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A CHARACTERIZATION OF 2-TREE PROBE INTERVAL GRAPHS

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Abstract

A graph is a *probe interval graph* if its vertices correspond to some set of intervals of the real line and can be partitioned into sets P and N so that vertices are adjacent if and only if their corresponding intervals intersect and at least one belongs to P . We characterize the 2-trees which are probe interval graphs and extend a list of forbidden induced subgraphs for such graphs created by Pržulj and Corneil in [*2-tree probe interval graphs have a large obstruction set*, *Discrete Appl. Math.* **150** (2005) 216–231].

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1. INTRODUCTION

A graph G is a *probe interval graph* if there is a partition of $V(G)$ into sets P and N and a collection $\{I_v : v \in V(G)\}$ of (open or closed) intervals of \mathbb{R} such that, for $u, v \in V(G)$, $uv \in E(G)$ if and only if $I_u \cap I_v \neq \emptyset$ and at least one of u, v belongs to P . The sets P and N are called the *probes* and *nonprobes*, respectively, and $\{I_v = (l_v, r_v) : v \in V(G)\}$ together with the partition into probes and nonprobes will be referred to in this paper as a *representation*. An *interval graph* is a probe interval graph with $N = \emptyset$, and this class of graphs has been studied extensively; see the texts by Fishburn [9], Golumbic [12], and Roberts [20] for introductions and other references.

The probe interval graph model was invented in order to aid with the task called *physical mapping* faced in connection with the human genome project, cf. work of Zhang and Zhang *et al.* [17, 22, 23]. In DNA sequencing projects, a *contig* is a set of overlapping DNA segments derived from a single genetic source. In order for DNA to be more easily studied, small fragments of it, called clones, are taken from multiple copies of the same genome. Physical mapping is the process of determining how DNA contained in a group of clones overlap without having to sequence all the DNA in the clones. Once the map is determined, the clones can be used as a resource to efficiently contain stretches of genome. If we are interested in overlap information between each pair of clones, we can use an interval graph to model this problem: vertices are clones and adjacency represents overlap. Using the probe interval graph model we can use any subset of clones, label them as probes, and test for overlap between a pair of clones if and only if at least one of them is a probe. This way there is flexibility, in contrast to the interval graph model, since all DNA fragments need not be known at time of construction of the probe interval graph model. Consequently, the size of the data set, which by nature can be quite large, is reduced.

We consider probe interval graphs as a combinatorial problem and focus on their structure. Here is a brief discussion of some of the recent results on the structure of probe interval graphs and where to find them. The paper by McMorris, Wang and Zhang [17] has results similar to those for interval graphs found in [11] by Fulkerson and Gross and [12] by Golumbic; e.g., probe interval graphs are weakly chordal, analogous to interval graphs being chordal, and, as maximal cliques are consecutively orderable in interval graphs, so-called quasi-maximal cliques are in probe interval graphs (see [17]). The neighborhood of graph classes surrounding probe interval graphs is discussed in [7] by Brown and Lundgren, [5] by Brown, Flink and Lundgren, [13] by Golumbic and Lipshteyn, and [14] by Golumbic and Trenk. Relationships between bipartite probe interval graphs, interval bigraphs and the complements of circular arc graphs are presented in [7].

One way to characterize and describe the structure of a hereditary class of

graphs is via a complete list of minimal forbidden induced subgraphs; we will call such a characterization a FISC (for forbidden induced subgraph characterization). A requisite for a FISC for a class of graphs is that any induced subgraph of a graph from that class still belongs to that class. We say a property of a class of graphs is *hereditary* if the property remains after taking vertex-deleted subgraphs, that is, if any induced subgraph keeps the property. A FISC is often tantamount to an efficient recognition algorithm and is potentially an elegant and concise way of completely describing the structure of the graph class, as in the celebrated result of Beineke for line graphs ([1]). FISCs have been obtained for probe interval graphs which are forests by Sheng in [21], for unit probe interval graphs which are forests by Brown, Lundgren, and Sheng in [8], for bipartite unit probe interval graphs by Brown and Langley in [6], and all of these FISCs are arguably concise. Conciseness, however, is not a possible characteristic for a FISC for probe interval graphs in general. Definitive evidence for this was established in [19] by Pržulj and Corneil where a large list of forbidden induced subgraphs for the probe interval graphs which are 2-trees was developed. This paper extends the results in [19] and, in a sense, completes them. Specifically, we add to Pržulj and Corneil's list of forbidden subgraphs and characterize the probe interval graphs which are 2-trees. Our characterization is not a FISC, but rather along the lines of the characterization *a cycle-free graph is an interval graph if and only if it is a caterpillar* which follows from [15].

The forbidden induced subgraphs developed by Pržulj and Corneil in [19] are those in Figure 1 together with T_3 and H in Figure 2. In this paper, we develop graphs which should be added to this list. Specifically, we develop a family of these new forbidden graphs in Section 3, and name six others in Section 5. But we make some observations about which graphs to include in the list, as there is a bit of inconsistency in the qualifications for the graphs included in the list of [19], at least as far as what one may want from a list of forbidden induced subgraphs for a class of graphs. The inconsistency is hard to avoid and may be a reflection of a choice by Pržulj and Corneil to try to keep the list as small as possible while keeping to the spirit of a FISC which, strictly speaking, needs to be carefully interpreted for 2-tree probe interval graphs. The property of being a probe interval graph is hereditary, but the property of being a 2-tree is not. $G_1 - v$, $G_2 - v$, and $G_3 - v$ in Figure 1 are not probe interval graphs but also not 2-trees; $G_1 - x$, $G_2 - x$, and $G_3 - x$ are probe interval graphs and 2-trees. We note that F_j may be extended beyond $j = 4$ and F_j , for any $j \in \{1, 2, 3, 4\}$, is not an induced subgraph of, say, F_7 and each F_k , for $k > 4$, is not a probe interval graph. So, not including F_k , for $k > 4$, in any list \mathcal{L} of forbidden subgraphs will yield the possibility of being given a 2-tree and a decision problem regarding membership to $\{2\text{-trees}\} \cap \{\text{probe interval graphs}\}$ which may yield a false negative if \mathcal{L} is the basis for the decision. We discuss this in more detail in Section 5.

We have obtained a characterization of 2-tree probe interval graphs which uses and develops concepts of generalized paths introduced by Beineke and Pippert in [2], and of specialized k -trees introduced by Proskurowski in [18]. This approach allows us to name precisely the class of graphs which are 2-tree probe interval.

