since $\overline{G} \in F_{23}$, by Lemma 4, $R^*(\overline{G}) = \overline{\overline{G}} = G$. Hence G is a super-radial graph. ∎

Lemma 14. *If $G \in F_{24}$, then G is not a super-radial graph.*

Proof. Since $G \in F_{24}, \overline{G} \in F_{22}$. Since $G \in F_{24}$, by definition $R^*(G) \in F_4$. Let H be a graph such that $R^*(H) = G$, which is not isomorphic to G. Suppose H is a self-centered graph. Then by Proposition 1, $R^*(H) = K_p$ and by our assumption $G = K_p$, which is a contradiction to $G \in F_{24}$. Suppose H with $d(H) = r(H) + 1$. Then by Lemma 4, $R^*(H) = \overline{H}$ and by our assumption $G = \overline{H}$ it implies $\overline{G} = H$. Since $d(H) = r(H) + 1$ implies $d(\overline{G}) = r(\overline{G}) + 1$, which is a contradiction to $\overline{G} \in F_{22}$.

Suppose H with $d(H) = 2r(H)$. Then $R^*(H) \in F_4$ and by our assumption $G \in F_4$, which is a contradiction to $G \in F_{24}$. Suppose H with $r(H)+2 \leq d(H) \leq 2r(H)-1$. Then by Lemma 6, $R^*(H) \in F_{22} \cup F_{23}$. By our assumption, $R^*(G) = G$ implies $G \in F_{22} \cup F_{23}$, which is a contradiction to $G \in F_{24}$. Suppose $H \in F_4$. Then $R^*(H) \in F_{11} \cup F_{12} \cup F_{22}$. By our assumption $R^*(H) = G, G \in F_{11} \cup F_{12} \cup F_{22}$, which is a contradiction to $G \in F_{24}$. Hence there is no graph H such that $R^*(H) = G$. Therefore $G \in F_{24}$ is not a super-radial graph. ■

Lemma 15. *If $G \in F_3$, then G is not a super-radial graph.*

Proof. Suppose $G \in F_3$ is a super-radial graph. Then there exists a graph H such that $R^*(H) = G$. If $H \in F_{11} \cup F_{12} \cup F_{22} \cup F_{23} \cup F_{24}$, then by previous argument $R^*(H) \in F_{11} \cup F_{22} \cup F_{23} \cup F_4$. By our assumption $R^*(H) = G, G \in F_{11} \cup F_{22} \cup F_{23} \cup F_4$, which is a contradiction to $G \in F_3$. Therefore $H \notin F_{11} \cup F_{12} \cup F_{22} \cup F_{23} \cup F_{24}$. Suppose $H \in F_3$ with $d(H) = r(H) + 1$. Then by Lemma 4, $R^*(H) = \overline{H}$. By our assumption $R^*(H) = G$ implies $G = \overline{H}$, which implies $\overline{G} = H$. Since $H \in F_3$ with $d(H) = r(H)+1, d(H) \geq 4$ and by Theorem B, $d(\overline{H}) \leq 2$. Since $G = \overline{H}, d(G) \leq 2$, which is a contradiction to $G \in F_3$.

Suppose $H \in F_3$ with H is self-centered graph. Then $R^*(H) = K_p$. By our assumption $R^*(H) = G, G = K_p$, which is a contradiction to $G \in F_3$. Suppose $H \in F_3$ with $d(H) = 2r(H)$. Then $R^*(H) \in F_4$. By our assumption $R^*(H) = G, G \in F_4$, which is a contradiction to $G \in F_3$. Suppose $H \in F_3$ with $r(H) + 2 \leq d(H) \leq 2r(H) - 1$. Then by Lemma 6, $R^*(H) \in F_{22} \cup F_{23}$. By our assumption $R^*(H) = G, G \in F_{22} \cup F_{23}$, which is a contradiction to $G \in F_3$. Suppose $H \in F_4$. Then $R^*(H) \in F_{11} \cup F_{12} \cup F_{22}$. By our assumption $R^*(H) = G, G \in F_{11} \cup F_{12} \cup F_{22}$, which is a contradiction to $G \in F_3$. By all the above arguments, there exists no graph H such that $R^*(H) = G$. Hence $G \in F_3$ is not a super-radial graph. ■

Lemma 16. *If $G \in F_4$ and $\overline{G} \in F_{11} \cup F_{22}$, then G is not a super-radial graph.*

Proof. Since $\overline{G} \in F_{11} \cup F_{22}$, then $R^*(\overline{G}) = K_p$. Suppose there exists a graph H such that $R^*(H) = G$, which is not isomorphic to $\overline{G}$.

Case (i). Suppose H is a self-centered graph. Then by Proposition 1, $R^*(H) = K_p$. By our assumption $R^*(H) = G, G = K_p$, which is a contradiction to $G \in F_4$.

Case (ii). Suppose H with $d(H) = r(H)+1$. Then by Lemma 4, $R^*(H) = \overline{H}$. By our assumption $R^*(H) = G$, $G = \overline{H}$ implies $\overline{G} = H$. By hypothesis $\overline{G} \in F_{11} \cup F_{22}$ implies $H \in F_{11} \cup F_{22}$, which is a contradiction to $d(H) = r(H) + 1$.

