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AN IMPLICIT WEIGHTED DEGREE CONDITION FOR HEAVY CYCLES[1]

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Abstract

For a vertex v in a weighted graph G, $id^w(v)$ denotes the implicit weighted degree of v. In this paper, we obtain the following result: Let G be a 2-connected weighted graph which satisfies the following conditions: (a) The implicit weighted degree sum of any three independent vertices is at least t; (b) $w(xz) = w(yz)$ for every vertex $z \in N(x) \cap N(y)$ with $xy \notin E(G)$; (c) In every triangle T of G, either all edges of T have different weights or all edges of T have the same weight. Then G contains either a hamiltonian cycle or a cycle of weight at least $2t/3$. This generalizes the result of Zhang *et al*. [9].

Keywords: weighted graph, hamiltonian cycles, heavy cycles, implicit degree, implicit weighted degree.

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1. Introduction

In this paper, we consider only finite, undirected and simple graphs. Notation and terminology not defined here can be found in [2]. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $H \subseteq G$. For a vertex $u \in V(G)$, $N_H(u) = \{v \in V(H) : uv \in E(G)\}$ and $d_H(u) = |N_H(u)|$. If $H = G$, we always use $N(u)$ and $d(u)$ in place of $N_G(u)$ and $d_G(u)$ respectively. $N_2(v) = \{u \in V(G) : d(u,v) = 2\}$, where $d(u,v)$ denotes the distance between vertices u and v in G

Based on the traditional definition of degree, Zhu, Li and Deng [10] introduced the concept of implicit degrees.

Definition [10]**.** Let v be a vertex of a graph G and $k = d(v) - 1$. Set $M_2 = \max\{d(u) : u \in N_2(v)\}$ and $m_2 = \min\{d(u) : u \in N_2(v)\}$. Suppose $d_1 \leq d_2 \leq d_3 \leq \cdots \leq d_k \leq d_{k+1} \leq \cdots$ is the degree sequence of vertices in $N(v) \cup N_2(v)$. If $N_2(v) \neq \emptyset$ and $d(v) \geq 2$, define

$$d^*(v) = \begin{cases} m_2, & \text{if} \quad d_k < m_2, \\ d_{k+1}, & \text{if} \quad d_{k+1} > M_2, \\ d_k, & \text{otherwise}, \end{cases}$$

then the *implicit degree* of v is defined as $id(v) = \max\{d(v), d^*(v)\}$. If $N_2(v) = \emptyset$ or $d(v) \leq 1$, then $id(v) = d(v)$.

Clearly, $id(v) \geq d(v)$ for each vertex v from the definition of implicit degree.

For a graph G, if we assign a non-negative number $w(e)$ to every edge e, then G is called a weighted graph and $w(e)$ is the weight of e. Clearly, an unweighted graph can be regarded as a weighted graph in which each edge is assigned a weight 1. The weight of a subgraph H of G and the weighted degree of a vertex v in G are defined as

$$w(H) = \sum\nolimits_{e \in E(H)} w(e) \quad \text{and} \quad d^w(v) = \sum\nolimits_{u \in N(v)} w(uv), \quad \text{respectively.}$$

Based on the idea of the definition of implicit degree, Li [7] gave the definition of implicit weighted degrees as follows.

Definition [7]**.** Let v be a vertex of a weighted graph G and $k = d(v) - 1$. Set $m_2^w = \min\{d^w(u) : u \in N_2(v)\}$ and $M_2^w = \max\{d^w(u) : u \in N_2(v)\}$. Suppose $d_1^w \leq d_2^w \leq \cdots \leq d_{k+1}^w \leq \cdots$ is the weighted degree sequence of vertices in $N(v) \cup N_2(v)$. If $N_2(v) \neq \emptyset$ and $d(v) \geq 2$, define

$$d^{w*}(v) = \begin{cases} m_2^w, & \text{if} \quad d_k^w < m_2^w, \\ d_{k+1}^w, & \text{if} \quad d_{k+1}^w > M_2^w, \\ d_k^w, & \text{otherwise}, \end{cases}$$

then the implicit weighted degree of v is defined as $id^w(v) = \max\{d^w(v), d^{w*}(v)\}$. If $N_2(v) = \emptyset$ or $d(v) \leq 1$, then $id^w(v) = d^w(v)$.

