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PAIRS OF EDGES AS CHORDS AND AS CUT-EDGES

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Abstract

Several authors have studied the graphs for which every edge is a chord of a cycle; among 2-connected graphs, one characterization is that the deletion of one vertex never creates a cut-edge. Two new results: among 3-connected graphs with minimum degree at least 4, every two adjacent edges are chords of a common cycle if and only if deleting two vertices never creates two adjacent cut-edges; among 4-connected graphs, every two edges are always chords of a common cycle.

Keywords: cycle, chord, cut-edge.

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1. Introduction

An edge ab is a *chord* of a cycle C if a and b are nonconsecutive vertices of C, and ab is a *cut-edge* of a connected graph if deleting ab creates a subgraph that is not connected (equivalently, if ab is in no cycle). Two edges are *adjacent* if they share an endpoint and are *nonadjacent* otherwise.

The 2-connected graphs such that every edge is a chord of a cycle were independently characterized, in rather different ways, in [4, 7]. Proposition 1 is rephrased from [7].

Proposition 1. *The following are equivalent for every 2-connected graph.*

(1a) *Every edge is a chord of a cycle.*

(1b) *Deleting one vertex never creates a cut-edge.*

Paralleling Proposition 1, Theorem 3 will show that, in a 3-connected graph with minimum degree at least 4, every two adjacent edges are chords of a common cycle if and only if deleting two vertices never creates two adjacent cut-edges. Theorem 5 will show that, in a 4-connected graph, every two edges are always chords of a common cycle.

If $S \subseteq V(G)$, then $G - S$ denotes the subgraph of G induced by $V(G) - S$, and $G - v$ denotes $G - \{v\}$ when $v \in V(G)$. For a vertex $v \notin S \subseteq V(G)$, a *$v$-to-$S$ path* is a v-to-w path where $w \in S$; for a subgraph H, a *v-to-H path* is a v-to-$V(H)$ path. Proposition 2 collects five properties of k-connected graphs that will be used in proofs.

Proposition 2. *For every k-connected graph with $k \geq 2$ the following hold.*

(a) *Every two vertices are the endpoints of k internally disjoint paths.*

(b) *If vertex $v \notin S \subseteq V(G)$ and $|S| \geq k$, then there exist k internally disjoint v-to-S paths $\pi_1, \ldots, \pi_k$ that have k distinct endpoints in S such that each $|V(\pi_i) \cap S| = 1$.*

(c) *Every k vertices are in a common cycle.*

(d) *If S is a set of vertex-disjoint paths that have a total of s edges and if T is a set of $t \geq 1$ vertices where $s + t = k$, then the paths in S and the vertices in T all lie in a common cycle.*

(e) *For every $k + 1$ vertices $v_0, \ldots, v_k$, there is v_0-to-v_k path through all of the vertices in $\{v_1, \ldots, v_{k-1}\}$.*

Proof. Property (a) is Menger's Theorem from [6]. Property (b) follows by creating a new vertex w that has neighborhood S, and then applying (a) to v and w in the larger k-connected graph. Property (c) is a standard result from [2]. Property (d) is from [1] (although Theorem 9 of [2] is the special case of (c) when S consists of two nonadjacent edges). Property (e) is from [8] (also see solution 6.68 in [5]). ■

2. Two Adjacent Chords

Observe that two adjacent edges ab and bc of a 4-connected graph are always chords of a common cycle, since b will be incident with two additional edges $bu, bv \notin \{ab, bc\}$, and so by Proposition 2(d) there will be a cycle C that contains bu and bv as well as a and c. Thus $a, b, c \in V(C)$ and $ab, bc \notin E(C)$, and so ab and bc are chords of C.

A *minimal edge cutset* (sometimes called an *edge cutset* or a *cocycle* or a *bond*) of a connected graph is an inclusion-minimal set of edges whose deletion would create a graph that is not connected. Thus, $\{e\}$ is a minimal edge cutset if and only if e is a cut-edge. Also, if $\{e,f\}$ is a minimal edge cutset, then neither e nor f is a cut-edge.

