Discussiones Mathematicae
Graph Theory 34 (2014) 683–690
doi:10.7151/dmgt.1752

THE CONNECTIVITY OF DOMINATION DOT-CRITICAL GRAPHS WITH NO CRITICAL VERTICES

Michitaka Furuya

Department of Mathematical Information Science,
Tokyo University of Science,
1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan

e-mail: michitaka.furuya@gmail.com

Abstract

An edge of a graph is called dot-critical if its contraction decreases the domination number. A graph is said to be dot-critical if all of its edges are dot-critical. A vertex of a graph is called critical if its deletion decreases the domination number.

In *A note on the domination dot-critical graphs*, Discrete Appl. Math. **157** (2009) 3743–3745, Chen and Shiu constructed for each even integer $k \geq 4$ infinitely many k-dot-critical graphs G with no critical vertices and $\kappa(G) = 1$. In this paper, we refine their result and construct for integers $k \geq 4$ and $l \geq 1$ infinitely many k-dot-critical graphs G with no critical vertices, $\kappa(G) = 1$ and $\lambda(G) = l$. Furthermore, we prove that every 3-dot-critical graph with no critical vertices is 3-connected, and it is best possible.

Keywords: dot-critical graph, critical vertex, connectivity.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

All graphs considered in this paper are finite, simple, and undirected. Let G be a graph. We let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For $u \in V(G)$, we let $N_G(u)$ and $N_G[u]$ denote the *open neighborhood* and the *closed neighborhood* of u, respectively; thus $N_G[u] = N_G(u) \cup \{u\}$. For $uv \in E(G)$, we let G/uv denote the graph obtained from G by contracting u and v into a single vertex x_{uv}. Formally, G/uv is the graph obtained by adding a new vertex x_{uv} to $G - \{u, v\}$ and joining x_{uv} to those vertices of $G - \{u, v\}$ which are adjacent to at least one of u and v in G. We let $\kappa(G)$ and $\lambda(G)$ denote the *connectivity* and the *edge-connectivity* of G, respectively. For $X \subseteq V(G)$, we let $G[X]$ denote the subgraph of G induced by X. For terms and symbols not defined here, we refer the reader to [3].

Let again G be a graph. For two subsets X, Y of $V(G)$, we say that X *dominates* Y if $Y \subseteq \bigcup_{x \in X} N_G[x]$. A subset of $V(G)$ which dominates $V(G)$ is called a *dominating set* of G. The minimum cardinality of a dominating set of G is called the *domination number* of G, and is denoted by $\gamma(G)$. A dominating set of G having cardinality $\gamma(G)$ is called a γ*-set* of G. An edge uv of G is said to be *dot-critical* if $\gamma(G/uv) < \gamma(G)$, and we say that G is *dot-critical* if every edge of G is dot-critical. If G is dot-critical and $\gamma(G) = k$, G is said to be k*-dot-critical*. A vertex u of G is said to be *critical* if $\gamma(G - u) < \gamma(G)$.

Burton and Sumner [1] posed a problem: For $k \geq 4$, what are the best upper bound for the diameter of a connected k-dot-critical graph with no critical vertices? Mojdeh and Mirzamani [5] conjectured that the diameter of connected k-dot-critical graphs with no critical vertices is at most $2k - 3$. Recently, the author and Takatou [4] showed that the conjecture is true. Before that time, Rad [6] proved the conjecture for 2-connected graphs is true, and he posed a new problem.

Problem 1 (Rad [6])**.** For an integer $k \geq 2$, is it true that a connected k-dot-critical graph with no critical vertices is 2-connected?

If Problem 1 is true, then the Mojdeh-Mirzamani conjecture follows from Rad's result. However, Chen and Shiu [2] gave its negative answer that for each even integer $k \geq 4$, there exist infinitely many k-dot-critical graphs G with no critical vertices and $\kappa(G) = 1$. (In fact, they constructed graphs with edge-connectivity exactly 1.) In Section 2, we extend their result by removing the parity condition of k and adding an edge-connectivity condition as follows.