Clearly, $id^w(v) \geq d^w(v)$ for every vertex v from the above definition.

Let $\alpha(G)$ be the independent number of a graph G. For a positive integer $k \leq \alpha(G)$, we define $\sigma_k(G) = \min\{\sum_{j=1}^{k} d(x_j) : x_1, x_2, \ldots, x_k$ are k independent vertices in $G\}$ and $\sigma_k^*(G) = \min\{\sum_{j=1}^{k} id(x_j) : x_1, x_2, \ldots, x_k$ are k independent vertices in $G\}$. For a weighted graph G, let $\sigma_k^w(G) = \min\{\sum_{j=1}^{k} d^w(x_j) : x_1, x_2, \ldots, x_k$ are k independent vertices in $G\}$ and $\sigma_k^{w*}(G) = \min\{\sum_{j=1}^{k} id^w(x_j) : x_1, x_2, \ldots, x_k$ are k independent vertices in $G\}$. If $k > \alpha(G)$, then they are all equal to $+\infty$.

A graph G is called hamiltonian if it has a hamiltonian cycle, i.e., a cycle that contains all vertices of G. There are many results about the existence of hamiltonian cycles and long cycles in graphs in terms of the degree sum of independent vertices. The following two theorems are famous.

Theorem 1 [4]**.** *If G is a 2-connected graph with $\sigma_1(G) \geq c/2$, then G contains either a hamiltonian cycle or a cycle of length at least c.*

Theorem 2 [8]**.** *If G is a 2-connected graph with $\sigma_2(G) \geq c$, then G contains either a hamiltonian cycle or a cycle of length at least c.*

After studying $\sigma_1(G)$ and $\sigma_2(G)$ conditions, Fournier and Fraisse [6] generalized them into degree conditions on more independent vertices, i.e., the $\sigma_k(G)$.

Theorem 3 [6]**.** *If G is a k-connected graph ($k \geq 2$) with $\sigma_{k+1}(G) \geq c$, then G contains either a hamiltonian cycle or a cycle of length at least $2c/(k+1)$.*

Zhang, Li and Broersma [9] gave a result about heavy cycles on weighted degree condition. The result extended Theorem 3 in the case $k = 2$ by adding two extra conditions.

(C1) $w(xz) = w(yz)$ for every vertex $z \in N(x) \cap N(y)$ with $xy \notin E(G)$;

(C2) For every triangle T of G, either all edges of T have different weights or all edges of T have the same weight.

Theorem 4 [9]**.** *Suppose G is a 2-connected weighted graph satisfying conditions* (C1) *and* (C2)*. If $\sigma_3^w(G) \geq t$, then G contains either a hamiltonian cycle or a cycle of weight at least $2t/3$.*

Motivated by the result of Theorem 4, we obtain the following result.

Theorem 5. *Let G be a 2-connected weighted graph satisfying conditions* (C1) *and* (C2)*. If $\sigma_3^{w*}(G) \geq t$, then G contains either a hamiltonian cycle or a cycle of weight at least $2t/3$.*

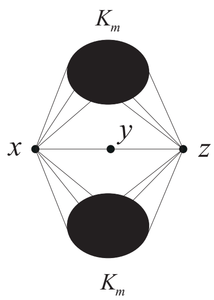

Figure 1. Example 1.

We give the proof of Theorem 5 in the next section. Here we give a graph G with $\sigma_3^{w*}(G) > \sigma_3^w(G)$. It is said that we can get a heavier cycle by using Theorem 5 than using Theorem 4.