Figure 1 illustrates several ideas that will occur in Theorem 3: Edges ab and bc cannot be chords of a common cycle C, since otherwise $E(C)$ would have to contain both bu and bv, which would prevent C from containing both a and c. Deleting the vertices u and v would create the two adjacent cut-edges ab and bc.

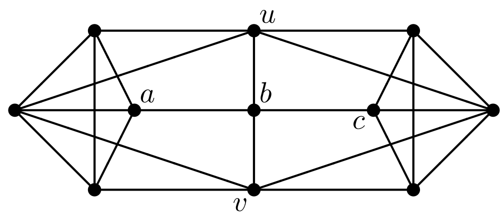

Figure 1. The adjacent edges ab and bc are not chords of a common cycle in this 3-connected graph with minimum degree 4.

Theorem 3. *The following are equivalent for every* 3*-connected graph with minimum degree at least* 4:

(3a) *Every two adjacent edges are chords of a common cycle.*

(3b) *Deleting two vertices never creates two adjacent cut-edges.*

Proof. Assume G is a 3-connected graph with minimum degree at least 4.

First suppose G satisfies condition (3a) and (arguing by contradiction) $S = \{v_1, v_2\} \subset V(G)$ where $G - S$ has adjacent cut-edges ab and bc. By (3a), ab and bc are chords of a cycle C of G, with a, b, c partitioning C into internally disjoint subpaths $C[a, b]$, $C[b, c]$, and $C[a, c]$ with the indicated endpoints. Since ab is a cut-edge of $G - S$, one of v_1, v_2 is an internal vertex of $C[a, b]$ and the other is an internal vertex of $C[a, c]$ (so that a is separated from bc when ab is deleted from $G - S$). Similarly, since bc is a cut-edge of $G - S$, one of v_1, v_2 is an internal vertex of $C[b, c]$ and the other is an internal vertex of $C[a, c]$. Therefore, one of v_1, v_2 would have to be in two of $C[a, b], C[b, c], C[a, c]$ (contradicting that these subpaths are internally disjoint).

Conversely, suppose G satisfies condition (3b) and (arguing by contradiction) the adjacent edges ab and bc of G are not chords of a common cycle. Let G' be the subgraph of G obtained by deleting ab and bc. The argument below will make repeated use of a, b, c not all being on a common cycle of G' (otherwise, such a cycle would also be a cycle of G that has chords ab and bc, contradicting (3b)). Thus, by Proposition 2(c), G' is not 3-connected. Since deleting b from the 3-connected graph G would leave a 2-connected graph and since b has degree at least 4 in G, deleting both ab and bc from G would leave a 2-connected graph. Therefore, G' is 2-connected (but not 3-connected), say with a separating set

$S = \{v_1, v_2\}$. Since S is not a separating set of the 3-connected graph G and $E(G') = E(G) - \{ab, bc\}$, and since (3b) implies that ab and bc are not both cut-edges of $G - S$, one of the following cases must occur.

Case 1. Exactly one of ab and bc is a cut-edge of $G - S$.

Case 2. $\{ab, bc\}$ is a minimal edge cutset of $G - S$.

Case 1. Exactly one of ab and bc is a cut-edge of $G-S$; to be specific, suppose ab (but not bc) is a cut-edge of $G - S$, with a in one connected component of $G' - S$ and b and c in the other. Since b has degree at least 4 in the 3-connected graph G, there is a cycle C of G by Proposition 2(d) such that C contains two edges incident with b different from ab and bc, and C also contains a. Thus, $a, b, v_1, v_2 \in V(C)$ and $ab, bc \notin E(C)$, which implies that C is also a cycle of G', and so $c \notin V(C)$. Vertices a, b, v_1, v_2 partition C into four subpaths $C[a, v_i]$ and $C[b, v_i]$ with the indicated endpoints.

By Proposition 2(b), G' has internally disjoint c-to-C paths π_1 and π_2 that have distinct endpoints in C with each $|V(\pi_i) \cap V(C)| = 1$. The two endpoints of π_1 and π_2 in C (call them w_1 and w_2, respectively) cannot be on the same a-to-b subpath of C (otherwise, the edges in $C \cup \pi_1 \cup \pi_2$ would contain a cycle of G' through all three of a, b, c); thus, in particular, $w_1 \neq b \neq w_2$. For each $i \in \{1, 2\}$, partition $C[b, v_i]$ into subpaths $C[b, w_i]$ and $C[v_i, w_i]$ with the indicated endpoints.