Theorem 2. *For integers $k \geq 4$ and $l \geq 1$, there exist infinitely many k-dot-critical graphs with no critical vertices, $\kappa(G) = 1$ and $\lambda(G) = l$.*

On the other hand, we prove the following theorem which affirms Problem 1 for $k \in \{2, 3\}$ in Section 3.

Theorem 3. *For $k \in \{2, 3\}$, every k-dot-critical graph with no critical vertices is 3-connected.*

Moreover, we show that Theorem 3 is best possible. In our argument, we make use of the following lemmas, which are proved in [1].

Lemma 4 (Burton and Sumner [1])**.** *A graph is 2-dot-critical with no critical vertices if and only if it is a complete multipartite graph whose partite sets contain at least three vertices.*

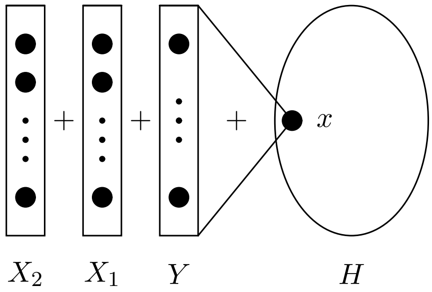

Figure 1. Graph $G(H, K; x, Y)$.

Lemma 5 (Burton and Sumner [1]). *Let G be a graph with no critical vertices, and let $e = uv \in E(G)$. Then e is dot-critical if and only if u and v belong to a common γ-set of G.*

Further, we frequently use the following lemma.

Lemma 6. *Let G be a graph with no critical vertices. If $S \subseteq V(G)$ dominates at least $|V(G)| - 1$ vertices of G, then $|S| \geq \gamma(G)$.*

Proof. If S is a dominating set of G, then $|S| \geq \gamma(G)$. Thus we may assume that S dominates exactly $|V(G)| - 1$ vertices of G (i.e. $|V(G) - (\bigcup_{x \in S} N_G[x])| = 1$). Write $V(G) - (\bigcup_{x \in S} N_G[x]) = \{y\}$. Then S is a dominating set of $G - y$. Since y is not a critical vertex of G, we have $|S| \geq \gamma(G - y) \geq \gamma(G)$. ■

2. DOT-CRITICAL GRAPHS WITH A CUTVERTEX AND GIVEN EDGE-CONNECTIVITY

In this section, we show Theorem 2 by constructing some dot-critical graphs.

We first give a general construction of dot-critical graphs G with no critical vertices and $\kappa(G) = 1$. Let H be a connected dot-critical graph with no critical vertices, and let x be a vertex of H. Let K be a complete bipartite graph with partite sets X_1 and X_2, and let Y be a non-empty set. We define the graph $G(H, K; x, Y)$ by $V(G(H, K; x, Y)) = V(H) \cup V(K) \cup Y$ and $E(G(H, K; x, Y)) = E(H) \cup E(K) \cup \{uy \mid u \in X_1 \cup \{x\}, y \in Y\}$ (see Figure 1).

Lemma 7. *If $H - x$ has no critical vertex and $|X_i| \geq 3$ for $i \in \{1, 2\}$, then $G = G(H, K; x, Y)$ is a dot-critical graph with no critical vertices and $\gamma(G) = \gamma(H) + 2$.*

Proof. We start with a claim.

Claim 8. *Let $S \subseteq V(G)$.*
(i) *If S dominates $V(H)$ in G, then $|S \cap V(H)| \geq \gamma(H)$.*

(ii) *If S dominates at least $|V(K)|-1$ vertices of K in G, then $|S\cap(V(K)\cup Y)| \geq 2$.*

Proof. (i) Recall that H contains no critical vertices. Since S dominates $V(H)$, $S\cap V(H)$ dominates $V(H)-\{x\}$ in H. This together with Lemma 6 implies that $|S\cap V(H)| \geq \gamma(H)$.

