

## ON THE EXISTENCE OF $(k,l)$ -KERNELS IN INFINITE DIGRAPHS: A SURVEY

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### Abstract

Let  $D$  be a digraph,  $V(D)$  and  $A(D)$  will denote the sets of vertices and arcs of  $D$ , respectively.

A  $(k,l)$ -kernel  $N$  of  $D$  is a  $k$ -independent (if  $u, v \in N$ ,  $u \neq v$ , then  $d(u, v), d(v, u) \geq k$ ) and  $l$ -absorbent (if  $u \in V(D) - N$  then there exists  $v \in N$  such that  $d(u, v) \leq l$ ) set of vertices. A  $k$ -kernel is a  $(k, k-1)$ -kernel.

This work is a survey of results proving sufficient conditions for the existence of  $(k,l)$ -kernels in infinite digraphs. Despite all the previous work in this direction was done for  $(2,1)$ -kernels, we present many original results concerning  $(k,l)$ -kernels for distinct values of  $k$  and  $l$ .

The original results are sufficient conditions for the existence of  $(k,l)$ -kernels in diverse families of infinite digraphs. Among the families that we study are: transitive digraphs, quasi-transitive digraphs, right/left pretransitive digraphs, cyclically  $k$ -partite digraphs,  $\kappa$ -strong digraphs,  $k$ -transitive digraphs,  $k$ -quasi-transitive digraphs.

**Keywords:** kernel,  $k$ -kernel,  $(k,l)$ -kernel, infinite digraph.

**2010 Mathematics Subject Classification:** 05C20, 05C63.

### 1. INTRODUCTION

In this work,  $D = (V(D), A(D))$  will denote a (possibly infinite) digraph without loops or multiple arcs in the same direction, with vertex set  $V(D)$  and arc

set  $A(D)$ . A *directed walk* is a sequence of vertices  $(v_0, v_1, \dots, v_n)$  such that  $(v_i, v_{i+1}) \in A(D)$  for every  $0 \leq i \leq n-1$ . If  $u$  and  $v$  are two vertices of a walk  $W$ , where  $u$  precedes  $v$  on  $W$ , the subsequence of  $W$  starting with  $u$  and ending with  $v$  is denoted by  $uWv$ . Union of walks will be denoted by concatenation or with  $\cup$ . As usual, a directed walk that does not repeat vertices will be called a *directed path*. A *ray* is an infinite sequence of different vertices  $(v_0, v_1, \dots)$  such that  $(v_i, v_{i+1}) \in A(D)$  for every  $i \in \mathbb{N}$ . A  *$uv$ -directed walk* is a directed walk with initial vertex  $u$  and terminal vertex  $v$ ; if a  $uv$ -directed walk exists in  $D$ , we say that  $u$  reaches  $v$  in  $D$ .

For a vertex  $v \in V(D)$ , we define the *out-neighbourhood* of  $v$  in  $D$  as the set  $N_D^+(v) = \{u \in V(D) : (v, u) \in A(D)\}$ ; when there is no possibility of confusion we will omit the subscript  $D$ . The elements of  $N^+(v)$  are called the *out-neighbours* of  $v$ , and the *out-degree* of  $v$ ,  $d_D^+(v)$ , is the number of out-neighbours of  $v$ . Definitions of in-neighbourhood, in-neighbours and in-degree of  $v$  are given analogously. We also define  $N^{+r}(v)$  ( $N^{-r}(v)$ ) as the set of vertices reached from (that reach)  $v$  at distance less than or equal to  $r$ . A vertex  $v$  is called a *sink* if  $d^+(v) = 0$  and a *source* if  $d^-(v) = 0$ .

A digraph is *unilateral* if for every pair of distinct vertices  $u, v \in V(D)$ , there exists a  $uv$ -directed path or a  $vu$ -directed path in  $D$ . A digraph is *strongly connected* (or *strong*) if for every pair of distinct vertices  $u, v \in V(D)$ , there exists a  $uv$ -directed path. A *strong component* (or *component*) of  $D$  is a maximal strong subdigraph of  $D$ . The *condensation* of  $D$  is the digraph  $D^*$  having the set of all strong components of  $D$  as its vertex set, and  $(S, T) \in A(D^*)$  if and only if there is an  $ST$ -arc in  $D$ . Clearly,  $D^*$  is an acyclic digraph (a digraph without directed cycles). Hence, if  $D$  is finite, then  $D^*$  has sinks and sources. A *terminal component* of  $D$  is a sink of  $D^*$ . An *initial component* of  $D$  is a source of  $D^*$ .

An arc  $(u, v) \in A(D)$  is called *asymmetric* (resp. *symmetric*) if  $(v, u) \notin A(D)$  (resp.  $(v, u) \in A(D)$ ). The *asymmetric part of  $D$* ,  $\text{Asym}(D)$ , is the subdigraph of  $D$  formed by the asymmetric arcs. A subdigraph of  $D$  (e.g. a directed path or a directed triangle) is asymmetric (symmetric) if all its arcs are asymmetric (symmetric). A digraph is *transitive* if  $(u, v), (v, w) \in A(D)$  implies that  $(u, w) \in A(D)$ . A digraph  $D$  is *asymmetrically transitive* if whenever  $(u, v), (v, w) \in \text{Asym}(D)$ , then  $(u, w) \in \text{Asym}(D)$ . If  $D$  is a digraph, the *dual* (or *converse*) of  $D$  is the digraph  $\overleftarrow{D}$  obtained from  $D$  by the reversal of every arc.

We will denote by  $G = (V(G), E(G))$  a possibly infinite undirected graph with vertex set  $V(G)$  and edge set  $E(G)$  without loops or multiple arcs. A *biorientation* of the graph  $G$  is a digraph  $D$  obtained from  $G$  by replacing each edge  $xy \in E(G)$  by either the arc  $(x, y)$  or the arc  $(y, x)$  or the pair of arcs  $(x, y)$  and  $(y, x)$ . A *semicomplete digraph* is a biorientation of a complete graph. An *orientation* of a graph  $G$  is an asymmetric biorientation of  $G$ ; thus, an *oriented graph* is an asymmetric digraph. A *tournament* is an orientation of a complete

graph. A *complete* digraph is a biorientation of a complete graph obtained by replacing each edge  $xy$  by the arcs  $(x, y)$  and  $(y, x)$ . An orientation of the digraph  $D$  is an asymmetric subdigraph of  $D$  obtained by deleting exactly one arc in every oppositely directed arc-pair; thus, an orientation of a digraph is an asymmetric digraph.

Let  $D$  be a digraph with vertex set  $V(D) = \{v_1, v_2, \dots, v_n\}$  and  $H_1, H_2, \dots, H_n$  a family of pairwise vertex disjoint digraphs. The *composition* of digraphs  $D[H_1, H_2, \dots, H_n]$  is the digraph having  $\bigcup_{i=1}^n V(H_i)$  as its vertex set and arc set  $\bigcup_{i=1}^n A(H_i) \cup \{(u, v) : u \in V(H_i), v \in V(H_j), (v_i, v_j) \in A(D)\}$ .

Let  $A$  and  $B$  be non-empty subsets of  $V(D)$ . If for every  $a \in A$  and every  $b \in B$  we have that  $(a, b) \in A(D)$ , we will write  $A \rightarrow B$ . When  $A = \{v\}$  for some  $v \in V(D)$ , we will simply write  $v \rightarrow B$ , and analogously if  $B = \{v\}$ . If  $S$  and  $T$  are subdigraphs of  $D$  we will abuse notation to write  $S \rightarrow T$  instead of  $V(S) \rightarrow V(T)$ . When  $A \rightarrow B$  and there are no  $BA$ -arcs in  $D$ , we will write  $A \mapsto B$ .

Let  $D$  be a digraph and  $S$  a subset of  $V(D)$ . We say that  $S$  is *k-independent* if for every pair of distinct vertices  $u$  and  $v$  of  $D$  we have  $d(u, v) \geq k$ . We say that  $S$  is *l-absorbent* (*l-dominating*) if for every vertex  $u \in V(D) \setminus S$  there exists  $v \in S$  such that  $d(u, v) \leq l$  ( $d(v, u) \leq l$ ). We can observe that a 2-independent set is an independent set in the usual sense, and a 1-dominating set is a dominating set in the usual sense. We say that  $S$  is absorbent if it is 1-absorbent. A  $(k, l)$ -kernel (introduced in [25]) is a  $k$ -independent and  $l$ -absorbent subset of  $V(D)$ . A  $k$ -kernel is simply a  $(k, k - 1)$ -kernel; thus, a 2-kernel is a kernel. Another special case of  $(k, l)$ -kernels that have been studied by some authors are the  $(2, 2)$ -kernels, or *quasi-kernels*, which were introduced by Chvátal and Lovász in [7] under the name “semi-kernels” (this notion later received a different meaning, as we will see shortly). Let us observe that a  $(k, l)$ -kernel is an  $(n, m)$ -kernel for every pair of integers  $n, m$  such that  $n \leq k$  and  $m \geq l$ .