Example. Let G be a graph shown in Figure 1, where K_m ($m \geq 2$) is a complete graph and x, z are adjacent to every vertex of each K_m, $N(y) = \{x, z\}$. We assign weight 2 to each edge of G. It is easy to verify that $\sigma_3^w(G) = 4m + 8$. And by the definition of implicit weighted degree, we get that $\sigma_3^{w*}(G) = 6(m+1)$.

2. Proof of Theorem 5

Our proof of Theorem 5 is based on the following three lemmas.

Lemma 6 [5]**.** *Let G ba a connected weighted graph satisfying conditions* (C1) *and* (C2)*. Then either*

(a) *all edges of G have the same weight, or*

(b) *G is a complete multipartite graph.*

Lemma 7 [3]**.** *Let G be a k-connected graph with at least three vertices. If $k \geq \alpha(G)$, then G is hamiltonian.*

Lemma 8. *Let G be a 2-connected graph such that $\sigma_3^*(G) \geq c$, then G contains either a hamiltonian cycle or a cycle of length at least $2c/3$.*

We give the proof of Lemma 8 in the next section.

***Proof of Theorem* 5.** Let G be a weighted graph satisfying the conditions of Theorem 5. If $\alpha(G) \leq 2$, then G is hamiltonian by Lemma 7.

Hence we assume $\alpha(G) \geq 3$. Then by Lemma 6, either all edges of G have the same weight or G is a complete multipartite graph.

First we suppose all edges of G have the same weight m. When $m = 0$, there is nothing to say. Suppose $m \neq 0$. By the definitions of implicit degree and implicit weighted degree, we have $id^w(v) = m(id(v))$ for each $v \in V(G)$. Hence, $\sigma_3^*(G) = \sigma_3^{w*}(G)/m \geq t/m$. Then, G contains either a hamiltonian cycle or a cycle C of length at least $2t/3m$ by Lemma 8. If G is not hamiltonian , then $w(C) = m \times |E(C)| \geq m \times (2t/3m) = 2t/3$.

Now, we assume that G is a complete multipartite graph and G is not hamiltonian . Let $|V(G)| = n$ and $V_1, V_2, \ldots, V_l$ be a partition of $V(G)$.

Claim 2.1 [5]**.** *If $x, y \in V_i$, then $w(xz) = w(yz)$ for every $z \in V(G) \setminus V_i$. In particular, $d^w(x) = d^w(y)$.*

Claim 2.2 [5]**.** *If G is not hamiltonian , then $|V_i| > n/2$ for some i with $1 \leq i \leq l$.*

We can assume, without loss of generality, that $|V_1| > n/2$ by Claim 2.2. Let $p = |V_1|$ and $q = n - p$. Since G is 2-connected, $2 \leq q < p$. And let $V_1 = \{v_1, v_2, \ldots, v_p\}$ and $V(G) \setminus V_1 = \{u_1, u_2, \ldots, u_q\}$.

Claim 2.3. *$id^w(x) = id^w(y)$ for any two distinct vertices $x, y \in V_i$.*

Proof. Since $x, y \in V_i$, $N(x) = N(y) = V(G) \backslash V_i$ and $N_2(x) \backslash \{y\} = N_2(y) \backslash \{x\} = V_i \setminus \{x, y\}$. By Claim 2.1, we know $d^w(x) = d^w(y)$ and the weighted degree sequences of vertices in $N(x) \cup N_2(x)$ and those of vertices in $N(y) \cup N_2(y)$ are the same. Then by the definition of implicit weighted degree, we can get that $id^w(x) = id^w(y)$. □

Claim 2.4. *$id^w(v) \geq t/3$ for each $v \in V_1$.*

Proof. Since $p > 2$, there are three vertices v_1, v_2 and v_3 in V_1 and $\{v_1, v_2, v_3\}$ is an independent set. Hence $id^w(v_1) + id^w(v_2) + id^w(v_3) \geq t$. We assume, without loss of generality, that $id^w(v_1) \geq t/3$. Then $id^w(v) = id^w(v_1) \geq t/3$ for each $v \in V_1$ by Claim 2.3. □

