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5-STARS OF LOW WEIGHT IN NORMAL PLANE MAPS WITH MINIMUM DEGREE 5

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Abstract

It is known that there are normal plane maps M_5 with minimum degree 5 such that the minimum degree-sum $w(S_5)$ of 5-stars at 5-vertices is arbitrarily large. In 1940, Lebesgue showed that if an M_5 has no 4-stars of cyclic type $(5,6,6,5)$ centered at 5-vertices, then $w(S_5) \leq 68$. We improve this bound of 68 to 55 and give a construction of a $(5,6,6,5)$-free M_5 with $w(S_5) = 48$.

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1. Introduction

A normal plane map (NPM for short) is a plane pseudograph in which loops and multiple edges are allowed, but the degree of each vertex and face is at least three. The degree of a vertex v is denoted by $d(v)$. A *k-vertex* is a vertex v with $d(v) = k$. A *k^+-vertex* (*k^--vertex*) is one of degree at least k (at most k). An NPM with minimum degree δ at least 5 is denoted by M_5. The *weight* of a subgraph of an NPM is the sum of degrees of its vertices. A k-star $S_k(v)$ is *minor* if its center v has degree (in the NPM) at most 5. All stars considered in this note are minor. By $w(S_k)$ we denote the minimum weight of minor k-stars in a given NPM.

In 1904, Wernicke [15] proved that every M_5 has a 5-vertex adjacent to a 6^--vertex. This result was strengthened by Franklin [8] in 1922 to the existence of a 5-vertex with two 6^--neighbors. In 1940, Lebesgue [14, p. 36] gave an approximate description of the neighborhoods of 5-vertices in M_5s. In particular, this description implies the results in [15, 8] and shows that there is a 5-vertex with three 8^--neighbors.

For M_5s, the bounds $w(S_1) \leq 11$ (Wernicke [15]) and $w(S_2) \leq 17$ (Franklin [8]) are tight. It was proved by Lebesgue [14, p. 36] that $w(S_3) \leq 24$, which was improved in 1996 by Jendrol' and Madaras [11] to the tight bound $w(S_3) \leq 23$. Furthermore, Jendrol' and Madaras [11] gave a precise description of minor 3-stars in M_5s.

For arbitrary NPMs, the following results concerning $(d-2)$-stars at d-vertices, $d \leq 5$, are known. Van den Heuvel and McGuinness [10] proved (in particular) that there is a vertex v such that either $w(S_1(v)) \leq 14$ with $d(v) = 3$, or $w(S_2(v)) \leq 22$ with $d(v) = 4$, or $w(S_3(v)) \leq 29$ with $d(v) = 5$. Balogh *et al.* [1] proved that there is a 5^--vertex adjacent to at most two 11^+-vertices. Harant and Jendrol' [9] strengthened these results by proving (in particular) that either $w(S_1(v)) \leq 13$ with $d(v) = 3$, or $w(S_2(v)) \leq 19$ with $d(v) = 4$, or $w(S_3(v)) \leq 23$ with $d(v) = 5$. Recently, we obtained a precise description of $(d-2)$-stars in NPMs [6].

For M_5s, Lebesgue [14, p. 36] proved $w(S_4) \leq 31$, which was improved by Borodin and Woodall [3] to the tight bound $w(S_4) \leq 30$. Note that $w(S_3) \leq 23$ easily implies $w(S_2) \leq 17$ and immediately follows from $w(S_4) \leq 30$ (in both cases, it suffices to delete a vertex of maximum degree from a minor star of the minimum weight). Recently, we obtained a precise description of 4-stars in M_5s [7].