In [33] von Neumann and Morgenstern introduce the concept of kernel of a digraph in the context of Game Theory. This concept attracted a lot of attention since kernels have a deep relation with perfect graphs and thus with the Strong Perfect Graph Conjecture, proposed by Berge and recently proved by Chudnovsky *et al.* [5]. But also kernels are related to winning strategies of some combinatorial games and even some applications can be found in Mathematical Logic. Also, the concept seems to have a lot of potential modeling real life situations, for example, if we have the map of a region modeled by a digraph (where vertices are locations and arcs are streets or highways between locations), and we want to find an optimal set of locations to build service centers (*e.g.* hospitals or schools) easily accesible to the whole population, a possible solution to this problem is to find a kernel in the digraph. The absorbence of the kernel will guarantee that from every location not in our kernel, one of the selected locations in the kernel

can be easily reached. From the independence we will know that we will avoid to build one center next to another. But we may want to have more precision and maybe set a maximum allowed distance for a location to reach one of our centers and also a minimum distance that our centers must have between them; this can be done using a  $(k, l)$ -kernel.

The main problems when we want to find a  $(k, l)$ -kernel are, first, that not every digraph has a  $(k, l)$ -kernel for given  $k$  and  $l$ . And that the problem of determining if a given digraph has a  $k$ -kernel is  $NP$ -complete. This was proved for kernels by Chvátal [6]. So, one of our main interests is to find sufficient conditions for a digraph to have a  $(k, l)$ -kernel. At the present moment there are some results in this direction, principally the work of Kucharska and Kwaśnik [24], Kwaśnik [25, 26], Galeana-Sánchez and Rincón [20], Galeana-Sánchez and Hernández-Cruz [14, 15, 16, 17, 18] and the work of Bród, Włoch and Włoch [4]. In a different direction we can highlight the work of Włoch and Włoch about how  $(k, l)$ -kernels are preserved in distinct types of products, for example with Szumny in [31, 32].

One concept related to the kernel of a digraph, and very useful when we are looking for a kernel, is the one of semi-kernel. A subset  $S \subseteq V(D)$  is a *semi-kernel* (or *local kernel*) of the digraph  $D$  if  $S$  is independent and  $S$  absorbs  $N^+(S)$ . This concept was introduced by Neumann-Lara [28] and generalized by Kucharska and Kwaśnik in [24] as follows: A subset  $S \subseteq V(D)$  is a *k-semi-kernel* of the digraph  $D$  if  $S$  is  $k$ -independent and for each  $u \in V(D) \setminus S$  such that  $d(S, u) \leq k - 1$ , it holds that  $d(u, S) \leq k - 1$ . Clearly, a semi-kernel is a 2-semi-kernel. In [16], the authors introduce a definition of  $(k, l)$ -*semi-kernel* of a digraph as a set  $S$ ,  $k$ -independent and such that, for every vertex  $v \in V(D) \setminus S$ , if  $d(S, v) \leq k - 1$  then  $d(v, S) \leq l$ . A *solution* of a digraph is a dual notion of a kernel, it is an independent, dominating (instead of absorbing) set. Clearly,  $N$  is a kernel of a digraph  $D$  if and only if  $N$  is a solution of the digraph  $\overline{D}$ . The concept of  $(k, l)$ -*solution* is analogously defined for a digraph.

In [7] we find the Chvátal-Lovász Theorem, one of the oldest existence results for  $(k, l)$ -kernels, stating that every finite digraph has a  $(2, 2)$ -kernel. It is proved in [15] that every finite transitive digraph has a  $(k, 1)$ -kernel for every integer  $k \geq 2$ . Also, in [4] it is proved that finite acyclic digraphs have  $k$ -kernel for every integer  $k \geq 2$ . It is also a very well known result that every finite semicomplete digraph has a  $k$ -kernel for every integer  $k \geq 3$ . In view of these results, it comes as a surprise the existence of an infinite transitive acyclic semicomplete digraph without a  $(k, l)$ -kernel for every pair of integers  $k, l$  such that  $k \geq 2$  and  $l \geq 1$ . The digraph  $D_{\mathbb{Z}} = (\mathbb{Z}, <)$  is the digraph with  $V(D_{\mathbb{Z}}) = \mathbb{Z}$  and such that  $(n, m) \in A(D_{\mathbb{Z}})$  if and only if  $n < m$ . Clearly,  $D_{\mathbb{Z}}$  is a tournament and thus a maximal independent set of  $D_{\mathbb{Z}}$  consists of a single vertex. It is also clear that  $D_{\mathbb{Z}}$  is transitive and acyclic. Nonetheless, if  $k \geq 2$  is an integer, then for every vertex

$n \in V(D_{\mathbb{Z}})$  and for every  $m > n$ ,  $m$  is not  $k$ -absorbed by  $n$ . In particular, for every  $n \in V(D_{\mathbb{Z}})$ ,  $n + 1$  is not  $k$ -absorbed by  $n$ , and thus, for every  $n \in V(D_{\mathbb{Z}})$ , we have that  $\{n\}$  is not a  $k$ -kernel of  $D$ . The digraph  $D_{\mathbb{Z}}$  was introduced in [11], where it is proved that each infinite tournament contains two distinct vertices  $x$  and  $y$  such that the set  $N^+(x) \cup N^-(y)$  is the whole vertex set.

It is clear that the behavior of infinite digraphs respect to  $(k, l)$ -kernels is different from the finite case. But, how different it is? In this paper we will explore similarities and differences between the finite and the infinite cases of certain families of digraphs. Surprisingly (again) some results are generalizable straightforward from their finite version; in other cases, adding a few new hypotheses will get the work done. Sometimes a weak analog is obtained for results that cannot be generalized.

In previous years, some work has been done in the direction of finding sufficient condition for the existence of kernels in infinite digraphs, let us mention the most remarkable results in this direction. The first results about the existence of kernels in an infinite digraph can be found in [28], where Neumann-Lara proved that every semi-kernel is contained in a maximal semi-kernel. Also he proved a very powerful result: If every induced subdigraph of  $D$  has a non-empty semi-kernel, then  $D$  is a *kernel-perfect digraph* (a digraph such that every induced subdigraph has a kernel).

A digraph such that for every vertex  $v \in V(D)$  we have that  $N^+(v)$  is a finite set is called *outwardly finite*. In [10], Duchet and Meyniel prove, using Gödel's Compactness Theorem, that an outwardly finite digraph  $D$  is kernel-perfect if and only if every finite induced subdigraph of  $D$  has a kernel. As a corollary of this result, they prove that if  $D$  is an outwardly finite digraph such that every odd directed cycle  $\mathcal{C}$  has the following property: "if all arcs of  $\mathcal{C}$  are incident to a subset  $T$  of vertices of  $\mathcal{C}$ , then some chord of  $\mathcal{C}$  has its head in  $T$ ", then  $D$  is kernel-perfect (let us recall that an arc of  $D$  is a *chord* of a directed cycle  $\mathcal{C}$  if it has its endpoints in  $\mathcal{C}$  but is not an arc of  $\mathcal{C}$ ).

In [30] Rojas-Monroy and Villarreal-Valdés prove that if every cycle and every ray of a digraph  $D$  has a symmetric arc, then  $D$  is a kernel-perfect digraph. An immediate consequence of this result is that every symmetric digraph is kernel-perfect.

A digraph is *right (left) pretransitive* if  $(u, v), (v, w) \in A(D)$  implies that  $(u, w) \in A(D)$  or  $(w, v) \in A(D)$  ( $(v, u) \in A(D)$ ). It is proved, also in [30], that if  $D$  is a right/left pretransitive digraph, such that every ray has a symmetric arc, then  $D$  is a kernel-perfect digraph. As an easy consequence it is proved that every transitive digraph such that every ray has a symmetric arc is kernel-perfect. A theorem generalizing the first result in Kernel Theory proved by von Neumann and Morgenstern states that every acyclic digraph without rays is a kernel-perfect digraph. Another generalization of a classical result, this time due to Richardson,

states that if  $D$  is a digraph such that  $D$  contains no rays and contains no odd cycles, then  $D$  is a kernel-perfect digraph.

Let  $D$  be a digraph and  $F$  be a subset of  $A(D)$ ; a set  $S \subseteq V(D)$  is a *semi-kernel modulo  $F$*  of  $D$  if  $S$  is an independent set of vertices such that, for every  $z \in V(D) \setminus S$  for which there exists an  $(S, z)$ -arc of  $D - F$ , there also exists a  $(z, S)$ -arc. In [13] Galeana-Sánchez and Guevara work with semi-kernels modulo  $F$ . The following result is proved: Let  $D$  be a digraph and  $D_1$  an asymmetrically transitive subdigraph of  $D$ . If  $D$  has no rays contained in  $\text{Asym}(D_1)$ ,  $D$  is  $\Gamma_{D_1}$  free, and every induced subdigraph of  $D$  has a nonempty semi-kernel modulo  $A(D_1)$ , then  $D$  is a kernel perfect digraph. In the previous statement,  $\Gamma_{D_1}$  is a set of 16 digraphs, that are forbidden to appear as induced subdigraphs of  $D$ . As a consequence of this result, they prove that every quasi-transitive digraph such that every directed triangle contained in  $D$  is symmetric and  $D$  has no asymmetric rays is a kernel-perfect digraph. Another consequence is that every infinite bipartite digraph has a kernel.