(ii) Since every vertex in Y is adjacent to exactly $|X_1|(\leq |V(K)|-3)$ vertices of K in G, if $S\cap Y \neq \emptyset$, then $|S\cap(V(K)\cup Y)| \geq 2$, as desired. Thus we may assume that $S\cap Y = \emptyset$. Then $S\cap V(K)$ dominates at least $|V(K)|-1$ vertices of K. Since K is a 2-dot-critical graph with no critical vertices by Lemma 4, $|S\cap(V(K)\cup Y)| = |S\cap V(K)| \geq \gamma(K) = 2$ by Lemma 6. □

We show that $\gamma(G) = \gamma(H)+2$. Let S be a γ-set of H, and let $u\in X_1$ and $y\in Y$. Note that $\{u,y\}$ dominates $V(K)\cup Y$. Hence $S\cup\{u,y\}$ is a dominating set of G, and so $\gamma(G)\leq |S|+2 = \gamma(H)+2$. Let S' be a γ-set of G. Since S' dominates $V(H)$ and $V(K)$ in G, $\gamma(G) = |S'| = |S'\cap V(H)|+|S'\cap(V(K)\cup Y)| \geq \gamma(H)+2$ by Claim 8. Consequently, we get $\gamma(G) = \gamma(H)+2$.

Next, we show that G has no critical vertex. Let $v\in V(G)$, and let S^* be a γ-set of $G-v$. We show that $|S^*| \geq \gamma(H)+2$. Since S^* dominates at least $|V(K)|-1$ vertices of K in G, $|S^*\cap(V(K)\cup Y)| \geq 2$ by Claim 8(ii), and hence $|S^*| = |S^*\cap V(H)|+|S^*\cap(V(K)\cup Y)| \geq |S^*\cap V(H)|+2$. Thus it suffices to show that $|S^*\cap V(H)| \geq \gamma(H)$. Since H has no critical vertex, if $S^*\cap V(H)$ dominates at least $|V(H)|-1$ vertices of H, then we have $|S^*\cap V(H)| \geq \gamma(H)$ by Lemma 6, as desired. Thus we may assume that $S^*\cap V(H)$ dominates at most $|V(H)|-2$ vertices of H. Since $S^*\cap V(H)$ dominates $V(H)-\{x,v\}$, this implies that $v\in V(H)$, $x\neq v$ and neither x nor v belongs to $S^*\cap V(H)$. In particular, $S^*\cap V(H)$ is a dominating set of $H-\{x,v\}$, and hence $|S^*\cap V(H)| \geq \gamma(H-\{x,v\})$. Since $H-x$ has no critical vertex, $\gamma(H-\{x,v\}) \geq \gamma(H-x) \geq \gamma(H)$. This leads to $|S^*\cap V(H)| \geq \gamma(H)$. Consequently, G has no critical vertex.

Finally we show that G is dot-critical. Let $e = vv' \in E(G)$. By Lemma 5, it suffices to show that there exists a dominating set of G with cardinality $\gamma(H)+2$ containing both v and v'. Since H is a dot-critical graph with no critical vertices, there exists a γ-set of H containing both x and x' where $x'\in N_H(x)$ by Lemma 5. In particular, there exists a γ-set T of H containing x. Let T' be a set which consists of a vertex in X_1 and a vertex in Y. Note that T' dominates $V(K)\cup Y$ in G.

Case 1. $v,v'\in V(H)$. Since H is a dot-critical graph with no critical vertices, there exists a γ-set S_1 of H containing both v and v' by Lemma 5. Then $S_1\cup T'$ is a dominating set of G with cardinality $\gamma(H)+2$ containing both v and v'.

Case 2. $v,v'\in V(K)\cup Y$. We can check that $\{v,v'\}$ dominates $V(K)\cup Y$ in G. Hence $T\cup\{v,v'\}$ is a dominating set of G with cardinality $\gamma(H)+2$ containing

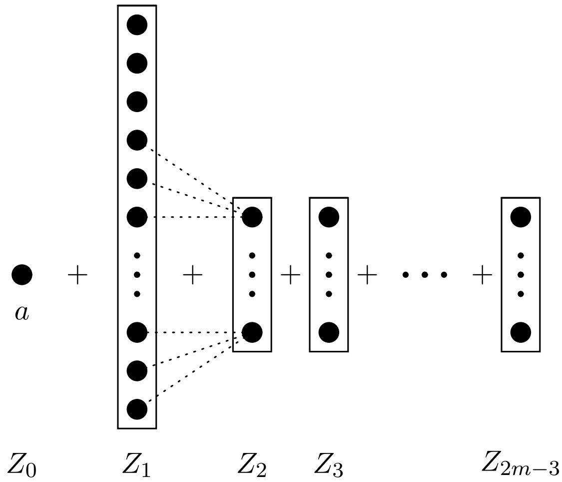

Figure 2. Graph $H_m(p)$ for odd integer m.

both v and v'.