Among all such cycles C and paths π_1, π_2 as just described, assume further that the two subpaths $C[b, w_i]$ have minimum lengths. That minimality implies that there is no path in G' between an internal vertex x of $C[v_i, w_i]$ and a vertex $y \neq w_i$ of $C[b, w_i]$ (such an x-to-y path could replace the x-to-y subpath of $C[b, v_i]$, and then the y-to-w_i subpath of $C[b, w_i]$ could be adjoined to π_i). Thus, every c-to-b path in G' intersects $\{w_1, w_2\}$. Moreover, there is no path in G' between an internal vertex of $C[v_i, w_i]$ and an internal vertex of $C[b, w_j]$ with $j \neq i$ (such a path would combine with π_1 and π_2 and subpaths of C to form a cycle of G' through all three of a, b, c). Thus, G' has no path between an internal vertex of $C[v_1, w_1]$ or $C[v_2, w_2]$ and an internal vertex of $C[b, w_1] \cup C[b, w_2]$, which implies that every a-to-b path in G' intersects $\{w_1, w_2\}$. Therefore, every a-to-b and every b-to-c path in G' intersects $\{w_1, w_2\}$, and so ab and bc would be adjacent cut-edges of $G - \{w_1, w_2\}$ (contradicting (3b)).

Case 2. $\{ab, bc\}$ is a minimal edge cutset of $G - S$, where a and c are in one connected component of $G' - S$, and b is in the other. The argument is essentially the same as for Case 1, except with the roles of vertices a and b interchanged (but with the role of the edge bc unchanged). There again is a cycle C with through a and b (and v_1, v_2, but not c) of G'. There are internally disjoint c-to-C paths π_1 and π_2 with endpoints w_1 and w_2 where, in this case, each $C[a, v_i]$ is partitioned into subpaths $C[a, w_i]$ and $C[v_i, w_i]$ with the two subpaths $C[a, w_i]$ having minimum lengths. Every a-to-b and every b-to-c path in G' again intersects $\{w_1, w_2\}$. Therefore, ab and bc would be adjacent cut-edges of $G - \{w_1, w_2\}$ (contradicting (3b)). ■

Being 3-connected with minimum degree at least 4 is a reasonable hypothesis for Theorem 3 for the following reasons. Being 4-connected would be too strong, since conditions (3a) and (3b) would always hold. The graph in Figure 2 is a 2-connected graph with minimum degree 4 that satisfies (3b) but not (3a). The graph formed by inserting all four diametrical chords into an 8-cycle is a 3-connected graph with minimum degree 3 that satisfies (3b) but not (3a).

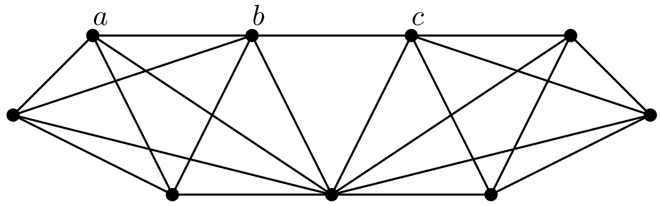

Figure 2. A 2-connected graph in which the adjacent edges ab and bc are not chords of a common cycle (in fact, bc is not a chord of a cycle).

3. TWO ARBITRARY CHORDS

Lemma 4. *In every 4-connected graph, every two nonadjacent edges are chords of a common cycle.*

Proof. Suppose G is a 4-connected graph and (arguing by contradiction) the nonadjacent edges ab and cd of G are not chords of a common cycle. Let G' be the subgraph of G obtained by deleting ab and cd. The argument below will make repeated use of a, b, c, d not all being on a common cycle of G' (otherwise, such a cycle would also be a cycle of G that has chords ab and cd, contradicting the assumption). Thus, by Proposition 2(c), G' is not 4-connected. Since deleting any two of a, b, c, d from the 4-connected graph G would leave a 2-connected graph, every two vertices of G will still be in a common cycle of G'. Therefore, G' is 2-connected (but not 4-connected), say with a minimum-cardinality separating set S where $|S| \in \{2, 3\}$ (and so G' is $|S|$-connected). Since S is not a separating set of the 4-connected graph G and $E(G') = E(G) - \{ab, cd\}$, one of the following cases must occur:

Case 1. Exactly one of ab and cd is a cut-edge of $G - S$.