Not exactly about sufficient conditions for the existence of  $(k, l)$ -kernels in digraphs, but also related, there is a paper by Erdős and Soukup. It is a well known result of Chvátal and Lovász [7] that every finite digraph has a  $(2, 2)$ -kernel. Once again, the digraph  $D_{\mathbb{Z}}$  is a counterexample for the infinite version of this result. In [11], Erdős and Soukup work in a very interesting conjecture stating that for any digraph  $D$ , there exist a partition  $(V_0, V_1)$  of  $V(D)$  such that  $D[V_0]$  has a  $(2, 2)$ -kernel and  $D[V_1]$  a  $(2, 2)$ -solution. This paper contains several results, generally about infinite digraphs which can be (algorithmically) constructed from tournaments and locally finite digraphs. Again, not exactly about sufficient conditions for the existence of  $(k, l)$ -kernels in digraphs, but related to, we can mention the work of Fraenkel [12], where he uses Game Theory tools to analyze the structure of the kernels of a digraph, and some of his results are valid in infinite digraphs.

The rest of the paper is structured as follows. In Section 2 we introduce a technique that will be used to generalize results valid for finite digraphs to infinite digraphs using ultrafilters and Gödel's Compactness Theorem. In Section 3 we prove a very important (and general) sufficient condition for a digraph to have  $(k, l)$ -kernel, that will be used later on in the article. In Section 4 we prove that every symmetric digraph has a  $k$ -kernel for every integer  $k \geq 2$  and propose two sufficient conditions for acyclic digraphs to have a  $k$ -kernel for every integer  $k \geq 2$ . In Section 5 we find a sufficient condition for transitive digraphs to have  $(k, 1)$ -kernel for every integer  $k \geq 2$ ; the condition is that every ray  $\{x_i\}_{i \in \mathbb{N}}$  has an arc of the form  $(x_j, x_i)$  with  $i < j$ . In Section 6 we prove that every cyclically  $k$ -partite strong digraph has at least  $k$  different  $k$ -kernels; also, we prove that if  $D$  is an unilateral digraph such that every directed cycle has length  $\equiv 0 \pmod{k}$ , and every directed cycle with one obstruction has length  $\equiv 2 \pmod{k}$ , then  $D$  is

cyclically  $k$ -partite and thus has a  $k$ -kernel. The main result of Section 7 is that every quasi-transitive digraph without rays has a  $(k, l)$ -kernel for every pair of integers  $k, l$  such that  $k \geq 4$  and  $l \geq 3$  or  $k = 3$  and  $l = 2$ . Section 8 is about pre-transitive digraphs. We prove that if  $D$  is an infinite right/left pretransitive strong digraph such that every directed triangle is symmetric, then  $D$  has a  $k$ -kernel for every  $k \in \mathbb{N}$ ,  $k \geq 2$ . The results of Sections 9 and 10 are straightforward generalizations from the finite case, we give sufficient conditions for the existence of  $(k, l)$ -kernels in terms of the circumference of a digraph in  $\kappa$ -strong digraphs and in locally in/out-semicomplete digraphs. Section 11 is the final section of the article, we work with two families of digraphs,  $k$ -transitive and  $k$ -quasi-transitive digraphs, generalizing transitive and quasi-transitive digraphs, respectively; the results of this section are similar to the results obtained for transitive and quasi-transitive digraphs. The main results are: If  $k \geq 2$  is an even integer and  $D$  is an infinite  $k$ -quasi-transitive digraph such that for every ray  $\{x_i\}_{i \in \mathbb{N}}$  there exists an arc  $(x_j, x_i)$  with  $i < j$ , then  $D$  has an  $(n, m)$ -kernel for every pair of integers  $n, m$  such that  $n \geq 2$ ,  $m \geq k + 1$ . And, if  $D$  is an infinite  $k$ -transitive digraph such that every ray  $(x_i)_{i \in \mathbb{N}}$  has an arc of the form  $(x_j, x_i)$  with  $i < j$ , then  $D$  has an  $(n, m)$ -kernel for every pair of integers  $n, m$  such that  $n \geq 2$ ,  $m \geq k - 1$ . Moreover, every  $(n, m)$ -kernel of  $D$  consists in choosing one vertex from every terminal component of  $D$ .

## 2. COMPACTNESS

As we mentioned earlier, in [10], Duchet and Meyniel prove that an outwardly finite digraph  $D$  is kernel-perfect if and only if every finite induced subdigraph of  $D$  has a kernel. Their proof is a direct application of Gödel's Compactness Theorem. They identify each vertex of a digraph  $D$  with a propositional variable. Then, an infinite set of propositional formulas  $\Sigma$  is constructed using these propositional variables in such a way that a truth function satisfying every formula in  $\Sigma$  codes a kernel for  $D$  (a vertex is in the kernel if the corresponding propositional variable receives the value 1). Finally, using the known results for the finite subdigraphs of  $D$ , for every finite subset  $\Gamma \subset \Sigma$ , a truth function is constructed for the propositional variables occurring in  $\Gamma$  such that all the formulas in  $\Gamma$  are satisfied.

This is a classic way of extending results known for finite graphs to infinite graphs, for example, The Four Color Theorem or Hall's Marriage Theorem can be extended in this way.

In this section, we introduce a technique that uses The Ultrafilter Lemma (suggested by Lajos Soukup), an equivalent form of Gödel's Compactness Theorem, to generalize results from finite digraphs to infinite digraphs. We begin by

recalling a definition.

**Definition.** Let  $\mathcal{Y}$  be a set. A set  $\mathbb{U}$  of subsets of  $\mathcal{Y}$  is an *ultrafilter* on  $\mathcal{Y}$  if the following conditions hold.

- (i) The empty set is not an element of  $\mathbb{U}$ .
- (ii) If  $X, Y \subseteq \mathcal{Y}$  such that  $X \subseteq Y$  and  $X \in \mathbb{U}$ , then  $Y \in \mathbb{U}$ .
- (iii) If  $X, Y \in \mathbb{U}$ , then  $X \cap Y \in \mathbb{U}$ .
- (iv) If  $X \subseteq \mathcal{Y}$ , then either  $X \in \mathbb{U}$  or  $\mathcal{Y} \setminus X \in \mathbb{U}$ .

Let us recall that the Ultrafilter Lemma is a weak form of the Axiom of Choice, also equivalent to the Boolean Prime Ideal Theorem, hence, the Axiom of Choice remains one of our main tools when dealing with infinite digraphs. Before stating the Ultrafilter Lemma, we need one further definition. If  $\mathcal{Y}$  is a set and  $X$  is a set of subsets of  $\mathcal{Y}$ , we say that  $X$  has the *finite intersection property* if the intersection over any finite subcollection of  $X$  is non-empty.

**Theorem 1 (The Ultrafilter Lemma).** *Let  $\mathcal{Y}$  be a set and  $X$  be a set of non-empty subsets of  $\mathcal{Y}$ . If  $X$  has the finite intersection property, then there is an ultrafilter  $\mathbb{U}$  on  $\mathcal{Y}$  such that  $X \subseteq \mathbb{U}$ .*

The following is a direct remark.

**Remark 2.** Let  $D$  be a digraph. If  $D$  has a  $(k, l)$ -kernel, then, for each finite  $X \subseteq V(D)$  there is a finite  $X \subseteq Y \subset V(D)$  such that  $D[Y]$  has a  $(k, l)$ -kernel.

The converse of the previous remark is not true in general, a counterexample can be found in [10]. Recall that a digraph is outwardly finite if the out-neighborhood of every vertex is finite; the following theorem shows that the converse of Remark 2 is true when restricted to the class of outwardly finite digraphs. Moreover, it generalizes the main result of [10]. Since this technique is introduced here, the proof of the following theorem will contain a lot of details that will be omitted in further applications. The proof of the following theorem is due to Lajos Soukup.

**Theorem 3.** *Let  $D = (V, A)$  be an outwardly finite digraph. If for every finite  $X \subset V(D)$  there is a finite  $X \subseteq Y \subset V(D)$  such that  $D[Y]$  has a  $(k, l)$ -kernel, then  $D$  has a  $(k, l)$ -kernel.*

**Proof.** We define the set

$$\mathcal{Y} = \{Y \subset V : Y \text{ is finite and } D[Y] \text{ has a } (k, l)\text{-kernel}\},$$

which is partially ordered by  $\subseteq$ .