Case 3. $|\{v, v'\} \cap V(H)| = |\{v, v'\} \cap (V(K) \cup Y)| = 1$. We may assume that $\{v, v'\} \cap V(H) = \{v\}$. Then this forces $v = x$ and $v' \in Y$. Let $u' \in X_1$. Then $T \cup \{v', u'\}$ is a dominating set of G with cardinality $\gamma(H) + 2$ containing both v and v'.

This completes the proof of Lemma 7. ■

***Proof of Theorem* 2.** We give two constructions of graphs $H_m(p)$ ($m \geq 2, p \geq 5$) depending on the parity of m.

Let $m \geq 2$ be an even integer. The following example can be found in [5]. Let $p \geq 5$ be an integer. Let $Z_0, \ldots, Z_{2m-3}$ be disjoint sets with $|Z_i| = p$ ($0 \leq i \leq 2m-3$). We define the graph $H_m(p)$ by $V(H_m(p)) = \bigcup_{0 \leq j \leq 2m-3} Z_j$ and $E(H_m(p)) = \bigcup_{0 \leq j \leq 2m-4} \{uv \mid u \in Z_j, v \in Z_{j+1}\}$.

Let $m \geq 3$ be an odd integer. The following example was constructed in [4]. Let $p \geq 5$ be an integer. Set $Z_0 = \{a\}$ and $Z_1 = \{b_{i,h} \mid 0 \leq i \leq p, 1 \leq h \leq 3\}$, and for each $2 \leq j \leq 2m-3$, set $Z_j = \left\{c_i^{(j)} \mid 1 \leq i \leq p\right\}$. We define the graph $H_m(p)$ by $V(H_m(p)) = \bigcup_{0 \leq j \leq 2m-3} Z_j$ and $E(H_m(p)) = \left(\bigcup_{0 \leq j \leq 2m-4} \{uv \mid u \in X_j, v \in X_{j+1}\}\right) - \left\{b_{i,h} c_i^{(2)} \mid 1 \leq i \leq p, 1 \leq h \leq 3\right\}$ (see Figure 2).

Then for integers $m \geq 2$ and $p \geq 5$, $H_m(p)$ is an m-dot-critical graph with no critical vertices and $H_m(p) - x$ has no critical vertex for every $x \in Z_{2m-3}$ (see [4, 5]). Furthermore, we can verify that $H_m(p)$ is p-edge-connected by a tedious argument (and we omit its details).

Fix two integers $k \geq 4$ and $l \geq 1$. Let p_1 and p_2 be integers with $p_1 \geq \max\{l, 5\}$ and $p_2 \geq \max\{l, 3\}$. Let K be a complete bipartite graph which is isomorphic to K_{p_2,p_2}, and let X_1 and X_2 be the partite sets of K. Let Y be a set with $|Y| = l$. We consider the graph $G = G(K, H_{k-2}(p_1); x, Y)$ where $x \in Z_{2(k-2)-3}$. Then by Lemma 7, G is a k-dot-critical graph with no critical vertices. Since G is connected and $G - x$ is disconnected, we have $\kappa(G) = 1$.

Claim 9. $\lambda(G) = l$.