For arbitrary NPMs, the problem of describing $(d-1)$-stars at d-vertices, $d \leq 5$, called *pre-complete stars*, appears difficult. As follows from the double n-pyramid, the minimum weight $w(S_{d-1})$ of pre-complete stars in NPMs can be arbitrarily large. Even when $w(S_{d-1})$ is restricted by appropriate conditions, the tight upper bounds on it are unknown. Borodin *et al.* [4, 5] proved (in particular) that if a planar graph with $\delta \geq 3$ has no edge joining two 4^--vertices, then there is a star $S_{d-1}(v)$ with $w(S_{d-1}(v)) \leq 38 + d(v)$, where $d(v) \leq 5$ (see [5, Theorem 2.A]). Jendrol' and Madaras [12] proved that if the weight of every edge in a planar graph with $\delta \geq 3$ is at least 9, then there is a pre-complete star in which every vertex has degree at most 20, where the bound of 20 is best possible.

The more general problem of describing d-stars at d-vertices, $d \leq 5$, called *complete stars*, at the moment seems intractable for arbitrary NPMs and difficult even for M_5s. In this note we make a modest contribution for the case of M_5s.

The following well-known construction shows that $w(S_5)$ is unbounded in M_5s. Take three concentric n-cycles $C^i = v_1^i \dots v_n^i$, where n is large and $1 \leq i \leq 3$, and join C^2 with C^1 by edges $v_j^2 v_j^1$ and $v_j^2 v_{j+1}^1$ whenever $1 \leq j \leq n$ (addition modulo n). The same is done with C^2 and C^3. Finally, join all vertices of C^1 to a new n-vertex, and do the same with C^3.

Definition. A 5-vertex v surrounded by vertices $v_1, \dots, v_5$ in a cyclic order is a $(5,6,6,5)$-*vertex* if there is a k, $k \leq 5$, such that $d(v_{k+1}) = d(v_{k+4}) = 5$, $d(v_{k+2}) \leq 6$, and $d(v_{k+3}) \leq 6$ (addition modulo 5).

Clearly, each 5-vertex in the M_5 constructed above is a $(5,6,6,5)$-vertex and, moreover, it has two 5-neighbors and two 6-neighbors. Lebesgue [14, p. 36] proved that if an M_5 has no $(5,6,6,5)$-vertices, then $w(S_5) \leq 68$.

The purpose of this note is to improve the bound of 68 to 55 (Theorem 1) and give a construction of a $(5,6,6,5)$-free M_5 with $w(S_5) = 48$ (see Figure 1). We first truncate all vertices of the dodecahedron, and then join the mid-points of the edges of each triangle and put a 2-vertex on each edge not incident with a triangle. Finally, we insert a 20-wheel inside every 20-face as shown in Figure 1. As a result, every 5-vertex has neighbors of degrees 5, 6, 7, 5, and 20 in this order.

Theorem 1. *If a normal plane map M_5 with minimum degree* 5 *does not contain* $(5,6,6,5)$*-vertices, then M_5 contains a* 5*-star of weight at most* 55 *with a* 5*-vertex as its center.*

Problem 2. Find best possible version of Theorem 1.

2. PROOF OF THEOREM 1

It suffices to prove the theorem for triangulations, since adding a diagonal edge into a non-triangular face of a normal plane map with $\delta = 5$ cannot create a new minor 5-star, nor can it reduce the weight of any existing minor 5-star. So