Also, for every finite  $X \subset V$ , we define the set

$$\mathcal{Y}_X = \{Y \in \mathcal{Y} : X \subseteq Y\}.$$



If  $X, X' \subset V$  are finite, then  $X \cup X'$  is also finite. Hence, there exists a finite  $X \cup X' \subseteq Y_0 \subset V$  such that  $D[Y_0]$  has a  $(k, l)$ -kernel. Clearly,  $Y_0 \in \mathcal{Y}_X \cap \mathcal{Y}_{X'}$ . Thus, the set  $\{\mathcal{Y}_X: X \subseteq V \text{ is finite}\}$  has the finite intersection property. Therefore, there exists an ultrafilter  $\mathbb{U}$  on  $\mathcal{Y}$  such that

$$\{\mathcal{Y}_X: X \subseteq V \text{ is finite}\} \subset \mathbb{U}.$$

For every  $Y \in \mathcal{Y}$ , let  $K_Y \subseteq Y$  be a  $(k, l)$ -kernel in  $D[Y]$ . We claim that

$$K = \{v \in V: \{Y \in \mathcal{Y}: v \in K_Y\} \in \mathbb{U}\}$$

is a  $(k, l)$ -kernel of  $D$ .

For the  $k$ -independence, let us suppose that  $u, v \in K$  are distinct vertices such that  $d(u, v) < k$ . Let  $P$  be an  $uv$ -path of minimum length in  $D$ . Since  $\mathbb{U}$  is closed under finite intersections, we can consider a set

$$Z \in \mathcal{Y}_{V(P)} \cap \{Y \in \mathcal{Y}: u \in K_Y\} \cap \{Y \in \mathcal{Y}: v \in K_Y\}.$$

But now we have reached a contradiction, since  $D[Z]$  contains  $P$  and has a  $(k, l)$ -kernel  $K$  such that  $u, v \in K$ . Hence,  $K$  is  $k$ -independent in  $D$ .

For the  $l$ -absorbence, assume that  $u \in V$ . Since  $D$  is outwardly finite, the set  $X = \{v \in V: d(u, v) \leq l\}$  is finite. Since for each  $Y \in \mathcal{Y}_X$  is true that  $X \cap K_Y \neq \emptyset$ , there exists  $x \in X$  such that

$$\{Y \in \mathcal{Y}_X: x \in K_Y\} \in \mathbb{U}.$$

Otherwise, since  $\mathbb{U}$  is an ultrafilter, for every  $x \in X$  we would have  $\{Y \in \mathcal{Y}_X: x \notin K_Y\} \in \mathbb{U}$ . Using again that  $\mathbb{U}$  is closed under finite intersections we can conclude that  $\{Y \in \mathcal{Y}_X: X \cap K_Y = \emptyset\} \in \mathbb{U}$ , contradicting  $\emptyset \notin \mathbb{U}$ . Finally,  $\mathbb{U}$  is upward closed, which implies  $\{Y \in \mathcal{Y}: x \in K_Y\} \in \mathbb{U}$ . Therefore,  $x \in K$  and  $d(u, v) \leq l$ . ■

As a final remark, we would like to observe that Theorem 3 is very powerful. It implies that every hereditary family of digraphs that has been proved to have a  $(k, l)$ -kernel when restricted to finite digraphs, will also have a  $(k, l)$ -kernel when restricted to outwardly finite digraphs.

### 3. TWO USEFUL LEMMAS IN INFINITE DIGRAPHS

We begin with a very simple result which is vastly used in its finite form in Kernel Theory. Nonetheless, in the infinite case we depend on the Axiom of Choice to prove it.

**Lemma 4.** *Let  $D$  be a digraph. If  $D$  has a non-empty  $(k, l)$ -semi-kernel, then  $D$  has a maximal  $(k, l)$ -semi-kernel.*

**Proof.** Since  $D$  has at least one non-empty  $(k, l)$ -semi-kernel, the set  $\mathcal{S}$  of all  $(k, l)$ -semi-kernels of  $D$  is non-empty. Clearly the set  $\mathcal{S}$  with the inclusion relation  $\subseteq$  is a partially ordered set  $(\mathcal{S}, \subseteq)$ . Let  $\{C_i\}_{i \in I} = \mathcal{C} \subseteq \mathcal{S}$  be a chain in  $(\mathcal{S}, \subseteq)$ . We will prove that  $\mathcal{C}$  has an upper bound, and by Zorn's Lemma we will have the existence of a maximal  $(k, l)$ -semi-kernel in  $D$ .

If  $C = \bigcup_{i \in I} C_i$ , then it is easy to observe that  $C \in \mathcal{S}$ . To prove that  $C$  is independent, let  $u, v \in C$ , then  $u \in C_i$  and  $v \in C_j$  for some  $i, j \in I$ . Since  $\mathcal{C}$  is a chain, we can assume without loss of generality that  $C_i \subseteq C_j$ , and then  $u, v \in C_j$ , but  $C_j$  is a  $(k, l)$ -semi-kernel and hence is  $k$ -independent, so  $d(u, v), d(v, u) \geq k$ , and then  $C$  is  $k$ -independent. To prove the second  $(k, l)$ -semi-kernel condition, let  $v \in C$  and  $u \in V(D)$  such that  $d(v, u) \leq k - 1$ . Recall that  $v \in C$  implies that  $v \in C_i$  for some  $i \in I$ . But  $C_i$  is a  $(k, l)$ -semi-kernel, thus  $d(v, u) \leq k - 1$  implies  $d(u, v) \leq l$ . Then,  $C \in \mathcal{S}$  and clearly  $C_i \subseteq C$  for every  $i \in I$ . ■

We say that a digraph  $D$  is *kernel perfect* if every induced subdigraph of  $D$  has a kernel. Our following lemma is inspired in a result of Víctor Neumann-Lara, [28], stating that if  $D$  is a digraph such that every induced subdigraph of  $D$  has a non-empty semi-kernel, then  $D$  is kernel-perfect. A finite version of this lemma was proved by the authors in [16]. It is worth noting that the digraph in Figure 1 does not have a 3-kernel, despite the fact that every induced subdigraph has a non-empty 3-semi-kernel. Hence, a direct generalization of the result of Neumann-Lara is not valid for  $(k, l)$ -kernels.

**Lemma 5.** *Let  $D$  be a digraph. If  $\{v\}$  is a  $(k, l)$ -semi-kernel of  $D$  for every  $v \in V(D)$ , then  $D$  has a  $(k, l)$ -kernel.*

**Proof.** By Lemma 4, we can consider a maximal  $(k, l)$ -semi-kernel of  $D$ , namely  $S \subseteq V(D)$ . If  $S$  is  $l$ -absorbent then  $S$  is a  $(k, l)$ -kernel of  $D$ , so let us assume that  $S$  is not  $l$ -absorbent, therefore there must exist a vertex  $v \in V(D) \setminus S$  such that  $d(v, S) > l$ . Let us observe that  $d(S, v) > k - 1$  because, by the second condition of  $(k, l)$ -semi-kernel,  $d(S, v) \leq k - 1$  implies that  $d(v, S) \leq l$  but  $v$  is not  $l$ -absorbed by  $S$ . We will consider two cases.

*Case 1.* If  $k - 1 \leq l$ , then  $k - 1 \leq l < d(v, S)$ . So, in view that  $d(S, v) > k - 1$ , we have that  $S' = S \cup \{v\}$  is a  $k$ -independent set. It is direct to verify that  $S'$  is a  $(k, l)$ -semi-kernel.

*Case 2.* If  $l < k - 1$ , then we can assume that  $d(v, S) \leq k - 1$ , otherwise  $S \cup \{v\}$  would be  $k$ -independent and we can proceed as in Case 1. So, since  $\{v\}$  is a  $(k, l)$ -semi-kernel, then  $d(S, v) \leq l < k - 1$ , a contradiction.

In both cases a contradiction arises from the assumption that  $S$  is not  $l$ -absorbent, so  $S$  must be  $l$ -absorbent and hence the desired  $(k, l)$ -kernel. ■

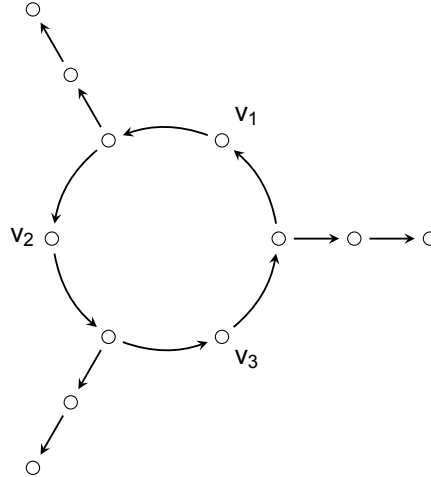


Figure 1. A digraph without a 3-kernel and such that every induced subdigraph has a 3-semi-kernel: At most one of the vertices  $v_1, v_2, v_3$  can be chosen to belong to a 3-kernel, hence one of them remains unabsorbed.

#### 4. ACYCLIC AND SYMMETRIC DIGRAPHS

The first sufficient condition for a finite digraph to have a kernel was given by von Neumann and Morgenstern in [33]. They proved that if  $D$  is a finite digraph without directed cycles, then  $D$  has a kernel. This result is easily generalizable for  $k$ -kernels in finite digraphs. As we have seen in connection with the digraph  $D_{\mathbb{Z}}$ , the direct generalization for infinite digraphs is not valid, so we propose two sufficient conditions for an acyclic digraph to have a  $k$ -kernel for every integer  $k \geq 2$ . To simplify the proof we will use a tool first defined by Kwaśnik, Włoch and Włoch in [27].