2. PRELIMINARIES

We discuss finite simple graphs and if G is a graph, we will denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. The *neighborhood* of a vertex v of graph G , that is, the set $\{u \in V(G) : uv \in E(G)\}$, will be denoted by $N_G(v)$. A subgraph of G on k vertices which is complete will be called a k -clique. Recall from the introduction the definition of a probe interval graph. We will be using the notation I_v for the interval corresponding to vertex v , and $I_v = (l_v, r_v)$, that is, l_v and r_v will be used to denote the left- and right-endpoints of the interval corresponding to vertex v . If G is a probe interval graph and $V(G)$ partitioned into P and N and vertices corresponding to $\mathcal{I} = \{I_v : v \in V(G)\}$, we refer to \mathcal{I} together with P and N as G 's *representation*.

A *2-tree* may be recursively defined as follows:

- K_2 is a 2-tree;
- Suppose G is a 2-tree; create G' by adding a vertex to G adjacent to both vertices of some K_2 of G .

In the direction of obtaining a FISC (forbidden induced subgraph characterization) for the 2-trees which are probe interval graphs Pržulj and Corneil in [19] established a list of 62 forbidden subgraphs. The graphs in Figure 1 plus the graphs T_3 and H in Figure 2 are the 62 graphs.

The graphs T_3 and H in Figure 2 are the two graphs which Sheng determined provide a FISC for cycle-free probe interval graphs, see [21], and T_2 will be referred to often in the development of our results.

The conciseness of the characterization for cycle-free probe interval graphs is encouraging in that it suggests one may find a concise characterization for a more general class of graphs than forests. This turns out to be the case to some extent. Indeed, in [8] a FISC for forests which are unit probe interval graphs (probe interval graphs which can be represented using only unit-length intervals, or intervals such that none properly contains another) was given which lists T_2 , two caterpillars and one infinite family of caterpillars, and, according to the results in [6], five more graphs suffice for a FISC for unit probe interval graphs which are bipartite. Dropping the unit restriction and generalizing beyond forests which are probe interval graphs has been less than encouraging, as far as a concise FISC is concerned. In [7] it is shown that any FISC for the bipartite

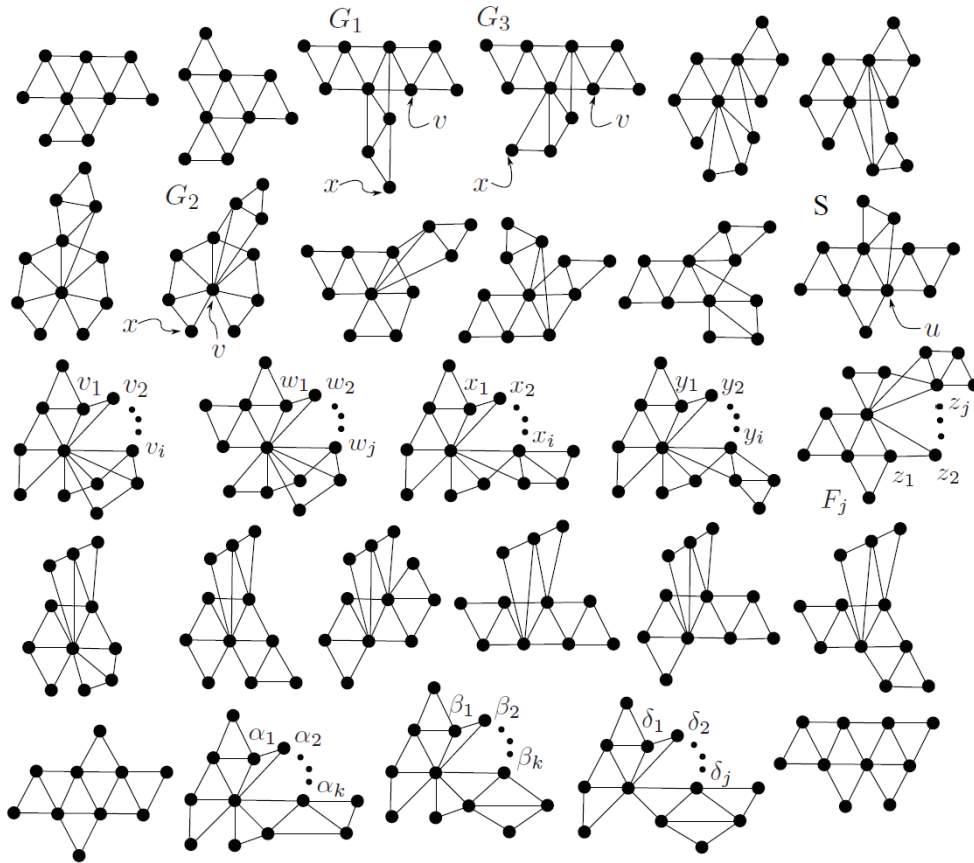
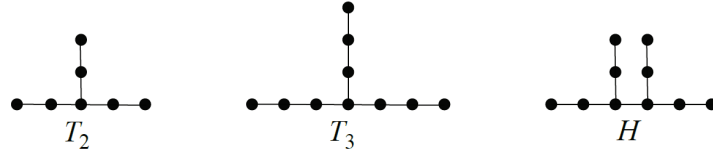


Figure 1. 60 of the forbidden induced subgraphs for 2-tree probe interval graphs determined by Pržulj and Corneil, where $i \in \{1, \dots, 6\}$, $j \in \{1, 2, 3, 4\}$, and $k \in \{1, \dots, 5\}$.

probe interval graphs in which the probe-nonprobe partition can correspond to the graph's bipartition (note that this need not be the case) includes sixteen graphs and twelve infinite families of graphs.

Recognition of probe interval graphs falls into two general categories: partitioned and non-partitioned, that is, G is given without knowing which vertices are to be probes and nonprobes, or G is given with the partition of $V(G)$ into probes and nonprobes. The best known recognition algorithm for the non-partitioned case is a polynomial time one, while for the partitioned case a linear-time recognition algorithm has been demonstrated by McConnell and Nussbaum in [16]. In [4] it is shown that the bipartite probe interval graphs enjoy a non-partitioned recognition algorithm which is linear in the number of vertices plus the number of edges, however, the structural results given show that any FISC will be quite complicated.

In this paper, we impose no restriction, such as unit, on the representation

Figure 2. T_2, T_3 , and H .

and generalize in a different direction than the bipartite one. As Pržulj and Cornil did, we consider the 2-trees which are probe interval graphs. We show that the list given in [19] (described above) should include, depending on how you count, at least seven more graphs. We present these graphs and discuss the issues surrounding the proper way to enumerate the size of the list in Section 5. In lieu of a FISC, which will have to overcome the nuance of describing a non-hereditary class, we use results of Flesch and Lundgren in [10] to give a characterization which precisely names the structure of 2-trees which are probe interval graphs. We will now describe this structure.

We recall a notion of Beineke and Pippert's in [2] which generalizes the idea of a path.