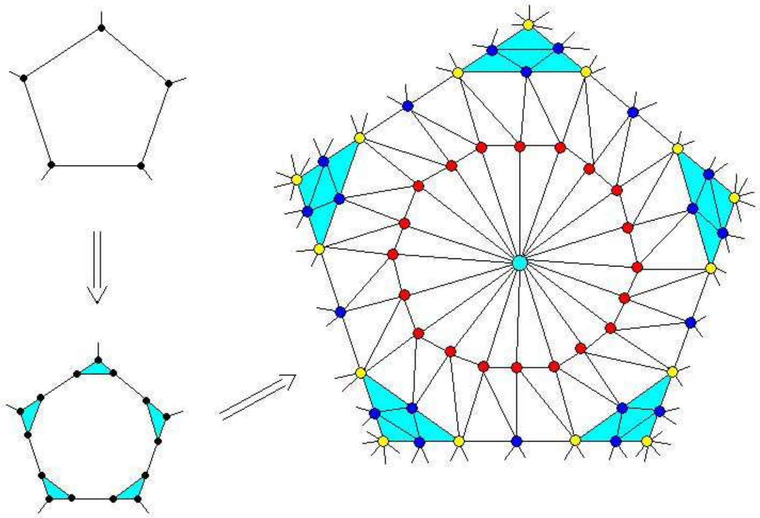

Figure 1. An M_5 without $(5,6,6,5)$-vertices such that $w(S_5)=48$.

suppose that a triangulation T, with its sets of vertices, edges, and faces denoted by V, E, and F, respectively, is a counterexample to Theorem 1. Euler's formula $|V|-|E|+|F|=2$ for T implies

$$\sum_{v\in V}(d(v)-6)=-12. \tag{1}$$

Assign an *initial charge* $\mu(v)=d(v)-6$ to each $v\in V$, so that only 5-vertices have negative initial charge. Also, put $\mu(f)=0$ for each $f\in F$. Using the properties of our T as a counterexample, we define a local redistribution of charges, preserving their sum, such that the final charge $\mu_3(x)$ is non-negative for all $x\in V\cup F$. This will contradict the fact that the sum of the final charges is, by (1), equal to -12. The technique of discharging is often used in solving structural and coloring problems on plane graphs.

Definition. For integer n, $n\geq 6$, put $\xi(n)=\frac{n-6}{n}$. For any 6^+-vertex v, put $\psi(v)=\xi(d(v))$.

In what follows, it is convenient to use the upwards convexity of the increasing function $\xi(n)$ for integer $n\geq 6$, which is checked easily.

Lemma 3. *For any integers p and q, where $6\leq p<q$, we have*
$$\xi(p)+\xi(q)\leq\xi(p+1)+\xi(q-1).$$

The *final charge* $\mu_3(v)$ of vertex v is defined by applying the rules R1–R3 as follows.

R1. Each 6^+-vertex v sends $\psi(v)$ to each incident 3-face.

R2. Let $f = xyz$ be a face such that $d(x) = 5$ and $d(z) \geq 6$. Then x receives from f the following charge:
(a) $\frac{\psi(z)}{2}$ if $d(y) = 5$, or
(b) $\psi(y) + \psi(z)$ if $d(y) \geq 6$.

The charge of x, where $x \in V \cup F$, after applying R1 and R2 is denoted by $\mu_2(x)$.

Definition. A 5-vertex v surrounded by vertices $v_1, \dots, v_5$ in a cyclic order is *bad* if $d(v_1) = d(v_2) = d(v_4) = 5$ and $d(v_3) = 7$.

Note that each bad 5-vertex v satisfies $d(v_5) \geq 29$ since $w(S_5) \geq 56$ by assumption.

R3. Suppose v is bad, and let the neighbors of v_2 and v_1 in the cyclic order be v, v_3, x, y, v_1 and v, v_2, y, z, v_5, respectively (see Figure 2).
(a) If $d(x) = 5$ (which means that v_2 is also bad), then v_1 gives $\frac{1}{14}$ to each of v and v_2.
(b) If $d(x) \geq 6$ and v_1 is also bad (that is $d(y) = 7$, and $d(z) = 5$), then v_2 gives $\frac{1}{14}$ to each of v and v_1.
(c) If $d(x) \geq 6$ but v_1 is not bad, then v_2 gives $\frac{1}{14}$ only to v.

Clearly, $\mu_3(v) = \mu_2(v) = 0$ for each 6^+-vertex v, and $\mu_3(f) = \mu_2(f) \geq 0$ for each face f. It remains to show that every 5-vertex receives at least 1 in total by R1–R3. If v is a bad 5-vertex, then $\mu_2(v) \geq 5 - 6 + \xi(7) + \xi(28) = -\frac{1}{14}$, which implies that $\mu_3(v) \geq 0$.