**Definition.** If  $D$  is a digraph and  $k \in \mathbb{N}$ , the  $k$ -closure of  $D$  is the digraph  $\mathcal{C}_k(D)$  such that  $V(\mathcal{C}_k(D)) = V(D)$  and  $A(\mathcal{C}_k(D)) = \{(u, v) : d_D(u, v) \leq k\}$ .

The finite version of the following lemma was first stated also by Kwaśnik, Włoch and Włoch in [27] and a proof can be found in [17]. The proof for finite digraphs also works for infinite digraphs, so it will be omitted.

**Lemma 6.** *If  $D$  is a digraph, then  $\mathcal{C}_k(D)$  has a kernel if and only if  $D$  has a  $(k + 1)$ -kernel.*

Now we use Lemma 6 to prove the following theorem; we would like to thank to Lajos Soukup for pointing out the necessity of transfinite recursion for this proof.

**Theorem 7.** *If  $D$  is an acyclic digraph without rays, then  $D$  has a  $k$ -kernel for every integer  $k \geq 2$ .*

**Proof.** The fact that  $D$  is acyclic implies that  $\mathcal{C}_k(D)$  is acyclic for every  $k \in \mathbb{N}$ . In virtue of Lemma 6 it suffices to prove that if  $D$  is acyclic and does not have rays, then  $D$  has a kernel.

Let us observe that if  $D$  is an acyclic digraph without rays, then for every vertex  $v \in V(D)$ , there exists a vertex  $w \in V(D)$  such that  $d(v, w) \in \mathbb{N}$  and  $d^+(w) = 0$ . It suffices to consider the terminal vertex in a directed path of maximum length with initial vertex  $v$ . So, by transfinite recursion, for each ordinal  $\alpha$  let us define a set  $S_\alpha \subseteq V(D)$  and a subdigraph  $D_\alpha$  of  $D$  as follows:

- $D_0 = D$ ,
- $S_0 = \{v \in V(D) : d_D^+(v) = 0\}$ ,
- $D_\alpha = D \left[ V(D) \setminus \bigcup_{\zeta < \alpha} (S_\zeta \cup N^-(S_\zeta)) \right]$ ,
- $S_\alpha = \{v \in V(D) : d_{D_\alpha}^+(v) = 0\}$ .

Since the sets  $S_\alpha$  are pairwise disjoint, there exists an ordinal  $\alpha$  such that  $S_\alpha = \emptyset$ . Then,  $D_\alpha = \emptyset$  as well, and so  $\bigcup_{\zeta < \alpha} S_\zeta$  is a kernel of  $D$ . ■

Although asking for a digraph not to have rays is not very restrictive in the cardinality sense, because a digraph without rays can have a vertex set of arbitrary cardinality, we think that is somewhat restrictive in a structural sense. Acyclic digraphs with finite out-neighborhood (outwardly finite) seem to have a richer global structure than acyclic digraphs without rays, and the outwardly finite hypothesis yields the possibility of using compactness.

**Theorem 8.** *If  $D$  is an outwardly finite acyclic digraph, then  $D$  has a  $k$ -kernel for every integer  $k \geq 2$ .*

**Proof.** It follows directly from Theorem 3 and the fact that every finite acyclic digraph has a  $k$ -kernel for every integer  $k \geq 2$ . ■

As we have already observed, in general, acyclic digraphs not necessarily have a  $k$ -kernel. If we forbid rays or infinite out-neighborhoods, then we can assure the existence of a  $k$ -kernel for every integer  $k \geq 2$ . We think that further restrictions on the existence of cycles in a digraph may lead to the existence of  $k$ -kernels for every integer  $k \geq 2$ , even when rays or vertices with infinite out-neighborhood are permitted. We propose the following conjecture.

**Conjecture 9.** *If  $D$  is a biorientation of an infinite tree, then  $D$  has a  $k$ -kernel for every  $k \geq 2$ .*

Conjecture 9 was proved true for  $k = 2$ , since an infinite tree is a bipartite digraph and thus it has a kernel [13].

Another sufficient condition for having a kernel is proved by Berge; he gave the finite version of the following theorem for  $k = 2$ , a proof can be found in [3].

**Theorem 10.** *If  $D$  is a symmetric digraph, then  $D$  has a  $k$ -kernel for every integer  $k \geq 2$ . Moreover, every maximal  $k$ -independent subset of  $D$  is a  $k$ -kernel.*

**Proof.** By Zorn's Lemma we can choose a maximal  $k$ -independent subset of  $V(D)$ , say  $N$ . We affirm that  $N$  is the desired  $k$ -kernel. By our choice  $N$  is  $k$ -independent. Let  $v \in V(D) \setminus N$  be an arbitrary vertex. Since  $N$  is a maximal  $k$ -independent subset, then  $N \cup \{v\}$  is not  $k$ -independent. If  $d(v, N) \leq k - 1$ , then  $v$  is  $(k - 1)$ -absorbed by  $N$ . So  $d(N, v) \leq k - 1$ , and then, there exists  $u \in N$  such that there exists an  $uv$ -directed path of length less than or equal to  $k - 1$ , but since  $D$  is symmetric there also exists a  $vu$ -directed path of length less than or equal to  $k - 1$ , and thus  $v$  is  $(k - 1)$ -absorbed by  $N$ . ■

We want to emphasize that Theorem 10 is equivalent to the Axiom of Choice, as it is shown in the following theorem communicated to us by Lajos Soukup.

**Theorem 11.** *If every symmetric digraph has a  $k$ -kernel for every integer  $k \geq 2$ , then every family of pairwise disjoint sets has a choice function.*

**Proof.** Let  $\{A_i : i \in I\}$  be a family of pairwise disjoint sets. Let

$$V = \bigcup \{A_i : i \in I\} \quad \text{and} \quad A = \bigcup \{[A_i \times A_i \setminus \Delta_{A_i}] : i \in I\},$$

where  $\Delta_{A_i} = \{(x, x) : x \in A_i\}$ . Hence,  $D = (V, A)$  is a symmetric digraph. Let us observe that  $K \subseteq V$  is a  $k$ -kernel if and only if  $|K \cap A_i| = 1$  for each  $i \in I$ . Clearly,  $K$  codes a choice function for  $\{A_i : i \in I\}$ . ■

## 5. TRANSITIVE DIGRAPHS

As we have already mentioned in previous sections, there are infinite transitive digraphs (even transitive tournaments) that do not have a  $k$ -kernel for any  $k \geq 2$ . In [30] and independently in [13] it is proved that if  $D$  is a transitive digraph such that every ray has at least one symmetric arc, then  $D$  has a kernel. We will generalize this result, weakening the condition of the existence of a symmetric arc in every ray and proving the existence of a  $(k, 1)$ -kernel for every integer  $k \geq 2$ . But before stating and proving our generalization we need to define a relation.

Let  $D$  be a digraph with set of strong components  $\mathfrak{C}$ . We define the relation  $\preccurlyeq$  on  $\mathfrak{C}$  in the following way. For every  $C_1, C_2 \in \mathfrak{C}$  we have that  $C_1 \preccurlyeq C_2$  if and only if there exists a  $C_1 C_2$ -directed path in  $D$ . In [30] it is observed that  $\langle \mathfrak{C}, \preccurlyeq \rangle$

is a reflexive partial order whose maximal elements (in the case that there exist) are the terminal strong components of  $D$ .

**Remark 12.** It is straightforward to observe that if  $D$  is a transitive digraph,  $C_1, C_2 \in \mathfrak{C}$  and  $C_1 \preceq C_2$ , then for every  $u \in V(C_1)$  and every  $v \in V(C_2)$ , we have that  $(u, v) \in A(D)$ .

**Theorem 13.** *Let  $D$  be a transitive digraph. Then  $D$  has a  $(k, 1)$ -kernel for every integer  $k \geq 2$  if and only if for every strong component  $S$  of  $D$ , there exists a terminal component  $T$  of  $D$  such that  $S \mapsto T$ .*

**Proof.** The theorem is obviously valid for finite digraphs, let us prove the infinite case.

For the “if” implication let  $S$  be a strong component of  $D$ . If  $T$  is a terminal strong component of  $D$  such that  $S \mapsto T$ , then every vertex in  $S$  will be absorbed by every vertex in  $T$ . Also, if we choose one vertex in every terminal strong component of  $D$ , the set of the chosen vertices will be  $k$ -independent for every integer  $k \geq 2$ , because every vertex is in a distinct terminal component. So, every set consisting of one vertex from every terminal component of  $D$  will be  $k$ -independent and absorbent.