Definition. A *2-path* of G is an alternating sequence of distinct 2- and 3-cliques of G , $(e_0, t_1, e_1, t_2, e_2, \dots, t_p, e_p)$, starting and ending with a 2-clique and such that t_i contains exactly two of the distinct 2-cliques: e_{i-1} and e_i ($1 \leq i \leq p$). The *length* of the 2-path is the number p of 3-cliques. The letters e and t are used to remind us of edges and triangles (K_3 s) and we will keep to this convention in the sequel.

We now classify a few structures in 2-trees and define some derived graphs we will use. A vertex v of a 2-tree G is a *2-leaf* of G if $N_G(v)$ is an edge of G . Let G be a 2-tree and define G^{1-} to be $G - P_G$, where P_G is the set of 2-leaves of G ; iteratively, $G^{2-} = G^{1-} - P_{G^{1-}}$. It will turn out that if G is a 2-tree probe interval graph, then G^{2-} must be a 2-path. For those graphs G where G^{2-} is a 2-path and for the purposes of our characterization, we need to classify certain vertices. These classifications will help with the partition of $V(G)$ into probes and nonprobes.

Suppose G is a 2-tree such that G^{2-} is the 2-path $(e_0, t_1, e_1, t_2, \dots, t_p, e_p)$, such that e_0 and e_p are defined in the following way. Let a_0 be a 2-leaf of G^{1-} such that $N_{G^{1-}}(a_0) \subset t_1$ and a_p be a 2-leaf of G^{1-} such that $N_{G^{1-}}(a_p) \subset t_p$. Define $e_0 = N_{G^{1-}}(a_0)$ and $e_p = N_{G^{1-}}(a_p)$. This will be our intended meaning for e_0 and e_p for the rest of the paper. Note that there may be an ambiguity in which edge of G^{2-} is to be e_0 or e_p , but this choice may always be made arbitrarily as it does not affect any results. We now describe two sets of 2-leaves in G and G^{1-} respectively.

- $\partial_1 G = \{v \in V(G) : v \text{ is a 2-leaf of } G, N_G(v) \neq e_i \text{ (} 0 \leq i \leq p)\}$;
- $\partial_2 G = \{v \in V(G^{1-}) : v \text{ is a 2-leaf of } G^{1-}, N_{G^{1-}}(v) \neq e_i \text{ (} 0 \leq i \leq p)\}$.

Interval p-graphs are the graphs in which vertices correspond to intervals (of, say, the real line) of p possible colors and edges correspond to nonempty intersections of differently colored intervals. Flesch and Lundgren showed in [10] that the class of 2-trees that have interval p -representations are spiny interior 2-lobsters, and in [3] it is shown that any p -chromatic probe interval graph is an interval p -graph. These relationships allow us to streamline our investigation and so we name the class of 2-trees in which probe interval graphs are necessarily confined: the spiny interior 2-lobsters.

Definition. A 2-tree G is a *2-lobster* if G^{2-} is a 2-path. A *spiny interior 2-lobster* is a 2-lobster G with $\partial_2 G = \emptyset$.

To bring us closer to the class of 2-trees which are probe interval graphs, we refine the classification of vertices in $\partial_1 G$ into the sets $\partial_1^1 G$ and $\partial_1^2 G$:

- $\partial_1^1 G = \{v \in \partial_1 G : N_G(v) \subseteq V(G^{2-})\}$;
- $\partial_1^2 G = \{v \in \partial_1 G : N_G(v) \not\subseteq V(G^{2-})\}$.

Now, certain vertices which are not adjacent to these 2-leaves will be particularly important and we identify those now. The sets we now define are sets of vertices of G^{2-} . To define the sets, we will speak of a vertex v being *grown from edge xy* , denoted $v \partial xy$, if v can be regarded as having been added to G in some step of the recursive construction of G via the edge xy . We will use this relation recursively as well, that is, $a \partial bc \partial cd$ denotes that $b \partial cd$ and then $a \partial bc$; note that a is not adjacent to d . For example, see the graph in Figure 4 called the 3-sun. In that graph, we may regard d as being grown from bf which may be regarded as grown from fc (or bc); that is, $d \partial bf \partial fc$ (or $d \partial bf \partial bc$), and $a \partial bc \partial fc$. The following definitions apply to spiny interior 2-lobster G .

- W^1 : a vertex v belongs to W^1 if some $x \in \partial_1^1 G$ satisfies $x \partial (t_i - v)$ for some t_i ($1 \leq i \leq p$).
- W^2 : a vertex v belongs to W^2 if some $y \in \partial_1^2 G$ satisfies $y \partial xz \partial e_i$, where $e_i = xv$, for some e_i ($1 \leq i \leq p - 1$).
- W^3 : vertex v belongs to W^3 if some $y \in \partial_1^2 G$ satisfies $y \partial xz \partial e_i$, where $e_i = xv$ ($i \in \{0, p\}$).
- $W^{3'}$: vertex v belongs to $W^{3'}$ if some pair $y, s \in \partial_1^2 G$ satisfy $y \partial xz \partial e_i$ and $s \partial xr \partial e_i$, where $i = 0$ or $i = p$, and $z \neq r$.

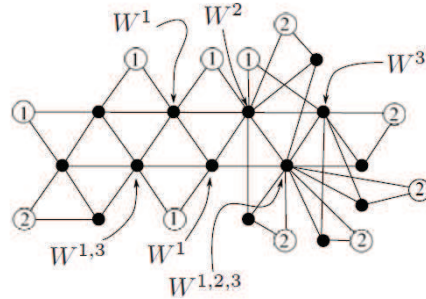


Figure 3. Examples of vertices of $\partial_1^1 G$, $\partial_1^2 G$, $W^1(G)$, $W^2(G)$, $W^3(G)$, and $W^{3'}(G)$. Vertices labeled with 1 belong to $\partial_1^1 G$, those labeled with 2 to $\partial_1^2 G$; vertices labeled with W^a , $W^{a,b}$, or $W^{a,b,c}$ belong to W^a , $W^a \cap W^b$, $W^a \cap W^b \cap W^c$, respectively.

- $W = W^1 \cup W^2 \cup W^3$.

In Figure 3 we illustrate these classifications of vertices. Notice that W^3 will never be empty, because G^{2-} eliminates two vertices on either end of the longest 2-path in G .

We are now ready to identify the class of 2-trees which are probe interval graphs: the sparse spiny interior 2-lobsters.

Definition. Let G be a spiny interior 2-lobster with G^{2-} the 2-path $(e_0, t_1, e_1, t_2, \dots, t_p, e_p)$. The following two conditions hold if and only if G is a *sparse spiny interior 2-lobster (ssi2-lobster)*:

1. No $t_i, 1 \leq i \leq p$, has two vertices in $W^1 \cup W^2 \cup W^{3'}$.
2. No $t_i, i \in \{1, p\}$, has three vertices x, y , and z such that $x, y \in W^3$ and if $e_0 = xy$ or $e_p = xy$ then $z \in W^1 \cup W^2 \cup W^{3'}$.