Case (iii). Suppose H with $r(H)+2 \leq d(H) \leq 2r(H)-1$. Then by Lemma 6, $R^*H) \in F_{22} \cup F_{23}$. By our assumption $R^*(H) = G$ implies $G \in F_{22} \cup F_{23}$, which is a contradiction to $G \in F_4$.

Case (iv). Suppose H with $d(H) = 2r(H)$. Then $d(H) - r(H) + 1 = 2r(H) - r(H) + 1 = r(H) + 1$. Clearly, every vertex with eccentricity $r(H)$ in H is isolated vertex in $R^*(H)$. Therefore, $R^*(H) \in F_4$.

In $\overline{R^*(H)}$, every isolated vertex in $R^*(H)$ is adjacent to all the vertices of $R^*(H)$. Therefore, $\overline{R^*(H)} \in F_{12}$. By our assumption $R^*(H) = G$. $\overline{R^*(H)} = \overline{G}$ implies $\overline{G} \in F_{12}$, which is a contradiction to $\overline{G} \in F_{11} \cup F_{22}$.

Case (v). Suppose $H \in F_4$. Then $R^*(H) \in F_{11} \cup F_{12} \cup F_{22}$. By our assumption $R^*(H) = G$ implies $G \in F_{11} \cup F_{12} \cup F_{22}$ which is a contradiction to $G \in F_4$.

Hence by all the above arguments, $G \in F_4$ and $\overline{G} \in F_{11} \cup F_{22}$ is not a super-radial graph. ■

Theorem 17. *A connected graph G is super-radial graph if and only if G has any one of the following properties.*

(i) $G \in F_{11}$,

(ii) $G \in F_{12}$ *with each component of* $\overline{G}$ *being complete,*

(iii) $G \in F_{22}$ *with* $\overline{G} \in F_{23}$,

(iv) $G \in F_{22}$ *and* $\overline{G} \in F_3$ *with* $d(\overline{G}) = r(\overline{G}) + 1$,

(v) $G \in F_{22}$ *and* $\overline{G} \in F_4$ *with each component of* $\overline{G}$ *being complete,*

(iv) $G \in F_{23}$ *with* $\overline{G} \in F_{23}$.

Proof. As the following table exhausts all the possibilities, we get the theorem.

	G	$\overline{G}$	**By Lemma/ Proposition**	**G is super-radial**
1	F_{11}	F_4	8	Yes
2	F_{12}	Each component of $\overline{G}$ is complete.	12(i)	Yes
		At least one component of $\overline{G}$ is not complete.	12(ii)	No
3	F_{22}	F_{22}	13(i)	No
		F_{23}	13(ii)	Yes
		F_{24}	13(iii)	No
		F_3 with $d(\overline{G}) = r(\overline{G}) + 1$	13(iv)	Yes
		F_3 with $d(\overline{G}) \neq r(\overline{G}) + 1$	13(iv)	No

		F_4 with each component of $\overline{G}$ beingcomplete	13(v)	Yes
		F_4 with at least one component of $\overline{G}$ being non complete	13(v)	No
4	F_{23}	F_{22}	14(i)	No
		F_{23}	14(ii)	Yes
5	F_{24}	F_{22}	15	No
6	F_3		16	No

■

Theorem 18. *A disconnected graph G is a super-radial graph if and only if* $\overline{G} \in F_{12}$.

Proof. Since G is disconnected, $\overline{G} \in F_{11} \cup F_{12} \cup F_{22}$. If $\overline{G} \in F_{11} \cup F_{22}$, then by Lemma 16, G is not a super-radial graph. If $\overline{G} \in F_{12}$, then by Lemma 4, $R^*(\overline{G}) = \overline{\overline{G}} = G$. That is $R^*(\overline{G}) = G$. Hence G is a super-radial graph. ■

The following examples show that the notion of super-radial graph is independent of radial graph, antipodal graph, eccentric graph and super-eccentric graph.

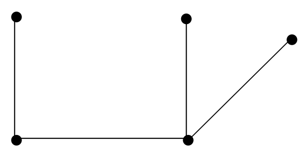


Figure 2. Super-radial graph but not antipodal graph.

Figure 3. Antipodal graph but not super-radial graph.

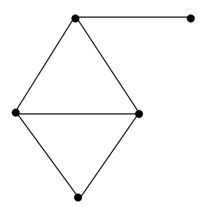


Figure 4. Super-radial graph but not eccentric graph.

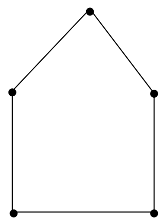


Figure 5. Eccentric graph but not super-radial graph.

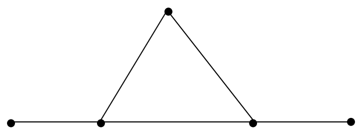


Figure 6. Super-radial graph but not radial graph.

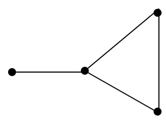


Figure 7. Radial graph but not super-radial graph.

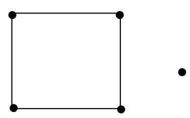


Figure 8. Super-radial graph but not super-eccentric graph.

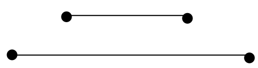


Figure 9. Super-eccentric graph but not super-radial graph.

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