For the “only if” part, let  $D$  be a transitive digraph that has a  $(k, 1)$ -kernel for every integer  $k \geq 2$ , and  $N$  be a  $(2, 1)$ -kernel of  $D$ . Let us assume that there exists a strong component  $S$  of  $D$  such that no terminal component of  $D$  can be reached from  $S$  by a directed path. Since  $N$  is a kernel of  $D$ , and by the transitivity of  $D$  it follows that there must exist a vertex  $v \in N$  such that  $S \mapsto v$ . Moreover, if  $R$  is the strong component of  $D$  containing  $v$ , then  $S \mapsto R$ . By our assumption,  $R$  is not a terminal component. If  $R'$  is a strong component reached by  $R$ , then  $R \mapsto R'$ , in particular  $v \mapsto R'$ . Now,  $R$  and  $R'$  are different strong components, thus, vertices in  $R'$  cannot be absorbed by  $v$ . But  $N$  is a kernel of  $D$ , and then, there must exist another vertex  $u \in N$  contained in a strong component  $R_1$  such that  $R' \mapsto u$ . But  $D$  is transitive, which implies that  $R \mapsto R_1$ , in particular,  $(v, u) \in A(D)$ , contradicting the independence of  $N$ . Thus, for every strong component  $S$  of  $D$ , there exists a terminal component  $T$  of  $D$  such that  $S \mapsto T$ . ■

**Corollary 14.** *Let  $D$  be a transitive digraph such that every ray  $(x_i)_{i \in \mathbb{N}}$  has an arc of the form  $(x_j, x_i)$  with  $i < j$ . Then  $D$  has a  $(k, 1)$ -kernel for every integer  $k \geq 2$ . Moreover, every  $(k, 1)$ -kernel of  $D$  consists in choosing one vertex from every terminal component of  $D$ .*

**Proof.** It suffices to prove that if  $C_0$  is a strong component of  $D$ , then there exists a terminal component  $T$  of  $D$  such that  $C_0 \preceq T$ . This is because, since  $D$  is a transitive digraph, Remark 12 implies  $C_0 \mapsto T$  and the result follows from the previous Theorem.

We will proceed by contradiction. Assume that for every  $C \in \mathfrak{C}$  such that  $C_0 \preceq C$  there exists  $C' \in \mathfrak{C}$  such that  $C' \neq C$  and  $C \preceq C'$ . In virtue of the Axiom of Choice we can build a sequence  $\{C_i\}_{i \in \mathbb{N}}$  satisfying  $C_0 \preceq C_1$  and, for every  $i < j$ ,  $C_i \neq C_{i+1}$  and  $C_i \preceq C_j$ . Appealing again to the Axiom of Choice, let us choose a vertex  $v_i \in V(C_i)$  for every  $i \in \mathbb{N}$ . Since  $D$  is transitive, by Remark 12,  $(v_i)_{i \in \mathbb{N}}$  is a ray in  $D$ . Moreover, if  $i < j$ ,  $(x_j, x_i) \notin A(D)$ . In the contrary case we would have that  $(x_j, x_i)$  is a  $C_j C_i$ -arc and thus a  $C_j C_i$  directed path, which by the definition of  $\preceq$  implies that  $C_j \preceq C_i$ . Since  $C_i \preceq C_j$  by the construction of  $\{C_n\}_{n \in \mathbb{N}}$  it would follow from the antisymmetry of  $\preceq$  that  $C_i = C_j$ . Again, by the construction of  $\{C_n\}_{n \in \mathbb{N}}$ , we know that  $j \neq i + 1$ , but this implies the existence of a directed cycle  $(C_i, C_{i+1}, \dots, C_j, C_i)$  in  $D^*$ , which is a contradiction, because  $D^*$  is acyclic. Therefore  $(x_i)_{i \in \mathbb{N}}$  is a ray in  $D$  such that  $(x_j, x_i) \notin A(D)$  for each  $i < j$ , a contradiction. Since the contradiction arises from assuming that for every  $C \in \mathfrak{C}$  such that  $C_0 \preceq C$  there exists  $C' \in \mathfrak{C}$  such that  $C' \neq C$  and  $C \preceq C'$ , there must exist a strong component  $T$  such that  $C_0 \preceq T$  and for every  $C' \in \mathfrak{C}$ ,  $C' = T$  or  $T \not\preceq C'$ . Hence,  $T$  is a  $\preceq$ -maximal element of  $\mathfrak{C}$ , and thus a terminal component of  $D$ .

As noted before, if  $N \subseteq V(D)$  has one vertex from every terminal component of  $D$ , then  $N$  is a  $(k, 1)$ -kernel for every integer  $k \geq 2$ . ■

Corollary 14 gives us a sufficient structural condition for an infinite transitive digraph to have a  $(k, l)$ -kernel. It is easy to observe that this condition is not necessary. As an example take the digraph  $D_{\mathbb{Z}}$  and add one new vertex  $v$  along with the arc  $(n, v)$  for every  $n \in \mathbb{Z}$ . This digraph clearly has the  $(k, 1)$ -kernel  $\{v\}$  for every integer  $k \geq 2$ , but does not fulfill the condition given in Corollary 14.

Although Theorem 13 characterizes transitive digraphs with  $(k, 1)$ -kernel for every integer  $k \geq 2$ , we do not have a characterization in terms of structural properties of the digraphs like in Corollary 14. So, we propose the following problem.

**Problem 15.** Find a structural characterization of transitive digraphs with  $(k, 1)$ -kernel for every integer  $k \geq 2$ .

## 6. CYCLICALLY $k$ -PARTITE DIGRAPHS

A digraph  $D$  is cyclically  $k$ -partite if there exists a partition of  $V(D)$  into  $k$  subsets,  $V_1, V_2, \dots, V_k$ , such that every arc of  $D$  is a  $V_i V_{i+1}$  arc (mod  $k$ ). One of the most general results for cyclically  $k$ -partite finite digraphs and  $k$ -kernels remains valid for infinite digraphs.

**Theorem 16.** *Let  $D = (V_1, \dots, V_k)$  be a cyclically  $k$ -partite digraph. If there exists  $i \in \{1, \dots, k\}$  such that all the sinks of  $D$  are contained in  $V_i$ , then  $V_i$  is a  $k$ -kernel of  $D$ .*

**Proof.** We may assume without loss of generality all the sinks of  $D$  belong to  $V_1$ . Since every arc of  $D$  is a  $V_i V_{i+1}$ -arc (mod  $k$ ), it is clear that  $V_i$  is  $k$ -independent for each  $i \in \{1, \dots, k\}$ . If  $u \in V(D) \setminus V_1$ , then  $u \in V_i$  for some  $i \in \{2, \dots, k\}$ . Since  $D$  is cyclically  $k$ -partite and every  $w \in V(D) \setminus V_1$  has  $d^+(w) \geq 1$ , there exists a  $uV_1$ -directed path of length less than or equal to  $k - 1$ . Hence,  $V_1$  is a  $k$ -kernel for  $D$ . ■

Let us recall a definition.

**Definition.** A closed walk  $\mathcal{C} = (x_0, \dots, x_n, x_{n+1} = x_0)$  is directed with an *obstruction* at vertex  $x_n$  if there exists a directed walk  $\mathcal{C}' = (x_0, x_1, \dots, x_n)$  and an arc  $(x_0, x_n) \in A(D) \setminus A(\mathcal{C}')$  such that  $\mathcal{C} = \mathcal{C}' \cup (x_0, x_n)$ .

Recall that a digraph  $D$  is unilateral if for every  $u, v \in V(D)$  there exists a  $uv$ -directed path or a  $vu$ -directed path. In [14], the following theorem is proved.

**Theorem 17.** *Let  $D$  be a finite unilateral digraph. If every directed cycle has length  $\equiv 0 \pmod{k}$  and every directed cycle with one obstruction has length  $\equiv 2 \pmod{k}$ , then  $D$  is cyclically  $k$ -partite. Thus,  $D$  has a  $k$ -kernel.*

This sufficient condition for a unilateral digraph to be cyclically  $k$ -partite is in fact a characterization. This result was obtained inspired in the generalization of Richardson's Theorem by Maria Kwaśnik. Richardsdon proved that if every directed cycle of a digraph  $D$  has even length, then  $D$  has a kernel. In [25], Kwaśnik generalizes this result in the following way.

**Theorem 18.** *Let  $D$  be a finite strong digraph. If every directed cycle of  $D$  has length  $\equiv 0 \pmod{k}$ , then  $D$  has a  $k$ -kernel.*

The most efficient way to prove Theorem 18 is to prove that, under such hypotheses, a digraph  $D$  is cyclically  $k$ -partite. In this section we present generalizations for Theorems 17 and 18. Moreover, in either case we prove the existence of a cyclic  $k$ -partition for the given digraph.

The following lemma will be stated without proof since the proof for the statement about finite digraphs also works for infinite digraphs.