Claim 2.5. *$id^w(v) = d^w(v)$ for any $v \in V_1$.*

Proof. Suppose there is some $v \in V_1$ such that $d^w(v) < id^w(v)$. Since G is a complete multipartite graph, $N_2(v) = V_1 \backslash \{v\}$ and $N(v) = V(G) \backslash V_1$. Since $|V_1| > n/2$, $|N_2(v)| \geq |N(v)|$. Let $k = d(v) - 1$, $M_2^w = \max\{d^w(u) : u \in N_2(v)\}$ and $m_2^w = \min\{d^w(u) : u \in N_2(v)\}$. Suppose $d_1^w \leq d_2^w \leq \cdots \leq d_{k+1}^w \leq \cdots$ is the weighted degree sequence of vertices of $N(v) \cup N_2(v)$. By Claim 2.1 and the definition of $id^w(v)$, we have $M_2^w = m_2^w$, $d_1^w \leq d_2^w \leq \cdots \leq d_{k+1}^w < id^w(v)$ and $id^w(v) \neq m_2^w$. Which is contrary to the definition of $id^w(v)$. So $id^w(v) = d^w(v)$ for any $v \in V_1$. □

By Claims 2.4 and 2.5, we can get that $d^w(v) \geq t/3$ for each $v \in V_1$. Now, we consider the cycle $C = v_1u_1v_2u_2 \cdots v_qu_qv_1$. Then Claim 2.1 implies

$$\begin{aligned} w(C) &= \sum_{i=1}^{q} w(v_iu_i) + \sum_{i=1}^{q-1} w(v_{i+1}u_i) + w(v_1u_q) \\ &= \sum_{i=1}^{q} w(v_1u_i) + \sum_{i=1}^{q-1} w(v_1u_i) + w(v_1u_q) \\ &= 2\sum_{i=1}^{q} w(v_1u_i) = 2d^w(v_1). \end{aligned}$$

Hence, $w(C) \geq 2t/3$. This completes the proof of Theorem 5. ■

3. Proof of Lemma 8

Let $P = x_1x_2 \cdots x_p$ be a path of a graph G, for any $I \subseteq V(P)$, define $I^- = \{x_i : x_{i+1} \in I\}$ and $I^+ = \{x_i : x_{i-1} \in I\}$. We use $L_P(x_1)$ to denote the vertex x_i such that $x_1x_i \in E(G)$ and $x_1x_j \notin E(G)$ for any $j > i$ on P. To prove Lemma 8 we need the following two lemmas.

Lemma 9 [1]**.** *Let G be a 2-connected non-hamiltonian graph and $P = x_1x_2 \cdots x_p$ be a longest path of G. Then G contains a cycle of length at least $d(x_1) + d(x_p)$.*

Lemma 10 [10]**.** *Let G be a 2-connected graph and $P = x_1x_2 \cdots x_p$ be a longest path of G. If $d(x_1) < id(x_1)$ and $x_1x_p \notin E(G)$, then either*

(1) *there is some $x_j \in N_P(x_1)^-$ such that $d(x_j) \geq id(x_1)$; or*

(2) *$N_P(x_1)^- = N_P(x_1) \cup \{x_1\} \setminus \{L_P(x_1)\}$ and $id(x_1) = \min\{d(v) : v \in N_2(x_1)\}$.*

Proof of Lemma 8. Let G be a 2-connected graph satisfying the conditions of Lemma 8 and suppose G is not hamiltonian. Then by Lemma 9, it suffices to prove that there exists a longest path $P = v_1v_2 \cdots v_p$ in G such that $d(v_1)+d(v_p) \geq 2c/3$.

Suppose to the contrary that $d(v_1) + d(v_p) < 2c/3$ for any longest path $P = v_1v_2 \cdots v_p$. Choose a longest path $P = v_1v_2 \cdots v_p$ such that $d(v_1) + d(v_p)$ is as large as possible. Then $N(v_1) \cup N(v_p) \subseteq V(P)$ and there is no cycle of length p in G. We can assume, without loss of generality, that $d(v_1) < c/3$. Since G is 2-connected, $|N_P(v_1)| \geq 2$ and $|N_P(v_p)| \geq 2$. Let $k = \max\{i : v_1v_i \in E(G)\}$ and $l = \min\{j : v_pv_j \in E(G)\}$. Then $3 \leq k \leq p-1$ and $2 \leq l \leq p-2$.