Proof. Let $F \subseteq E(G)$ with $|F| \leq l - 1$. First, we show that for each $u \in V(G)$, there exists a path of $G - F$ joining u and x. Since $H_{k-2}(p_1)$ is l-edge-connected, if $u \in V(H_{k-2}(p_1))$, then there exists a path of $H_{k-2}(p_1) - F$ joining u and x. Thus we may assume that $u \in V(K) \cup Y$. Since $|F| \leq l - 1$ and $|Y| = l$, $N_{G-F}(x) \cap Y \neq \emptyset$. Let $v \in N_{G-F}(x) \cap Y$. Since $G[V(K) \cup Y]$ is isomorphic to K_{p_2,p_2+l}, $G[V(K) \cup Y]$ is l-edge-connected. Hence there exists a path P of $G[V(K) \cup Y] - F$ joining u and v. By combining P with the edge vx, we can construct a path of $G - F$ joining u and x. Consequently, there exists a path of $G - F$ joining u and x for $u \in V(G)$, and hence $G - F$ is connected. Since F is arbitrary, this implies that G is l-edge-connected. On the other hand, since the set F' of edges between x and Y satisfies that $|F'| = l$ and $G - F'$ is disconnected, G is not $(l + 1)$-edge-connected. Therefore we have $\lambda(G) = l$. □

Since p_1 and p_2 are arbitrary, there exist infinitely many connected k-dot-critical graphs G with no critical vertices, $\kappa(G) = 1$ and $\lambda(G) = l$. Therefore Theorem 2 holds. ■

3. Dot-critical Graphs with Small Domination Number

In this section, we prove Theorem 3 and its best possibility. By Lemma 4, every 2-dot-critical graph with no critical vertices is 3-connected. Thus it suffices to show the following theorem.

Theorem 10. *Every 3-dot-critical graph with no critical vertices is 3-connected.*

Proof. Let G be a 3-dot-critical graph with no critical vertices.

Claim 11. *The graph G is connected.*

Proof. Suppose that G is disconnected. Then there exists a component C of G with $\gamma(C) = 1$. Let $u \in V(C)$ be a vertex which dominates $V(C)$. If $V(C) = \{u\}$, then u is a critical vertex of G, which contradicts the assumption that G has no critical vertex. Thus $N_C(u) \neq \emptyset$. Let $v \in N_C(u)$. By Lemma 5, there exists a γ-set S of G containing both u and v. Then $S - \{v\}$ is a dominating set of G with cardinality 2, which is a contradiction. □

Let X be a minimum cutset of G. Suppose that $|X| \leq 2$.

Claim 12. *The graph $G - X$ contains no isolated vertex.*

Proof. Suppose that $G - X$ contains an isolated vertex u. Let $x \in X$. By the minimality of X, $ux \in E(G)$. By Lemma 5, there exists a γ-set S of G containing both u and x. Then $S - \{u\}$ dominates $V(G) - (X - \{x\})$. In particular, $S - \{u\}$ dominates at least $|V(G)| - 1$ vertices of G, which contradicts Lemma 6. □

Let C_1 and C_2 be two vertex-disjoint non-empty graphs such that $V(C_1) \cup V(C_2) = V(G) - X$ and there exists no edge of G between $V(C_1)$ and $V(C_2)$ (i.e. C_i is a graph which consists of the union of some components of $G - X$).

Claim 13. *Let $i \in \{1, 2\}$.*

(i) *There exists a vertex of G which dominates $V(C_i)$.*

(ii) *Every vertex of G dominating $V(C_i)$ belongs to $V(C_i) - (\bigcup_{x \in X} N_G(x))$.*

Proof. (i) Let $u \in V(C_{3-i})$. By Claim 12, $N_{C_{3-i}}(u) \neq \emptyset$. Let $v \in N_{C_{3-i}}(u)$. By Lemma 5, there exists a γ-set S of G containing both u and v. Then the unique vertex in $S - \{u, v\}$ dominates $V(C_i)$.

(ii) Let w_i be a vertex which dominates $V(C_i)$. By (i), there exists a vertex w_{3-i} which dominates $V(C_{3-i})$. If $w_i \in \bigcup_{x \in X} N_G[x]$, then w_i dominates a vertex in X, and hence $\{w_1, w_2\}$ dominates at least $|V(G)| - 1$ vertices of G, which contradicts Lemma 6. Thus $w_i \notin \bigcup_{x \in X} N_G[x]$. Consequently, we have $w_i \in V(C_i) - (\bigcup_{x \in X} N_G(x))$. □