**Lemma 19.** *Let  $D$  be a digraph. If every directed cycle of  $D$  has length  $\equiv 0 \pmod{k}$ , then every directed closed walk of  $D$  has length  $\equiv 0 \pmod{k}$ .*

**Theorem 20.** *Let  $D$  be a strong digraph. If every directed cycle in  $D$  has length  $\equiv 0 \pmod{k}$ , then  $D$  is cyclically  $k$ -partite.*



**Proof.** Since  $D$  is strong, for every  $u, v \in V(D)$ ,  $d(u, v) \in \mathbb{N}$ . Let  $v \in V(D)$  be a fixed vertex and  $V_i = \{u \in V(D) : d(v, u) \equiv i \pmod{k}\}$  for each  $1 \leq i \leq k$ . We affirm that  $\{V_i\}_{i=1}^k$  is a cyclic partition of  $V(D)$ . First we will prove that  $\{V_i\}_{i=1}^k$  is a partition. Clearly  $V_i \cap V_j = \emptyset$  if and only if  $i \neq j$ , because  $d(v, u)$  is uniquely determined for every  $u \in V(D)$ . Also, it follows from the first observation of the proof that  $\bigcup_{i=1}^k V_i = V(D)$ . It remains to prove that  $V_i \neq \emptyset$  for every  $1 \leq i \leq k$ . So, it follows from the fact that  $D$  is strong that  $d^-(v) \geq 1$ . Thus, let  $u \in N^-(v)$  be an in-neighbour of  $v$ . Again by the strongness of  $D$ , there exists a  $vu$ -directed path, say  $\mathcal{C} = (v = x_0, x_1, \dots, x_n = u)$ , and then  $\mathcal{C}' = \mathcal{C} \cup (u, v)$  is a directed cycle in  $D$ . Without loss of generality we can choose  $\mathcal{C}$  to realize the distance from  $v$  to  $u$ . It follows by the main hypothesis of the theorem that  $n \equiv 0 \pmod{k}$ . Since  $\mathcal{C}$  is a  $uv$ -directed path of minimum length, we have that  $d(v, x_i) \equiv i \pmod{k}$  for  $1 \leq i \leq k$ . Thus  $V_i \neq \emptyset$  for every  $1 \leq i \leq k$ . We have already proved that  $\{V_i\}_{i=1}^k$  is a partition, let us prove that it is cyclic.

Let  $(u, w) \in A(D)$  be an arbitrary arc and let us assume that  $u \in V_i$  and  $w \in V_j$  for some  $1 \leq i, j \leq k$ . We will prove that  $j \equiv i + 1 \pmod{k}$ . Let  $\mathcal{C}$  and  $\mathcal{D}$  be  $vu$  and  $vw$ -directed paths of minimum length, respectively. By the strongness of  $D$  it also exists a  $wv$ -directed path, say  $\mathcal{D}'$ . Clearly  $\mathcal{D} \cup \mathcal{D}'$  and  $\mathcal{C} \cup (u, w) \cup \mathcal{D}'$  are closed directed walks in  $D$ . Hence, by Lemma 19, we have that  $\ell(\mathcal{D} \cup \mathcal{D}') \equiv \ell(\mathcal{C} \cup (u, w) \cup \mathcal{D}') \equiv 0 \pmod{k}$ . But  $\ell(\mathcal{D} \cup \mathcal{D}') = \ell(\mathcal{D}) + \ell(\mathcal{D}')$  and  $\ell(\mathcal{C} \cup (u, w) \cup \mathcal{D}') = \ell(\mathcal{C}) + 1 + \ell(\mathcal{D}')$ . Therefore  $\ell(\mathcal{D}) \equiv \ell(\mathcal{C}) + 1$ . Since  $\mathcal{C}$  and  $\mathcal{D}$  realize the distances from  $v$  to  $u$  and from  $v$  to  $w$  respectively, we have that  $d(u, v) \equiv i \pmod{k}$  and  $d(w, v) \equiv j \equiv i + 1$ .

Thus, every arc of  $D$  is a  $V_i V_{i+1}$ -arc  $\pmod{k}$  and hence  $\{V_i\}_{i=1}^k$  is a cyclic  $k$ -partition. ■

**Corollary 21.** *Let  $D$  be a strong digraph. Then  $D$  is bipartite if and only if every directed cycle has even length.*

**Theorem 22.** *Let  $D$  be a strong digraph. If every directed cycle of  $D$  has length  $\equiv 0 \pmod{k}$ , then  $D$  has at least  $k$  distinct  $k$ -kernels.*

**Proof.** By Theorem 20,  $D$  is cyclically  $k$ -partite with partition  $\{V_i\}_{i=1}^k$ . By Theorem 16, every  $V_i$  is a  $k$ -kernel for  $D$ . ■

The proof of Theorem 20 is valid whether we are considering finite or infinite digraphs. On the other hand, the proof we propose for Theorem 24 relies heavily in Theorem 17. The reason is that the widely known characterization of finite unilateral digraphs does not work for infinite digraphs: There are unilateral digraphs without a directed spanning walk. Less important to our concern, but also worth of mention, the characterization of strong digraphs neither works in the infinite case. There are infinite strong digraphs without a closed directed spanning walk. An example of this fact is the digraph depicted in Figure 2.

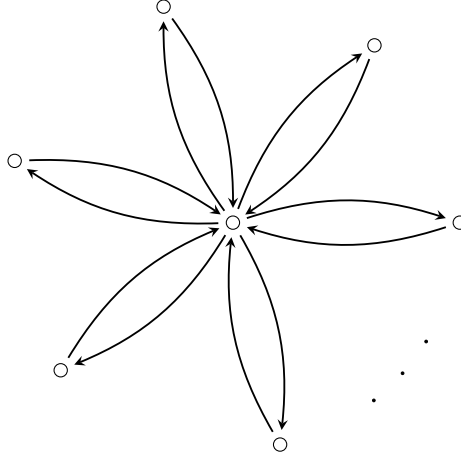


Figure 2. A flower with an infinite number of petals, example of an infinite strong digraph without a closed directed spanning walk.

So, we will proceed using Gödel's Compactness Theorem. The following lemma is a necessary tool for our theorem.

**Lemma 23.** *Let  $D$  be an infinite unilateral digraph. Then, for each finite  $X \subset V(D)$  there is a finite  $X \subseteq Y \subset V(D)$  such that  $D[Y]$  has a directed spanning walk, and so,  $D[Y]$  is unilateral.*

**Proof.** By induction on  $|X|$ . If  $X = \{u, v\}$ , then there is a walk  $P$  from  $u$  to  $v$  or from  $v$  to  $u$ . So,  $Y = V(P)$  works.

Assume that  $X = X' \cup \{v\}$ . By the induction hypothesis, we can assume that  $D[X']$  has a spanning walk  $W = (u_0, \dots, u_n)$ . If there is a walk  $P$  from  $v$  to  $u_0$  or from  $u_n$  to  $v$ , then  $X' \cup V(P)$  works.

So, we can assume that there is a  $u_0v$ -walk,  $P_0$ , and a  $vu_n$ -walk,  $Q_n$ . Thus, there is an  $0 \leq i < n$  such that there is a walk  $P_i$  from  $u_i$  to  $v$  and there is a walk  $Q_{i+1}$  from  $v$  to  $u_{i+1}$ . Hence,  $V((u_0Wu_i) \cup P_i \cup Q_{i+1} \cup (u_{i+1}Wu_n))$  works as  $Y$ . ■

**Theorem 24.** *Let  $D$  be a unilateral digraph. If every directed cycle of  $D$  has length  $\equiv 0 \pmod{k}$  and every directed cycle of  $D$  with one obstruction has length  $\equiv 2 \pmod{k}$ , then  $D$  is cyclically  $k$ -partite.*

**Proof.** Using ultrafilters. Define  $\mathcal{Y}$  as the set

$$\mathcal{Y} = \{Y \subset V : D[Y] \text{ has a spanning directed walk}\}.$$

For a finite  $X \subset V$  define  $\mathcal{Y}_X$  as the set

$$\mathcal{Y}_X = \{Y \in \mathcal{Y} : X \subseteq Y\},$$

and let  $\mathbb{U}$  be an ultrafilter on  $\mathcal{Y}$  such that

$$\{\mathcal{Y}_X : X \subset V \text{ is finite}\} \subset \mathbb{U}.$$

For each  $Y \in \mathcal{Y}$ , let  $f_Y : Y \rightarrow \{1, \dots, k\}$  be a cyclic  $k$ -partition of  $D[Y]$ .

Define the function  $f : V \rightarrow \{1, \dots, k\}$  as follows:

$$f(v) = i \text{ if and only if } \{Y \in \mathcal{Y} : f_Y(v) = i\} \in \mathbb{U}.$$

Then  $f$  is a cyclic  $k$ -partition of  $D$ . ■

When dealing with cyclically  $k$ -partite digraphs we are interested not only in the existence of  $k$ -kernels, but in the general structure of this family as well. Cyclically  $k$ -partite strong and unilateral finite digraphs have been characterized in terms of the length of directed cycles and directed cycles with one obstruction respectively. A characterization of cyclically  $k$ -partite digraphs (without connectivity restraints) by Hell and Nešetřil, in terms of graph homomorphisms and algorithms, can be found in [23]. We think that it would be enlightening to give a characterization of cyclically  $k$ -partite digraphs analogous to the existing ones for strong and unilateral digraphs, at least to identify or characterize the cyclically  $k$ -partite digraphs with  $k$ -kernel. From the  $k$ -kernel point of view there is a lot of work to do with this family. There are few general sufficient conditions for the existence of  $k$ -kernels in cyclically  $k$ -partite digraphs. We have already seen that if all the sinks of a cyclically  $k$ -partite digraph are contained in the same part, then the digraph has a  $k$ -kernel. In [22], it is proved that if all the sinks of a finite cyclically 3-partite digraph are contained in at most two parts of the cyclic partition, then  $D$  has a 3-kernel. It is also proved that the 3-kernel problem restricted to the class of finite cyclically 3-partite digraphs is NP-complete. In this context we state the following problem.