Case 1. $N(v_1) = \{v_2, v_3, \ldots, v_k\}$.
Since $G - v_k$ is connected, there exists an edge $v_rv_s \in E(G)$ with $r < k < s$. Choose such an edge v_rv_s such that s is as large as possible. By the choice of P, we have $N(v_r) \subseteq V(P)$. Let

$$P_1 = v_rv_{r-1} \cdots v_1v_{r+1}v_{r+2} \cdots v_p.$$

Since G has no cycle of length p, $v_rv_p \notin E(G)$. Next, we are going to look for a special vertex $v_j \in V(P)$ such that $\{v_r, v_j, v_p\}$ is an independent set of G.

Subcase 1.1. There is some j with $r+1 \leq j \leq s-1$ such that $v_rv_{j+1} \in E(G)$ and $v_rv_j \notin E(G)$. We consider the longest path $P_2 = v_jv_{j-1}\cdots v_{r+1}v_1v_2\cdots v_rv_{j+1}$ $v_{j+2}\cdots v_p$. Since G has no cycle of length p, we have $v_jv_p \notin E(G)$. Then we have found the vertex v_j.

Subcase 1.2. $v_rv_i \in E(G)$ for each i with $r+1 \leq i \leq s$. By the choice of P, $r \geq 4$ and there must exist some j with $2 \leq j \leq r-2$ such that $v_rv_j \notin E(G)$. (Since, otherwise, P_1 is a longest path with $d(v_r) + d(v_p) > d(v_1) + d(v_p)$, a contradiction.) We consider the longest path $P_3 = v_jv_{j-1}\cdots v_1v_{j+1}v_{j+2}\cdots v_p$. Then $v_jv_p \notin E(G)$. So we find the vertex v_j.

Claim 3.1. *$id(v_r) \leq d(v_1)$ and $id(v_j) \leq d(v_1)$.*

Proof. Suppose $id(v_r) > d(v_1)$. We consider the longest path P_1 defined as before. If $d(v_r) = id(v_r)$, then $d(v_r) + d(v_p) > d(v_1) + d(v_p)$, a contradiction.

Suppose $d(v_r) < id(v_r)$. For convenience, we set $P_1 = y_1y_2\cdots y_p$, where $y_1 = v_r, y_2 = v_{r-1}, \ldots, y_p = v_p$. Since $v_rv_s \in E(G)$ and $v_rv_j \notin E(G)$, we have $N_{P_1}(v_r)^- \neq N_{P_1}(v_r) \cup \{v_r\} \setminus \{L_{P_1}(v_r)\}$. Hence, by Lemma 10, there exists some $y_a \in N_{P_1}(y_1)^-$ such that $d(y_a) \geq id(y_1)$. Then $y_ay_{a-1}\cdots y_1y_{a+1}y_{a+2}\cdots y_p$ is a longest path of G with $d(y_a) + d(y_p) \geq id(y_1) + d(y_p) = id(v_r) + d(v_p) > d(v_1) + d(v_p)$, which is contrary to the choice of P. Hence $id(v_r) \leq d(v_1)$.

Suppose $id(v_j) > d(v_1)$. If v_j is got in Subcase 1.1, we consider P_2. If v_j is got in Subcase 1.2, we consider P_3 (where P_2 and P_3 are defined as before). No matter what cases happen, we have $d(v_r, v_j) = 2$. By the choice of P, we get that $d(v_r) \leq d(v_1)$. So $id(v_j) \neq \min\{d(u) : u \in N_2(v_j)\}$. By similar arguments as above, we get contradictions. So $id(v_j) \leq d(v_1)$. □

Since $\{v_r, v_j, v_p\}$ is an independent set of G, we have $id(v_r) + id(v_j) + id(v_p) \geq c$. So $id(v_p) \geq c - id(v_r) - id(v_j) \geq c - 2d(v_1)$ by Claim 3.1. If $d(v_p) = id(v_p)$, then $d(v_1) + d(v_p) = d(v_1) + id(v_p) \geq c - d(v_1) > 2c/3$, a contradiction. Hence we can assume $d(v_p) < id(v_p)$.