Our next goal is to show that every non-bad 5-vertex v satisfies $\mu_2(v) \geq 0$, and then we will complete the proof of Theorem 1 by checking that in fact $\mu_3(v) \geq 0$.

Remark 4. Suppose a 5-vertex v is adjacent to a 6^+-vertex w; then v receives from w by R1 and R2 one of the charges $\psi(w)$, $\frac{3\psi(w)}{2}$, or $2\psi(w)$ depending on the number of 5-vertices in the two 3-faces incident with edge vw: three, two, or one, respectively.

Definition. The *type of a* 5*-vertex* v is the vector $(d_1, \dots, d_5)$ of the degrees of the neighbors of v in the non-decreasing order. (So, d_1 is the smallest degree among the neighbors of v, d_2 is the second smallest, and so on.)

Lemma 5. *If* v *is a non-bad* 5*-vertex, then* $\mu_2(v) \geq 0$.

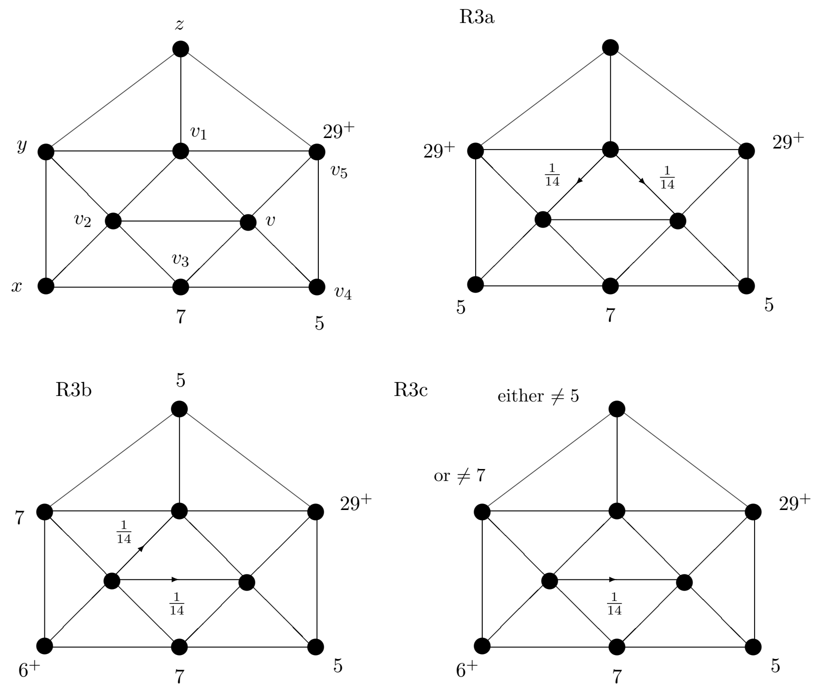

Figure 2. Rule R3.

Proof. Clearly, $\mu_2(v) = -1 + \mu^+_{1,2}(v)$, where $\mu^+_{1,2}(v)$ denotes the total donation to v from its neighbors by R1 and R2. Thus it suffices to prove that $\mu^+_{1,2}(v) \geq 1$. It will be helpful to note that

$$\frac{3}{2}\xi(27) = \frac{3}{2} \cdot \frac{21}{27} = \frac{7}{6} > \frac{8}{7} > 1. \tag{2}$$

Let v be of type $(d_1, \ldots, d_5)$; then $d_1 + \cdots + d_5 \geq 56 - 5 = 51$ since T is a counterexample to Theorem 1.

Now our proof splits into four cases.