3. FOUNDATION FOR THE CHARACTERIZATION

We recall some definitions and results from [21] and [19] which will be useful for our characterization.

An *asteroidal triple (AT)* in a graph G is a set of three vertices with the property that between each pair of vertices there is a path connecting them which does not intersect the neighborhood of the third. A collection of sets $\{X, Y, Z\}$ is an *asteroidal collection (AC)* if for each $x \in X$, each $y \in Y$, and each $z \in Z$, the triple $\{x, y, z\}$ is an asteroidal triple. Each of these sets X, Y , and Z is called an *asteroidal set (AS)*. The following results in [19] and [21] are for determining the probe-nonprobe partition in a probe interval graph.

Lemma 1 [21]. *If G is a probe interval graph, then at least one vertex of every asteroidal triple must be a nonprobe.*

Lemma 2 [21]. *If T_2 is an induced subgraph of a probe interval graph, then the vertex of degree 3 in the induced T_2 must be a nonprobe.*

Corollary 3 [19]. *At least one asteroidal set of an asteroidal collection in a probe interval graph G must consist entirely of nonprobes. Thus at least one asteroidal set of a probe interval graph must be an independent set.*

Lemma 4 [19]. *In every asteroidal triple of a probe interval graph G there must exist a nonprobe vertex u such that there exists a path between the other two asteroidal triple vertices that avoids $N(u)$ and has a nonprobe internal vertex.*

Corollary 5 [19]. *Up to isomorphism, there exists only one probe-nonprobe partition of vertices of the 3-sun of Figure 4.*

To conserve notation, in all of what follows, we will drop the $V(\cdot)$ notation and simply write t_i or e_i and use $+$ and $-$ to denote \cup and \setminus ; so for example, $t_i - e_i + v$ means $V(t_i) \setminus V(e_i) \cup \{v\}$, where v is a vertex.

Lemma 6. *Let G be a spiny interior 2-lobster with a probe interval representation. Any $w_x \in W^1$ must be a nonprobe.*

Proof. Let G be a spiny interior 2-lobster with a probe interval representation with $G^{2-} = (e_0, t_1, e_1, t_2, \dots, t_p, e_p)$, and let $x \in \partial_1^1 G$ with $N_G(x) = t_i - w_x$. The graph G^{2-} is a subgraph of a 2-path of length $p + 4$ in G ; label this 2-path $G^{2-*} = (e_{-2}, t_{-1}, e_{-1}, t_0, e_0, t_1, \dots, e_p, t_{p+1}, e_{p+1}, t_{p+2}, e_{p+2})$. Let $a \in t_{i-2} - e_{i-2}$, $b \in t_{i-1} - e_{i-1}$, $c \in t_{i+1} - e_i$, $d \in t_{i+2} - e_{i+1}$, $r \in t_i - e_i$, $s \in t_i - e_{i-1}$. With $X = \{x\}$, $Y = \{a, b\}$, and $Z = \{c, d\}$, we have $\{X, Y, Z\}$ is an AC. By Corollary 3, since $ab \in E(G)$ and $cd \in E(G)$, the AS $X = \{x\}$ must contain all nonprobes. Now consider the subgraph H induced by the vertices $\{b, c, r, s, x, w_x\}$, which is isomorphic to the 3-sun. By Corollary 5, since x must be a nonprobe, w_x must also be a nonprobe. ■

The next lemma is to the end of showing that any vertex in W^2 must be a nonprobe.

Lemma 7. *Up to isomorphism, there are exactly three probe-nonprobe partitions of vertices of the probe interval graph Q in Figure 4.*

Proof. Label the vertices of the probe interval graph Q as in the Figure 4 and reserve the sets P and N for the probes and nonprobes, respectively. The vertices $\{a, g, h\}$ form an AT. By Lemma 4, there must exist a nonprobe AT vertex u with a path between the other two AT vertices that avoids $N_Q(u)$ and has a nonprobe

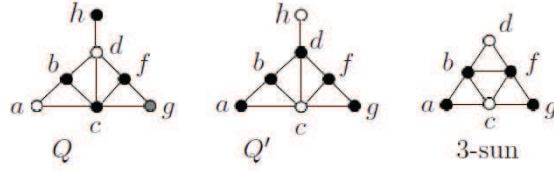


Figure 4. Graphs Q and the 3-sum with their possible (P, N) -partitions up to isomorphism. The white vertices are nonprobes, black are probes, and the grey can be either a probe or nonprobe.

internal vertex. Assume that h is the AT vertex with this property. The only vertex in the neighborhood of h is d , so the internal vertex that must also be a nonprobe can be either b, f , or c .

Assume that b is this nonprobe vertex, and thus a and c must be probes. In a probe interval representation of G , I_h and I_c must not overlap, since they are not adjacent and $c \in P$. Furthermore, d is adjacent to both h and c . Now, without loss of generality, let $r_h < l_c$ and hence $l_d < r_h$ and $r_d > l_c$. Now both b and f are adjacent to d but not adjacent to one another or to h . Assume that f is a probe, so I_f and I_b must not overlap. If $I_f \subset I_d$, then $I_g \cap I_d \neq \emptyset$, which is a contradiction since d is a probe and d and g are not adjacent. Thus, since f is not adjacent to h , we must have $r_b < l_f$. However, this forces $I_a \cap I_d \neq \emptyset$, since a is adjacent to both c and b , which is a contradiction.

Let us assume then that f is a nonprobe and that $r_b < r_f$. Since a and g are adjacent to c and not to d , then $r_d < l_a$ and $r_d < l_g$. Thus, we have $I_a \cap I_f \neq \emptyset$, which is a contradiction. We also get a contradiction if $f \in N$ and $r_f < r_b$, so if $h \in N$ then $b \in P$.

Similar arguments yield contradictions if we assume h and f must be the nonprobes that satisfy Lemma 4, so it is also true that if $h \in N$, then $f \notin N$. Hence it must be the case that $N = \{h, c\}$ and $P = \{a, b, d, f, g\}$ (see Q' in Figure 4). With this probe-nonprobe partition and the interval assignment $I_c = (0, 6), I_a = (0, 2), I_b = (1, 2.5), I_d = (2, 4), I_f = (3.5, 5), I_g = (4, 6), I_h = (2.5, 3.5)$, we can see that this is a probe interval representation.

Now we start with the AT nonprobe vertex being a , which forces $b, c \in P$. By Lemma 4, either d or f must be a nonprobe. Assume that f is a nonprobe, which means that d and g are probes. We also know from above that h must be a probe if f is a nonprobe, and thus the only vertices that are nonprobes are a and f . Since $N_Q(a) \not\subseteq N_Q(f)$ and $N_Q(f) \not\subseteq N_Q(a)$, neither $I_a \subseteq I_f$ nor $I_f \subseteq I_a$. We know that I_a and I_f must overlap, or else the probe interval representation would be an interval representation, which is a contradiction since G contains an AT. Without loss of generality, let $l_a < l_f$. Since $b \in N_Q(a)$, but $b \notin N_Q(f)$, we know that $r_b < l_f$. However, d is adjacent to both b and f , so $l_d < r_b$ and $r_d > l_f$. This forces $I_a \cap I_d \neq \emptyset$, a contradiction. Therefore, d must be the nonprobe vertex.