By Claim 13, for each $i \in \{1, 2\}$, there exists a vertex $w_i \in V(C_i) - (\bigcup_{x \in X} N_G(x))$ which dominates $V(C_i)$. Let $i \in \{1, 2\}$ and $w \in N_{C_i}(w_i)$. We show that w is adjacent to all vertices in X. By Lemma 5, there exists a γ-set S of G containing both w_i and w. Then the unique vertex a in $S - \{w_i, w\}$ dominates $V(C_{3-i})$. By Claim 13(ii), $a \in V(C_{3-i}) - (\bigcup_{x \in X} N_G(x))$. Since S is a dominating set of G and neither w_i nor a belongs to $\bigcup_{x \in X} N_G(x)$, w is adjacent to all vertices in X. Recall that w_i dominates $V(C_i)$. Since i and w are arbitrary, every vertex in X dominates $(V(C_1) \cup V(C_2)) - \{w_1, w_2\}$. Let $x \in X$ and $x' \in V(C_1) - \{w_1\}$. Then $\{x, x'\}$ is a dominating set of $G - w_2$, which contradicts Lemma 6. This completes the proof of Theorem 10. ■

Next, we construct for $k \in \{2, 3\}$, infinitely many k-dot-critical graphs with no critical vertices and connectivity exactly 3. Let $p \geq 4$ be an integer. Set $X_0 = \{x\}$, $X_1 = \{y_{i,h} \mid 1 \leq i \leq 3, 1 \leq h \leq 3\}$, $X_2 = \{z_1, z_2, z_3\}$ and $X_3 = \{w_i \mid 1 \leq i \leq p\}$. We define the graph $H'_2(p)$ by $V(H'_2(p)) = X_2 \cup X_3$ and $E(H'_2(p)) = \{uv \mid u \in X_2, v \in X_3\}$ (i.e. $H'_2(p) = K_{3,p}$). Furthermore, we define the graph $H'_3(p)$ by $V(H'_3(p)) = \bigcup_{0 \leq j \leq 3} X_j$ and $E(H'_3(p)) = \left(\bigcup_{0 \leq j \leq 2}\{uv \mid u \in X_j, v \in X_{j+1}\}\right) - \{y_{i,h} z_i \mid 1 \leq i \leq 3, 1 \leq h \leq 3\}$ (see Figure 3). Then for each $k \in \{2, 3\}$, it is

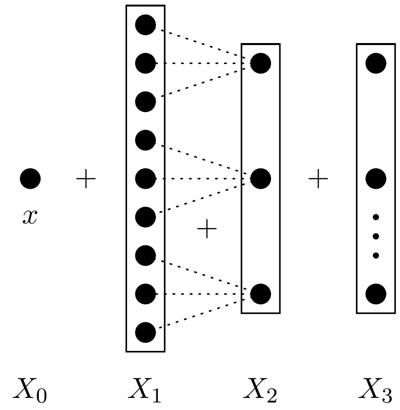

Figure 3. Graph $H'_3(p)$.

easy to verify that $H'_k(p)$ is a k-dot-critical graph with no critical vertices and $\kappa(H'_k(p)) = 3$. Therefore Theorem 3 is best possible.

Acknowledgment

The author is grateful to referees for their careful reading.

References

[1] T. Burton and D.P. Sumner, *Domination dot-critical graphs*, Discrete Math. **306** (2006) 11–18.
doi:10.1016/j.disc.2005.06.029

[2] X. Chen and W.C. Shiu, *A note on the domination dot-critical graphs*, Discrete Appl. Math. **157** (2009) 3743–3745.
doi:10.1016/j.dam.2009.07.014

[3] R. Diestel, Graph Theory 4th Edition (Verlag, Heidelberg, Springer, 2010).

[4] M. Furuya and M. Takatou, *Upper bound on the diameter of a domination dot-critical graph*, Graphs Combin. **29** (2013) 79–85.
doi:10.1007/s00373-011-1095-1

[5] D.A. Mojdeh and S. Mirzamani, *On the diameter of dot-critical graphs*, Opuscula Math. **29** (2009) 165–175.

[6] N.J. Rad, *On the diameter of a domination dot-critical graph*, Discrete Appl. Math. **157** (2009) 1647–1649.
doi:10.1016/j.dam.2008.10.015

Received 8 July 2013
Revised 20 September 2013
Accepted 30 September 2013