Case 1. $6 \leq d_1$. By Remark 4, $\mu^+_{1,2}(v) = 2(\psi(d_1) + \cdots + \psi(d_5))$. This is smallest when $d_1 = \cdots = d_4 = 6$ and $d_5 = 27$, as otherwise it can be made smaller by using Lemma 3, or just by reducing d_5 if $d_5 > 27$. Thus $\mu^+_{1,2}(v) \geq 2\xi(27) > 1$ by (2).

Case 2. $5 = d_1 < d_2$. By Remark 4, $\mu^+_{1,2}(v) \geq \frac{3}{2}(\xi(d_2) + \cdots + \xi(d_5))$, which is smallest when $d_2 = \cdots = d_4 = 6$ and $d_5 = 28$. Thus $\mu^+_{1,2}(v) \geq \frac{3}{2}\xi(28) > 1$ by (2).

Case 3. $5 = d_1 = d_2 < d_3$. If the two 5-neighbors of v form a 3-face with v, then $\mu^+_{1,2}(v) \geq \frac{3}{2}(\xi(d_3) + \xi(d_4) + \xi(d_5))$, which is smallest when $d_3 = d_4 = 6$ and $d_5 = 29$. Thus $\mu^+_{1,2}(v) \geq \frac{3}{2}\xi(29) > 1$ by (2).

So assume that the two 5-neighbors of v do not form a 3-face with v. By Remark 4, $\mu^+_{1,2}(v) \geq \frac{3}{2}(\xi(a)+\xi(b))+\xi(c)$, where a, b, c is a permutation of d_3, d_4, d_5 such that the a-vertex and b-vertex form a 3-face with v. Recall that v is not a $(5, 6, 6, 5)$-vertex by assumption. So if $d_3 = d_4 = 6$ then $c = 6$ and $\max\{a, b\} \geq 29$; thus $\mu^+_{1,2}(v) \geq \frac{3}{2}\xi(29) > 1$ by (2). Otherwise, $d_4 \geq 7$ and $\mu^+_{1,2}(v) \geq \frac{3}{2}\xi(7)+\xi(28) = \frac{3}{2} \cdot \frac{1}{7} + \frac{22}{28} = 1$.

Case 4. $5 = d_1 = d_2 = d_3 < d_4$. Then $d_4 + d_5 \geq 36$. If $6 \leq d_4 \leq 7$, then the d_4-vertex and the d_5-vertex form a 3-face with v, since v is neither bad nor a $(5, 6, 6, 5)$-vertex by assumption; thus $\mu^+_{1,2}(v) \geq \frac{3}{2}\xi(d_5) \geq \frac{3}{2}\xi(29) > 1$ by (2). If $d_4 \geq 8$, then $\mu^+_{1,2}(v) \geq \xi(8) + \xi(28) = \frac{29}{28} > 1$. ■

Lemma 6. *If v is a non-bad 5-vertex, then $\mu_3(v) \geq 0$.*

Proof. If v gives nothing to bad 5-vertices by R3, then $\mu_3(v) = \mu_2(v) \geq 0$ by Lemma 5. So we may assume that v gives either $\frac{1}{14}$ or $\frac{1}{7}$ in total to bad 5-vertices by R3. It suffices to prove that $\mu^+_{1,2}(v) \geq \frac{8}{7}$.

We may assume that v is either the vertex v_1 in R3(a) or the vertex v_2 in R3(b) or R3(c). In both cases, $d(x) + d(y) \geq 34$ since $w(S_5(v_2)) \geq 56$. In the first case, $d(x) = 5$ and $d(y) \geq 29$ (see Figure 2), and so $\mu^+_{1,2}(v) \geq 2\xi(29) > \frac{8}{7}$ by (2). In the second case, $\mu^+_{1,2}(v) \geq \frac{3}{2}(\xi(6) + \xi(28)) = \frac{3}{2}\xi(28) > \frac{8}{7}$ by (2). ■

Thus we have proved $\mu_3(x) \geq 0$ for every $x \in V \cup F$, which contradicts (1) and completes the proof of Theorem 1.

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