Hence either $N = \{a, d\}$, $P = \{b, c, f, h, g\}$ or $N = \{a, d, g\}$, $P = \{b, c, f, h\}$ (see Q in Figure 4). With either (P, N) -partition and the interval assignment $I_c = (2, 6), I_a = (2, 3), I_b = (2, 3), I_d = (1, 4), I_f = (3, 5), I_g = (4, 6), I_h = (0, 2)$, we can see that both are probe interval representations.

A similar argument yields a contradiction if we assume the AT nonprobe vertex is g . However, these probe-nonprobe partitions are isomorphic to the ones above. Thus there are exactly three probe-nonprobe partitions of vertices of probe interval graph Q in Figure 4 up to isomorphism. ■

Now we can prove that the vertices of W^2 must be nonprobes.

Lemma 8. *Let G be a spiny interior 2-lobster with a probe interval representation. Any $w_y \in W^2$ must be a nonprobe.*

Proof. Let G be a spiny interior 2-lobster with probe interval representation, P the set of probes, N the nonprobes, and $G^{2-} = (e_0, t_1, e_1, t_2, \dots, t_p, e_p)$. Let $y \in \partial_1^2$ such that $N_{G^{2-}}(y) = e_i - w_y$ for $w_y \in W^2(G)$; from the definition of W^2 , $i \neq 0, p$. The graph G^{2-} is a subgraph of a 2-path of length $p + 4$ in G ; label this 2-path $G^{2-*} = (e_{-2}, t_{-1}, e_{-1}, t_0, e_0, t_1, \dots, e_p, t_{p+1}, e_{p+1}, t_{p+2}, e_{p+2})$. Let $a \in t_{i-2} - e_{i-2}$, $b \in t_{i-1} - e_{i-1}$, $c \in t_{i+2} - e_{i+1}$, $d \in t_{i+3} - e_{i+2}$, $r \in t_{i+1} - e_i$, and $N_G(y) = e_i + u - w_y$. We have three cases to consider. Either both $b, c \notin N_G(w_y)$, exactly one of b or $c \notin N_G(w_y)$, or $b, c \in N_G(w_y)$.

Case 1. The vertices $b, c \notin N_G(w_y)$. If both $b, c \notin N_G(w_y)$, then the graph induced by vertices $\{b, s, w_y, r, c, u, y\}$ is T_2 . By Lemma 2, w_y must be a nonprobe.

Case 2. Exactly one of b or $c \in N_G(w_y)$. Without loss of generality, assume $b \in N_G(w_y)$ and $c \notin N_G(w_y)$. Label vertex v such that $v \in t_i$, but $v \notin e_{i-1}$, which implies that $v \in e_i$. With $Y = \{y\}$, $X = \{a, b\}$, and $Z = \{c, d\}$, $\{X, Y, Z\}$ is an AC. Since $ab \in E(G)$ and $cd \in E(G)$, the AS $Y = \{y\}$ must contain all nonprobes by Corollary 3. We know that $y \notin N_G(r)$ since $r \notin e_i$. We know that $u, y, r, c, v \notin N_G(b)$ because $b \notin e_{i-1}$. Thus the graph H induced by the vertices $\{b, w_y, r, c, v, u, y\}$ is isomorphic to Q in Figure 4, with $w_y \cong d$ and $y \cong a$. By Lemma 7 and the fact that $y \in N$, w_y is a nonprobe.

Case 3. The vertices $b, c \in N_G(w_y)$. Let $e_i = w_y f$; with $Y = \{y\}$, $X = \{a, b\}$, and $Z = \{c, d\}$, $\{X, Y, Z\}$ is an AC. Since $ab \in E(G)$ and $cd \in E(G)$, the AS $Y = \{y\}$ must contain all nonprobes by Corollary 3. Now consider the subgraph H induced by the vertices $\{b, s, f, y, w_y, r, c\}$, which is isomorphic to Q' in Figure 4 with $w_y \cong c$ and $y \cong h$. By Lemma 7 and the fact that $y \in N$, the vertex w_y is a nonprobe. ■

Now, to the end of proving our main result, we develop a new family of forbidden subgraphs not given in [19].

Lemma 9. *The graphs $H_i, i \in \mathbb{Z}^+$ are not probe interval graphs.*

Proof. Label $H_i, i \in \mathbb{Z}^+$ as in Figure 5. The subgraph induced by the vertices $\{w_y, w_x, r, g, y, f, h\}$ is T_2 . Thus by Lemma 2, w_y is a nonprobe. Now consider the subgraph induced by $\{r, b, x, d_1, w_x, w_y\}$, which is a 3-sun with w_y as a nonprobe. By Corollary 5, b must also be a nonprobe. Lastly, with $X = \{x\}, Y = \{r\}$, and $Z = \{g, y\}$, $\{X, Y, Z\}$ is an AC. By Lemma 3 and the fact that g and y are adjacent, either r or x must be a nonprobe. However, we already said that b must be a nonprobe, and both r and x are adjacent to b , which is a contradiction. Therefore, no $H_i, i \in \mathbb{Z}^+$ is a probe interval graph. ■

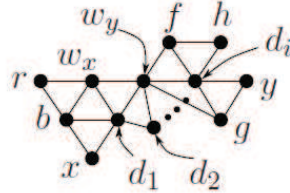


Figure 5. $H_i, i \in \mathbb{Z}^+$.

4. CHARACTERIZATION

We have established most of the structural results sufficient for our characterization of 2-tree probe interval graphs.