Case 2. $\{ab, cd\}$ is a minimal edge cutset of $G - S$.

Case 3. ab and cd are both cut-edges of $G - S$.

Case 1. Exactly one of ab and cd is a cut-edge of $G - S$; to be specific, suppose ab (but not cd) is a cut-edge of $G - S$ where, without loss of generality, a is in one connected component of $G-S$ and b,c,d are in the other. If $|S|=2$, then a and c would be in different connected components of $G-(S\cup\{b\})$ (contradicting that G is 4-connected). Therefore, $|S|=3$ and G' is 3-connected. By Proposition 2(a), G' has three internally disjoint a-to-b paths π_1,π_2,π_3. Let $\Theta=\pi_1\cup\pi_2\cup\pi_3$. If c and d were both on the same path π_i, then π_i together with either one of the other two a-to-b paths in Θ would form a cycle of G' through all four of a,b,c,d. Similarly, if c and d were on two separate paths π_i and π_j, then $\pi_i\cup\pi_j$ would be a cycle of G' through all four of a,b,c,d. Therefore, c and d cannot both be in $V(\Theta)$.

Suppose for the moment that $c\in V(\Theta)$ (and so $d\notin V(\Theta)$); without loss of generality, suppose $c\in V(\pi_1)$. By Proposition 2(b), the 3-connected graph G' has internally disjoint d-to-Θ paths τ_1,τ_2,τ_3 that have distinct endpoints (say t_1,t_2,t_3, respectively) in Θ with each $V(\tau_i)\cap\Theta=\{t_i\}$. Each t_i is in one of the four following paths: the a-to-c subpath of π_1, the c-to-b subpath of π_1, the path π_2, or the path π_3. If, say, t_1 and t_2 are in the same one of these four paths, then subpaths of that path π_i through t_1 and t_2 would combine with $\tau_1\cup\tau_2$ and a path $\pi_j\neq\pi_i$ to form a cycle of G' through all four of a,b,c,d. If, say, t_1 is in the a-to-c subpath of π_1 and t_2 is in the c-to-b subpath of π_1 and t_3 is in π_3, then the a-to-t_2 subpath of π_1, the t_2-to-t_3 path $\tau_2\cup\tau_3$, and the t_3-to-b subpath of π_3 would combine with π_2 to form a cycle of G' through all four of a,b,c,d. If, say, $t_2\in V(\pi_2)$ and $t_3\in V(\pi_3)$, then π_1 would combine with the b-to-t_2 subpath of π_2, the t_2-to-t_3 path $\tau_2\cup\tau_3$, and the t_3-to-a subpath of π_3 to form a cycle of G' through all four of a,b,c,d. Thus and similarly, no matter where t_1,t_2,t_3 are located in Θ, there would be a cycle of G' through all four of a,b,c,d.

Therefore, $c\notin V(\Theta)$ and, similarly, $d\notin V(\Theta)$. By Proposition 2(b), G' again has internally disjoint d-to-Θ paths τ_1,τ_2,τ_3 that have distinct endpoints (say t_1,t_2,t_3, respectively) in Θ with each $V(\tau_i)\cap V(\Theta)=\{t_i\}$. Let $H=\Theta\cup\tau_1\cup\tau_2\cup\tau_3$. By the argument in the preceding paragraph, assume that no two of t_1,t_2,t_3 are in the same π_i, and so, without loss of generality, suppose each $t_i\in V(\pi_i)$ and let H_i be the subgraph of H formed by $\pi_i\cup\tau_i$. Vertex $c\notin V(H)$ (otherwise, much as in the preceding paragraph, H would contain a cycle of G' through all four of a,b,c,d). Thus, by Proposition 2(b), G' has internally disjoint c-to-H paths $\sigma_1,\sigma_2,\sigma_3$ that have distinct endpoints (say s_1,s_2,s_3, respectively) in H with each $V(\sigma_i)\cap V(H)=\{s_i\}$.