Claim 3.2. *$N(v_p) = \{v_l, v_{l+1}, \ldots, v_{p-1}\}$ and $d(v_i) < id(v_p)$ for any $v_i \in N_P(v_p)^+$.*

Proof. Suppose to the contrary that there exists some $v_i \in N_P(v_p)^+$ such that $d(v_i) \geq id(v_p)$. Then $v_1v_2\cdots v_{i-1}v_pv_{p-1}\cdots v_i$ is a longest path of G different from P with $d(v_1) + d(v_i) \geq d(v_1) + id(v_p) > d(v_1) + d(v_p)$, which is contrary to the choice of P. Therefore, Claim 3.2 holds by Lemma 10. □

By Claim 3.2, we can check that $l \geq s$, otherwise, $v_1v_2\cdots v_rv_sv_{s+1}\cdots v_pv_{s-1}$ $v_{s-2}\cdots v_{r+1}v_1$ is cycle of length p, a contradiction. Since $G-v_l$ is connected, there exists an edge $v_{r'}v_{s'} \in E(G)$ such that $s' < l < r'$. Choose such an edge $v_{r'}v_{s'}$ such that s' is as small as possible. We can get that $v_rv_{r'} \notin E(G)$ and $v_jv_{r'} \notin E(G)$ for $v_rv_{r-1}\cdots v_1v_{r+1}v_{r+2}\cdots v_{r'-1}v_pv_{p-1}\cdots v_{r'}$ and $v_jv_{j-1}\cdots v_{r+1}v_1v_2\cdots v_rv_{j+1}v_{j+2}$ $\cdots v_{r'-1}v_pv_{p-1}\cdots v_{r'}$ (v_j is got in Subcase 1.1) or $v_jv_{j-1}\cdots v_1v_{j+1}v_{j+2}\cdots v_{r'-1}v_p$ $v_{p-1}\cdots v_{r'}$ (v_j is got in Subcase 1.2) are longest paths of G. So $\{v_r, v_j, v_{r'}\}$ is an independent set of G. Hence, $id(v_r) + id(v_j) + id(v_{r'}) \geq c$. Then, by Claim 3.1, $id(v_{r'}) \geq c - id(v_r) - id(v_j) \geq c - 2d(v_1)$.

Considering the following longest path

$$P' = v_{r'}v_{r'+1}\cdots v_pv_{r'-1}v_{r'-2}\cdots v_{s'+1}v_{s'}\cdots v_1,$$

we can claim $d(v_{r'}) < id(v_{r'})$. If not, $d(v_{r'})+d(v_1) \geq c-2d(v_1) = c-d(v_1) > 2c/3$, a contradiction.

We can get that $v_{r'}$ is not adjacent to some vertex $v_d \in N(v_p)$. If not, we have $d(v_{r'}) > d(v_p)$. Then, for the longest path P', $d(v_{r'}) + d(v_1) > d(v_p) + d(v_1)$, which is contrary to the choice of P. Now $v_{r'}v_d \notin E(G)$ and $v_{r'}v_{s'} \in E(G)$, which imply that $N_{P'}(v_{r'})^- \neq N_{P'}(v_{r'}) \cup \{v_{r'}\} \setminus \{L_{P'}(v_{r'})\}$. For convenience, let $P' = x_1x_2\cdots x_p$ such that $x_1 = v_{r'}, x_2 = v_{r'+1}, \ldots, x_p = v_1$. By Lemma 10, there exists some vertex $x_b \in N_{P'}(x_1)^-$ such that $d(x_b) \geq id(x_1)$. Then $x_bx_{b-1}\cdots x_1x_{b+1}x_{b+2}\cdots x_p$ is a longest path of G with $d(x_b) + d(x_p) \geq id(x_1) + d(x_p) = id(v_{r'}) + d(v_1) \geq c - d(v_1) > 2c/3$, which is contrary to the choice of P.