Lemma 10. *Let G be a 2-tree. If G is an ssi2-lobster, then G is a probe interval graph.*

Proof. Suppose G is a sparse spiny interior 2-lobster such that $G^{2-} = (e_0, t_1, e_1, \dots, t_p, e_p)$. Reserve the sets P and N for the probes and nonprobes, respectively. By definition of e_0 and e_p , we know that both contain at least one vertex in W^3 . If e_0 contains exactly one vertex in W^3 , label it w . Similarly if e_p contains exactly one vertex in W^3 , label it a . If both vertices of e_0 (or e_p) are in $W^3 - (W^1 \cup W^2 \cup W^{3'})$, then label the vertex in e_0 and e_1 as w (or e_p and e_{p-1} as a). If not then by the first condition of an ssi2-lobster, both vertices of e_0 (and e_p) are not in the set $W^1 \cup W^2 \cup W^{3'}$. Thus choose a vertex $w \in e_0$ such that $w \in W^3 - (W^1 \cup W^2 \cup W^{3'})$, and choose a vertex $a \in e_p$ such that $a \in W^3 - (W^1 \cup W^2 \cup W^{3'})$. Let $y \in \partial_1^2$ such that $N_G(y) = \{N_{G^{1-}}(z), z\} - \{w\} = \{e_0, z\} - \{w\}$. Label $e_0 = wx$, and notice that $N_{G^{1-}}(z) = \{w, x\}$ and $y \in N_G(x)$. Similarly let $d \in \partial_1^2$ such that $N_G(d) = \{N_{G^{1-}}(c), c\} - \{a\} = \{e_p, c\} - \{a\}$, and label $e_p = ab$. Now we let $e_{-2} = yz, t_{-1} = \{x, y, z\}, e_{-1} = zx, t_0 = \{z, w, x\}, t_{p+1} = \{a, b, c\}, e_{p+1} = bc, t_{p+2} = \{b, c, d\}$, and $e_{p+2} = cd$.

Make each vertex $v \in W^1 \cup W^2 \cup W^{3'} \cup (W^3 - \{w, a\})$ and each vertex $z \in \partial_1 - \{y, d\}$ a nonprobe. If w or a is in $W^1 \cup W^2 \cup W^{3'}$ then it will be a nonprobe; however, if it is just in $W^3 - W^{3'}$ then it will be a probe. By the first condition of an ssi2-lobster, no clique contains two vertices from $W^1 \cup W^2 \cup W^{3'}$, and by the second condition if a clique contains a vertex from $(W^3 - \{w, a\})$ then it does not contain a vertex from $W^1 \cup W^2 \cup W^{3'}$. Hence no clique in G^{2-} contains two nonprobe vertices; therefore, by the definition of a 2-tree, no two nonprobe vertices of G^{2-} are adjacent. Furthermore, no two vertices of ∂_1 are adjacent, and if a vertex of ∂_1 is adjacent to a vertex of $W^1 \cup W^2 \cup W^{3'} \cup (W^3 - \{w, a\})$ then there is a clique with two vertices in $W^1 \cup W^2 \cup W^{3'}$ and G is not an ssi2-lobster. Thus this probe-nonprobe partition has no adjacencies between nonprobe vertices. Now we must assign intervals using this partition.

Let $G^{2-*} = (e_{-2}, t_{-1}, e_{-1}, t_0, e_0, t_1, e_1, \dots, t_p, e_p, t_{p+1}, e_{p+1}, t_{p+2}, e_{p+2})$, and for each $v \in G^{2-*}$ assign an ordered pair (m, n) such that t_m is the first clique that contains v and t_n is the last. Assign the interval $I_{v_{m,n}} = (m, n + \frac{1}{2})$ to each $v \in G^{2-*}$. Notice that the interval $(i + \frac{1}{2}, i + 1)$ contains only the vertices from e_i , and the interval $(i, i + \frac{1}{2})$ contains only the vertices from t_i . For each $e_i \in G^{2-*}$, let $M_i = \{v \in V(G) : v \notin V(G^{2-*}) \text{ and } N_{G^{2-*}}(v) = e_i\}$ and enumerate the vertices of M_i as $x_{(i,1)}$ to $x_{(i,|M_i|)}$. Assign the interval $I_{x_{(i,j)}} = (i + \frac{1}{2} + \frac{j-1}{2|M_i|}, i + \frac{1}{2} + \frac{j}{2|M_i|})$ to each $x_{(i,j)} \in M_i$ for all M_i . For each $x_{(i,j)} \in M_i$ let $M_{i,j} = \{v \in \partial_1^2 : x_{(i,j)} \in N(v)\}$ and enumerate the vertices of $M_{(i,j)}$ as $y_{(i,j,1)}$ to $y_{(i,j,|M_{(i,j)}|)}$. Assign the interval $I_{y_{(i,j,k)}} = (i + \frac{1}{2} + \frac{j-1}{2|M_i|}, i + \frac{1}{2} + \frac{j}{2|M_i|})$ to each $y_{(i,j,k)} \in M_{(i,j)}$. Recall that these vertices are nonprobes, so their intersecting intervals do not yield adjacencies. Since the spiny interior 2-lobster is sparse, each $y_{(i,j,k)} \in M_{(i,j)}$ is not adjacent to a distinct vertex of e_i , which is a nonprobe. Therefore, no adjacency results between each $y_{(i,j,k)}$ and its non-adjacency in e_i .

For each $t_i, 1 \leq i \leq p$, let $Q_i = \{v \in \partial_1^1 : N_G(v) \subset t_i\}$, and let $Q_0 = \{v \in \partial_1^2 : N_G(v) = zw\}$ and $Q_{p+1} = \{v \in \partial_1^2 : N_G(v) = ac\}$. Enumerate the vertices of $Q_i, 0 \leq i \leq p+1$ as $c_{(i,1)}$ to $c_{(i,|Q_i|)}$. Assign the interval $I_{c_{(i,j)}} = (i, i + \frac{1}{2})$. Recall that these vertices are nonprobes, so their intersecting intervals do not yield adjacencies. Since G is an ssi2-lobster, the vertex $w_i \in t_i$ with $w_i \notin N_{G^{2-*}}(c_{(i,j)})$ is the same for all j , and w_i is a nonprobe since it is in W^1 . Therefore no adjacency results between each $c_{(i,j)}$ and w_i for all j . Thus G has a probe interval representation. ■

Theorem 11. *If G be a 2-tree, then G is a probe interval graph if and only if it is an ssi2-lobster.*

Proof. Let G be a 2-tree that has a probe interval representation with P the set of probes and N the nonprobes. Assume that it is not an ssi2-lobster. Thus it is either not a spiny interior 2-lobster or it is a spiny interior 2-lobster that is not

sparse. If it is not a spiny interior 2-lobster, then it is not an interval p -graph, [10]. Since p -chromatic probe interval graphs are contained in interval p -graphs, it is not a probe interval graph.

Therefore, assume that G is a spiny interior 2-lobster that is not sparse. Let $G^{2-} = (e_0, t_1, e_1, \dots, t_p, e_p)$. If the first condition of being an ssi2-lobster is violated, then either there is a t_i with two vertices w_x and w_y in $W - W^3$, there is a t_i with two vertices w_x and w_y with $w_x \in W - W^3$ and $w_y \in W^{3'}$, or there is a t_i with two vertices w_x and w_y such that $w_x, w_y \in W^{3'}$. If the second condition of being an ssi2-lobster is violated, then there is a $t_i, i \in \{1, p\}$, with three vertices w_x, w_y , and w_z such that $w_x, w_z \in W^3$ and if $e_0 = w_x w_z$ or $e_p = w_x w_z$, then $w_y \in W^1 \cup W^2 \cup W^{3'}$.