Suppose for the moment that two of s_1,s_2,s_3 are in the same subgraph H_i; without loss of generality, say $s_1,s_2\in V(H_3)$. Each of s_1 and s_2 is in one of the three following paths: the a-to-t_3 subpath of π_3, the t_3-to-b subpath of π_3, or the t_3-to-d path τ_3. In each of the resulting nine possibilities, all or part of the s_1-to-s_2 subpath of H_3 could be replaced with $\sigma_1\cup\sigma_2$ to form an a-to-c-to-d path that would combine with subpaths of $H_1\cup H_2$ to form a cycle of G' through all four of a,b,c,d.

By the preceding paragraph, suppose no two of s_1, s_2, s_3 are in the same subgraph H_i of H; specifically suppose f is a permutation of $\{1, 2, 3\}$ such that each s_i is in $H_{f(i)}$. Each s_i might be in the a-to-$t_{f(i)}$ subpath of $\pi_{f(i)}$ or in the $t_{f(i)}$-to-b subpath of $\pi_{f(i)}$ or the $t_{f(i)}$-to-d path $\tau_{f(i)}$. In each of the resulting cases, two of the paths $\sigma_1, \sigma_2, \sigma_3$ would combine with a subgraph of H to form a cycle of G' through all four of a, b, c, d.

Case 2. $\{ab, cd\}$ is a minimal edge cutset of $G-S$. Without loss of generality, say $G' - S$ has connected components H°_{ac} and H°_{bd} where vertices a, c are in the subgraph H_{ac} of G' that is induced by $V(H^\circ_{ac}) \cup S$ and vertices b, d are in the subgraph H_{bd} of G' that is induced by $V(H^\circ_{bd}) \cup S$.

First suppose $|S| = 2$ with $S = \{v_1, v_2\}$. By Proposition 2(a), there is a set $\Sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ of four internally disjoint a-to-c paths in G, at most one of which can contain both the edges ab and cd.

Claim. *H_{ac} contains a v_1-to-v_2 path through a and c inside H_{ac}.*

Proof. First suppose three paths in Σ contain none of v_1, v_2, b, d. Apply Proposition 2(b) to $v = v_1$ (respectively, to $v = v_2$) and the union S of the vertex sets of those three paths from Σ to obtain v_1-to-S paths π_{11}, π_{12} (and v_2-to-S paths π_{21}, π_{22}) in the 2-connected graph G'. The union of those three paths from Σ and the paths $\pi_{11}, \pi_{12}, \pi_{21}, \pi_{22}$ will contain a v_1-to-v_2 path through a and c inside H_{ac}.

Now suppose instead that one path in Σ, say σ_1, contains v_1 but not v_2 and two other paths $\sigma_2, \sigma_3 \in \Sigma$ contain none of v_1, v_2, b, d. Apply Proposition 2(b) to $v = v_2$ and $S = V(\sigma_1) \cup V(\sigma_2) \cup V(\sigma_3)$ to obtain v_2-to-S paths π_1, π_2 in the 2-connected graph G' where each π_i has endpoint $p_i \in S$ with $V(\pi_i) \cap S = \{p_i\}$. Each p_i is in one of the four following paths: the a-to-v_1 subpath of σ_1, the v_1-to-c subpath of σ_1, the path σ_2, or the path σ_3. If p_1 and p_2 are both in the same one of these four paths, then one of the paths π_1 and π_2 will combine with subpaths of σ_1, σ_2, and σ_3 to form a v_1-to-v_2 path through a and c inside H_{ac}. Each of the remaining six possibilities with p_1 and p_2 in the different ones of those four paths will similarly lead to a v_1-to-v_2 path through a and c inside H_{ac}.

Finally, if path $\sigma_1 \in \Sigma$ contains v_1 but not v_2 and $\sigma_2 \in \Sigma$ contains v_2 but not v_1 and $\sigma_3 \in \Sigma$ contains none of v_1, v_2, b, d, then the v_1-to-a subpath of σ_1 followed by σ_3 followed by the c-to-v_2 subpath of σ_2 will be a v_1-to-v_2 path through a and c inside H_{ac}.