Case 2. $N(v_1) \neq \{v_2, v_3, \ldots, v_k\}$.
Choose $v_j \notin N(v_1)$ with $j < k$ such that j is as large as possible. Then $v_1v_i \in E(G)$ for each i with $j < i \leq k$. Then $\{v_1, v_j, v_p\}$ is an independent set of G. If $d(v_1) < id(v_1)$, then there exists some $v_d \in N_P(v_1)^-$ such that $d(v_d) \geq id(v_1) > d(v_1)$ by Lemma 10. Then $v_dv_{d-1}\cdots v_1v_{d+1}v_{d+2}\cdots v_p$ is a longest path of G with $d(v_d) + d(v_p) > d(v_1) + d(v_p)$, a contradiction.

Next we assume $id(v_1) = d(v_1)$. By similar arguments as in Claim 3.1, we have $id(v_j) \leq d(v_1)$. If $d(v_p) = id(v_p)$, then $id(v_1) + id(v_j) + id(v_p) \geq c$, which implies $id(v_p) \geq c - id(v_1) - id(v_j) \geq c - 2d(v_1)$. Hence $d(v_1) + d(v_p) = d(v_1) + id(v_p) \geq c - d(v_1) > 2c/3$, a contradiction.

Suppose $d(v_p) < id(v_p)$. If $N(v_p) \neq \{v_l, v_{l+1}, \ldots, v_{p-1}\}$, then there is a vertex $v_f \in N_P(v_p)^+$ such that $d(v_f) \geq id(v_p)$ by Lemma 10. So $v_fv_{f+1}\cdots v_pv_{f-1}v_{f-2}\cdots$ v_1 is a longest path with $d(v_1) + d(v_f) \geq d(v_1) + id(v_p) > d(v_1) + d(v_p)$, which is contrary to choice of P.

Hence $N(v_p) = \{v_l, v_{l+1}, \ldots, v_{p-1}\}$. By the 2-connectivity of G, we know that there exists an edge $v_rv_s \in E(G)$ with $r < l < s$. We can easily verify that $\{v_1, v_j, v_s\}$ is an independent set of G. Then $id(v_1) + id(v_j) + id(v_s) \geq c$. Therefore, $id(v_s) \geq c - id(v_1) - id(v_j) \geq c - 2d(v_1)$.

Now for the longest path $P_5 = v_s v_{s+1} \cdots v_p v_{s-1} v_{s-2} \cdots v_1$, we claim that there exists a vertex $v_i \in N(v_p)$ such that $v_i v_s \notin E(G)$. Otherwise, $d(v_s) > d(v_p)$ and hence $d(v_1) + d(v_s) > d(v_1) + d(v_p)$, which is contrary to the choice of P.

Since $v_i v_s \notin E(G)$ and $v_s v_r \in E(G)$, $N_{P_5}(v_s)^- \neq N_{P_5}(v_s) \cup \{v_s\} \setminus \{L_{P_5}(v_s)\}$. For convenience, we let $P_5 = z_1 z_2 \cdots z_p$ with $z_1 = v_s, z_2 = v_{s+1}, \ldots, z_p = v_1$. By Lemma 10, there is a vertex $z_g \in N_{P_5}(z_1)^-$ such that $d(z_g) \geq id(z_1)$. Then $z_g z_{g-1} \cdots z_1 z_{g+1} z_{g+2} \cdots z_p$ is a longest path of G with $d(z_g) + d(z_p) \geq id(z_1) + d(z_p) = id(v_s) + d(v_1) \geq (c - 2d(v_1)) + d(v_1) > 2c/3$, a contradiction.

Now we complete the proof of Lemma 8. ■

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