Case 1. There is a t_i with two vertices w_x and w_y in $W - W^3$. In this case, w_x and w_y are nonprobes by Lemmas 6 and 8, which is a contradiction since $w_x w_y \in E(G)$.

Case 2. There is a t_i with two vertices w_x and w_y such that $w_x \in W - W^3$ and $w_y \in W^{3'}$. Let $N_G(y) = \{N_{G^{2-}}(z), z\} - \{w_y\} = \{e_p, z\} - \{w_y\}$ and $N_G(s) = \{N_{G^{2-}}(r), r\} - \{w_y\} = \{e_p, r\} - \{w_y\}$ for $y, s \in \partial_1^2 G$ and distinct $z, r \in V(G)$. Since w_x and w_y are both in t_i , $w_y \in e_j, i \leq j \leq p$. If $w_y \notin e_{i-1}$, there is a vertex g such that $e_{i-1} \subset N_G(g)$ and $w_y \notin N_G(g)$. Let $e_{i-1} = w_x h$. The subgraph H induced by the vertices $\{w_y, h, g, y, s, z, r\}$ is T_2 with w_y the vertex of degree 3, and hence a nonprobe by Lemma 2 (see A_1 in Figure 6). By Lemmas 6 and 8, w_x is a nonprobe, which is a contradiction since $w_x w_y \in E(G)$.

Therefore, let us assume that $w_y \in e_{i-1}$. Since w_y is in both e_{i-1} and e_i , this precludes w_x from being in W^1 since vertices in this set must be in two consecutive e_j s. This means that $w_x w_y = e_i$ for $i \neq 0$ and $i \neq p$. Let $e_p = w_y f$ and $b \in e_{i-1}$ and consider the subgraph H induced by the vertices $\{w_y, f, z, y, r, s, b\}$, which is isomorphic to Q in Figure 4 with $w_y \cong d$ (see A_2 in Figure 6). By Lemmas 6 and 8, w_x is a nonprobe. Since $w_x b \in E(G)$, b must be a probe. Thus by Lemma 7, w_y must be a nonprobe, a contradiction.

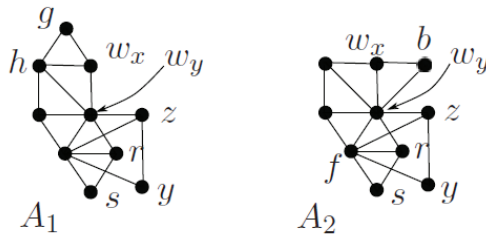


Figure 6. Labeled examples for Case 2 in the proof of Theorem 11.

Case 3. There is a t_i with two vertices w_x and w_y such that $w_x, w_y \in W^{3'}$.

Let $N_G(y) = \{N_{G^2-}(z), z\} - \{w_y\} = \{e_i, z\} - \{w_y\}$ and $N_G(s) = \{N_{G^2-}(r), r\} - \{w_y\} = \{e_i, r\} - \{w_y\}$ for $y, s \in \partial_1^2 G$ and distinct $z, r \in V(G)$, ($i = 0$ or $i = p$), and $N_G(x) = \{N_{G^2-}(v), v\} - \{w_x\} = \{e_i, v\} - \{w_x\}$ and $N_G(c) = \{N_{G^2-}(d), d\} - \{w_x\} = \{e_i, d\} - \{w_x\}$ for $x, c \in \partial_1^2 G$ and distinct $v, d \in V(G)$, ($i = 0$ or $i = p$). First consider the case that neither $w_x w_y \neq e_0$ nor e_p . Then the subgraph H induced by the vertices $\{w_y, y, z, s, r, w_x, v\}$ is isomorphic to T_2 with w_y the vertex of degree 3. Similarly, the subgraph H induced by the vertices $\{w_x, v, d, x, c, w_y, z\}$ is isomorphic to T_2 with w_x the vertex of degree 3 (see A_3 in Figure 7). By Lemma 2, both w_x and w_y must be nonprobes, which is a contradiction since they are adjacent.

Therefore, let us assume without loss of generality that $w_x w_y = e_p$ and that $w_x \notin e_{p-1}$. Let $a \in t_p$ such that $a \neq w_x, w_y$, and notice that there exists a vertex b such that $a, w_y \in N_G(b)$, but $b \notin N_G(w_x)$. Thus the subgraph H induced by the vertices $\{w_x, v, x, c, d, a, b\}$ is T_2 with w_x the vertex of degree 3. By Lemma 2, w_x must be a nonprobe. If $N_{G^1-}(z) = N_{G^1-}(r) = e_0$, then $e_0 = w_y f$ and the subgraph H induced by the vertices $\{w_y, z, y, r, s, d, f\}$ is isomorphic to Q in Figure 4 (see A_4 in Figure 7). Since $d \in N_G(w_x)$ and w_x is a nonprobe, d must be a probe. By Lemma 7, this forces w_y to be a nonprobe, a contradiction.

If $N_{G^1-}(z) = N_{G^1-}(r) = e_p$, then let g be a vertex such that $g \notin t_p$ and $g \in N_G(b)$. We know such a vertex exists since G^{2-} has at least one t_i . The subgraph H induced by the vertices $\{w_y, y, z, s, r, w_x, b\}$ is isomorphic to Q in Figure 4. By Lemma 7 and the fact that w_x is a nonprobe, b must also be a nonprobe. If $g \in N_G(w_y)$, consider the subgraph H induced by the vertices $\{w_y, y, z, s, r, w_x, g\}$, which is again isomorphic to Q in Figure 4. Hence, by Lemma 7, w_x and g must be nonprobes, but this is a contradiction since $bg \in E(G)$. If $g \notin N_G(w_y)$ consider the subgraph H induced by the vertices $\{w_y, y, z, s, r, w_x, b, g\}$. The set $\{g, y, s\}$ is an AT (see A_5 in Figure 7). By Lemma 1, one of these vertices must be a nonprobe. Since g is adjacent to b , which is a nonprobe, the nonprobe AT vertex must be either y or s . However, both of these vertices are adjacent to w_x , which has already been shown to be a nonprobe, a contradiction.

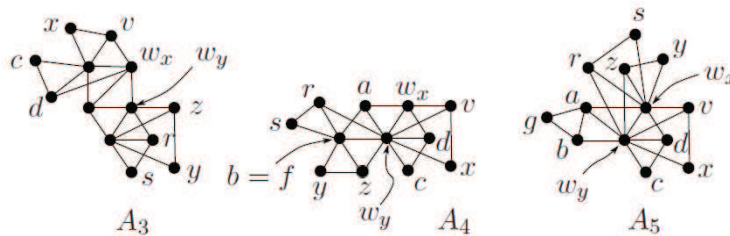


Figure 7. Labeled examples for Case 3 in the proof of Theorem 11.