Therefore, H_{ac} does contain a v_1-to-v_2 path through a and c, as claimed. □

Similarly, $H_{b,d}$ contains a v_1-to-v_2 path through b and d. But this contradicts that those two internally disjoint paths would form a cycle of G' through all four of a, b, c, d.

To finish Case 2, now suppose $|S| = 3$, say with $S = \{v_1, v_2, v_3\}$. By Proposition 2(c), for every $x \in \{a, b, c, d\}$ there is a cycle $C_{\bar{x}}$ of the 3-connected graph G' such that $C_{\bar{x}}$ contains the three vertices in $\{a, b, c, d\} - \{x\}$ (with two of the three in one of H_{ac} and H_{bd}, and one in the other), but does not contain x. Although $C_{\bar{x}}$ might contain three vertices of S, exactly two of v_1, v_2, v_3 will have one neighbor along $C_{\bar{x}}$ in H°_{ac} and the other neighbor along $C_{\bar{x}}$ in H°_{bd}. There will be four pairs $C_{\bar{x}}, C_{\bar{y}}$ of such cycles that have $x \in \{a, c\}$ and $y \in \{b, d\}$. Since S contains only three pairs of vertices, there is an $x \in \{a, c\}$ and a $y \in \{b, d\}$ such that $C_{\bar{x}}$ and $C_{\bar{y}}$ both contain the same pair $v_i, v_j \in S$, with each of v_i and v_j having one neighbor along $C_{\bar{x}}$ from H°_{ac} and one neighbor along $C_{\bar{y}}$ from H°_{bd}. But this contradicts that the v_i-to-v_j subpath of $C_{\bar{x}}$ through b and d inside of H_{bd} and the v_i-to-v_j subpath of $C_{\bar{y}}$ through a and c inside of H_{ac} would be internally disjoint paths that form a cycle of G' through all four of a, b, c, d.

Case 3. Both ab and cd are cut-edges of $G - S$. The assumption that G is 4-connected implies $|S| = 3$, say with $S = \{v_1, v_2, v_3\}$. Without loss of generality, suppose $G' - S$ has connected components H°_a, H°_{bd}, and H°_c where vertex a is in the subgraph H_a of G' induced by $V(H^{\circ}_a) \cup S$, vertices b, d are in the subgraph H_{bd} of G' induced by $V(H^{\circ}_{bd}) \cup S$, and vertex c is in the subgraph H_c of G' induced by $V(H^{\circ}_c) \cup S$. Argue as in the final, $|S| = 3$ paragraph of the argument in Case 2, except now with $H_a \cup H_c$ in the role previously played by $H_{a,c}$. This leads to four cycles $C_{\bar{x}}$, each containing exactly two of v_1, v_2, v_3 that have one neighbor along $C_{\bar{x}}$ in $H_a \cup H_c$ and the other in H_{bd}. There will again be an $x \in \{a, c\}$ and a $y \in \{b, d\}$ such that $C_{\bar{x}}$ and $C_{\bar{y}}$ both contain the same pair $v_i, v_j \in S$. But this again contradicts that the v_i-to-v_j subpath of $C_{\bar{x}}$ through b and d inside of H_{bd} and the v_i-to-v_j subpath of $C_{\bar{y}}$ through a and c inside of $H_a \cup H_c$ would form a cycle of G' through all four of a, b, c, d. ■

Figure 3 shows a 3-connected graph with minimum degree 4 that has two nonadjacent edges that are not chords of a common cycle.

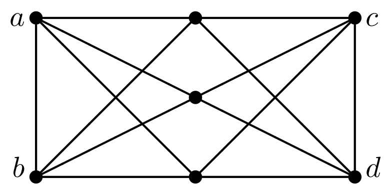

Figure 3. The nonadjacent edges ab and cd are not chords of a common cycle in this 3-connected graph.

Theorem 5. *In every* 4*-connected graph, every two edges are chords of a common cycle.*

Proof. Suppose G is a 4-connected graph, which implies that deleting two vertices will never create a cut-edge. Thus G satisfies condition (3b) and, since G has minimum degree at least 4, Theorem 3 implies that every two adjacent edges are chords of a common cycle. Lemma 4 implies the same is true for every two nonadjacent edges. ∎

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