Case 4. There is a $t_i, i \in \{1, p\}$, with three vertices w_x, w_y , and w_z such that

$w_x, w_z \in W^3$ and if $e_0 = w_x w_z$ or $e_p = w_x w_z$ then $w_y \in W^1 \cup W^2 \cup W^{3'}$. Without loss of generality, let $e_p = w_x w_z$, $e_{p-1} = w_x w_y$, $N_G(x) = \{N_{G^{2-}}(v), v\} - \{w_x\} = \{e_p, v\} - \{w_x\}$ and $N_G(z) = \{N_{G^{2-}}(d), d\} - \{w_z\} = \{e_p, d\} - \{w_z\}$ for $x, z \in \partial_1^2 G$ and $v, d \in V(G)$. We first consider the case that $v = d$. If $w_y \in W^{3'}$, let $N_G(y) = \{N_{G^{2-}}(c), c\} - \{w_y\} = \{e_0, c\} - \{w_y\}$ and $N_G(s) = \{N_{G^{2-}}(r), r\} - \{w_y\} = \{e_0, r\} - \{w_y\}$ for $y, s \in \partial_1^2 G$ and distinct $c, r \in V(G)$. The subgraph induced by all the vertices of G^{2-} and $\{r, s, c, y, v, x, z\}$ is isomorphic to H_i in Figure 5 for some $i \in Z^+$, and hence by Lemma 3, G is not a probe interval graph.

Now assume $v = d$ and $w_y \in W^1 \cup W^2$, and recall that by Lemmas 6 and 8, w_y must be a nonprobe. The graph H induced by the vertices $\{w_y, w_z, w_x, v, z, x\}$ is isomorphic to the 3-sun. With w_y already a nonprobe, we know that v must also be a nonprobe by Corollary 5. Since w_y is in $W^1 \cup W^2$, the clique t_{p-1} must exist on G^{2-} , so without loss of generality we let $t_{p-1} = \{a, w_y, w_x\}$. The set $\{a, z, x\}$ is an AT. By Lemma 1, one of these vertices must be a nonprobe, but since a is adjacent to w_y , the nonprobe must be either z or x . However, both z and x are adjacent to v , which has already been shown to be a nonprobe, a contradiction.

Now consider the case $v \neq d$. If $w_y \in W^{3'}$, the subgraph H induced by the vertices $\{w_y, c, y, r, s, w_z, x\}$ is isomorphic to T_2 with w_y the vertex of degree 3, and by Lemma 2, w_y must be a nonprobe. By Lemmas 6 and 8, if $w_y \in W^1 \cup W^2$, then w_y is a nonprobe. In either case, there is a vertex r that is either a 2-leaf of G^{1-} or in t_{p-1} such that $w_x, w_y \in N_G(r)$. Now consider the subgraph H induced by the vertices $\{w_x, w_y, w_z, d, x, z, r\}$, which is isomorphic to Q in Figure 4. By Lemma 7, w_y cannot be a nonprobe, a contradiction.

Every case yields a contradiction, so if G is not a sparse spiny interior 2-lobster, then it is not a probe interval graph.

Now let G be a ssi2-lobster. By Lemma 10, G has a probe interval representation. ■

5. EXTENDED LIST OF FORBIDDEN SUBGRAPHS

We now present graphs which cannot be induced subgraphs of any 2-tree probe interval graph and are not included in the list, call it \mathcal{L} , of forbidden induced subgraphs developed by Pržulj and Corneil in [19]. There is some inconsistency in what qualifies a graph for membership to \mathcal{L} , but in so far as we believe we have captured the intention behind \mathcal{L} , we'll add to it and create \mathcal{L}^* , where \mathcal{L}^* should be sufficient to decide whether any given 2-tree is a probe interval graph. \mathcal{L} will not suffice for this because if we are given F_5 from Figure 1 there is no induced subgraph to find in F_5 from the list which will indicate F_5 is not a probe interval

graph. Clearly $\mathcal{L} \subseteq \mathcal{L}^*$.

Recall H_i of Figure 5 and that any H_i is not a probe interval graph by Lemma 3. Pržulj and Corneil exclude graphs F_k , for $k > 4$ because vertices z_5, \dots, z_k can be removed and not create a probe interval graph. So F_5, F_6 and so on are not minimal. Regarding H_i , from H_k ($k > 1$) we may remove d_2, \dots, d_k and still have a graph which is not a probe interval graph. So, in the spirit of Pržulj and Corneil, we include H_1 in \mathcal{L} . But any H_k for $k > 1$ is not a probe interval graph yet has no subgraph isomorphic to any of $\mathcal{L} \cup \{H_1\}$, so H_2, H_3 and so on should be included in \mathcal{L}^* .

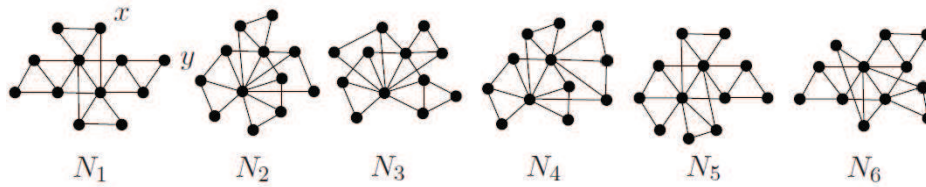


Figure 8. New forbidden subgraphs.

Now consider the graphs N_i , $1 \leq i \leq 6$, in Figure 8. Each N_i is not a probe interval graph since each has t_1 with two vertices in W^2 . If v is any vertex of N_i , for $1 \leq i \leq 6$, $N_i - v$ is not a 2-tree or has a probe interval representation. For example, $N_1 - x$ is neither a probe interval graph, nor is it a 2-tree; $N_1 - y$, however, is an ssi2-lobster and is therefore a probe interval graph. So each N_i is not minimal in the typical way one seeks in a FISC, but we believe the list has properties consistent with the spirit of \mathcal{L} . To wit, G_1, G_2, G_3 in Figure 1 have vertices whose removal will produce a graph which is not a probe interval graph and not a 2-tree, and vertices whose removal will leave a 2-tree which is a probe interval graph. So N_1, N_2, \dots, N_6 should be included in \mathcal{L} .

Coming back to \mathcal{L}^* , consider the graph S in Figure 1. It can be regarded as a family of graphs, since a k -fan can be inserted at u and, for each k , give a graph which is not a probe interval graph with no induced subgraph from \mathcal{L} (details are in [19].) Similar arguments can be made about the eight classes of graphs in Figure 1 and so should be included in \mathcal{L}^* . In all we have 10 infinite classes of graphs in \mathcal{L}^* : the eight in Figure 1 plus S extended as just discussed, plus H_i , $i \geq 1$. Therefore, \mathcal{L} , the list of forbidden induced subgraphs for 2-tree probe interval graphs in the spirit of Pržulj and Corneil, contains at least 69 graphs, and \mathcal{L}^* contains 27 graphs and 10 families.

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