**Problem 25.** For a given integer  $k \geq 2$ , find an integer  $\varphi(k)$  such that every cyclically  $k$ -partite digraph with all its sinks contained in at most  $\varphi(k)$  parts of the cyclic partition has a  $k$ -kernel.

As we have already observed,  $\varphi(2) = 2$  and, for finite digraphs,  $\varphi(3) = 2$ .

## 7. QUASI-TRANSITIVE DIGRAPHS

We begin this section with the definition of a quasi-transitive digraph.

**Definition.** A digraph  $D$  is *quasi-transitive* if  $(u, v), (v, w) \in A(D)$  implies  $(u, w) \in A(D)$  or  $(w, u) \in A(D)$ .

Since their introduction by Ghouila-Houri in [21], quasi-transitive digraphs have been largely studied. Two of the most important results about quasi-transitive digraphs are the one of Ghouila-Houri, who characterized asymmetric quasi-transitive digraphs as precisely those digraphs which underlying graph can receive a transitive orientation; and a recursive structural characterization of this family due to Bang-Jensen and Huang found in [2]. This structural characterization was a central part of the proof that the authors gave for the finite version of the principal result of this section. Nonetheless, the aforementioned characterization only works for finite digraphs, and since a fundamental part of the proof is done by induction, we were unable to find an analogous characterization theorem for infinite digraphs. In the present article, a different approach is considered. We will prove that every quasi-transitive strong digraph has a  $(k, l)$ -semi-kernel and use local properties of quasi-transitive digraphs to prove that this  $(k, l)$ -quasi-kernel is also a  $(k, l)$ -kernel.

We begin with some technical results.

**Lemma 26.** *Let  $D$  be a quasi-transitive digraph and  $u, v \in V(D)$ . If  $d(v, u) = 2$  or  $4 \leq d(v, u) \in \mathbb{N}$ , then  $d(u, v) = 1$ . If  $d(v, u) = 3$ , then  $d(u, v) \leq 3$ .*

**Proof.** The proof of the finite case remains valid since it is proved by induction on  $d(v, u) \in \mathbb{N}$ . ■

**Lemma 27.** *Let  $D$  be a quasi-transitive digraph. If  $A$  and  $B$  are strong components of  $D$  such that there is an  $AB$ -arc in  $D$ , then  $A \mapsto B$ . Hence, the condensation of  $D$ ,  $D^*$ , is transitive.*

**Proof.** Since there is an  $AB$ -arc in  $D$ , for every  $a \in V(A)$  and every  $b \in V(B)$ , there is an  $ab$ -directed path in  $D$ . The proof of the finite case of the first part of the lemma remains valid since it is proved by induction on  $d(a, b) \in \mathbb{N}$  that  $(a, b) \in A(D)$ . To prove that  $D^*$  is transitive, let  $(A, B)$  and  $(B, C)$  be arcs of  $D^*$ . Then, by the first part of the lemma, there are vertices  $a \in V(A)$ ,  $b \in V(B)$  and  $c \in V(C)$ , such that  $(a, b), (b, c) \in A(D)$ . Since  $D$  is quasi-transitive,  $(a, c) \in A(D)$  or  $(c, a) \in A(D)$ . But since  $D^*$  is acyclic,  $(c, a) \notin A(D)$ . Thus,  $(a, c) \in A(D)$  and this implies that  $(A, C) \in A(D^*)$ . ■

The following lemma is “one half” of Bang-Jensen and Huang’s characterization theorem. The strong case is the one that could not be obtained for infinite digraphs. In the following lemma,  $T[Q_i]_{i \in I}$  denotes the composition of the family of digraphs  $\{Q_i\}_{i \in I}$  over the digraph  $T$ .

**Lemma 28.** *Let  $D$  be a non-strong digraph. Then  $D$  is quasi-transitive if and only if there exist an acyclic transitive digraph  $T$  with vertex set  $V(T) = \{v_i\}_{i \in I}$  and a family of strong quasi-transitive digraphs  $\{Q_i\}_{i \in I}$  such that  $D = T[Q_i]_{i \in I}$ .*

**Proof.** The sufficiency is straightforward to verify. For the necessity, let  $D$  be a quasi-transitive digraph. Let  $T$  be the condensation of  $D$ , i.e.,  $T = D^*$ . Hence,  $T$  is acyclic. Recall that  $V(T) = \{C_i\}_{i \in I}$  is the set of strong components of  $D$ . Since  $D$  is quasi-transitive,  $C_i$  is a quasi-transitive strong digraph for every  $i \in I$ . So, let  $Q_i = C_i$  for every  $i \in I$ . It follows from the definition of  $D^*$  and Lemma 27 that  $D = T[Q_i]_{i \in I}$  and  $T$  is transitive. ■

From this point on, our results aim to prove that every quasi-transitive digraph has a  $(k, l)$ -semi-kernel. Maybe some lemmas like the following one may look a bit odd.

**Lemma 29.** *Let  $D$  be a quasi-transitive digraph. Then, for every directed cycle  $\mathcal{C}$  of  $D$ , there are at least two arcs of  $\mathcal{C}$ , say  $(u_1, v_1), (u_2, v_2) \in A(\mathcal{C})$  such that  $d(v_i, u_i) \leq 2$ ,  $i \in \{1, 2\}$ .*

**Proof.** By induction on  $\ell(\mathcal{C})$ . If  $\ell(\mathcal{C}) = 2$  or  $\ell(\mathcal{C}) = 3$ , the result is clear. Let  $\mathcal{C} = (x_0, x_1, \dots, x_n = x_0)$  be a directed cycle of length  $n \geq 4$  in  $D$ . Since  $D$  is quasi-transitive and  $(x_0, x_1), (x_1, x_2) \in A(D)$ , then  $(x_0, x_2) \in A(D)$  or  $(x_2, x_0) \in A(D)$ . In the latter case it is clear that  $d(x_1, x_0), d(x_2, x_1) \leq 2$  and we are done. In the former case, let us apply the induction hypothesis to the cycle  $\mathcal{C}' = (x_0, x_2) \cup (x_2 \mathcal{C} x_0)$ , which has length  $n - 1$ , to obtain two arcs with the desired condition in  $A(\mathcal{C}')$ . Since  $A(\mathcal{C}' - (x_0, x_2)) \subset A(\mathcal{C})$ , if the two arcs obtained from the induction hypothesis are different from  $(x_0, x_2)$  we are done. Let us assume that one of the arcs is  $(x_0, x_2)$ . Hence,  $d(x_2, x_0) \leq 2$ . If  $d(x_2, x_0) = 1$ , it is the case we have already analyzed. So  $d(x_2, x_0) = 2$ . Let  $v \in V(D)$  be a vertex such that  $(x_2, v), (v, x_0) \in A(D)$ . If  $v = x_1$ , then the arcs  $(x_0, x_1), (x_1, x_2) \in A(D)$  are symmetric and we are done. If not, we have that  $(x_1, x_2), (x_2, v) \in A(D)$ . Since  $D$  is quasi-transitive,  $(x_1, v) \in A(D)$ , which implies that  $d(x_1, x_0) \leq 2$ ; or  $(v, x_1) \in A(D)$ , which implies that  $d(x_2, x_1) \leq 2$ . In either case we reach the desired conclusion. ■

**Lemma 30.** *If  $D$  is a quasi-transitive digraph without rays, then for every  $S \subseteq V(D)$  there exists a vertex  $v \in S$  such that, if  $u \in S$  and  $(v, u) \in A(D)$ , then  $d_D(u, v) \leq 2$ .*

**Proof.** Let  $S$  be a subset of  $V(D)$  and suppose that there is no vertex in  $S$  with the desired property. Then for every  $x \in S$  there exists a vertex  $y \in S$  such that  $(x, y) \in A(D)$  and  $d_D(y, x) > 2$ . Since every vertex in  $S$  has at least one neighbour in  $S$  with this property and there are not rays in  $D$ , then there must exist a directed cycle  $\mathcal{C} = (x_0, x_1, \dots, x_n = x_0)$  such that  $V(\mathcal{C}) \subseteq S$

and  $d_D(x_i, x_{i-1}) > 2$  with  $1 \leq i \leq n$ . This contradicts Lemma 29. Since the contradiction arose from the assumption that there is no vertex in  $S$  with the desired property, then there must exist at least one such vertex. ■

**Lemma 31.** *Let  $D$  be a quasi-transitive digraph without rays. If  $D$  has a non-empty 3-semi-kernel, then  $D$  has a 3-kernel.*

**Proof.** Since  $D$  has a non-empty 3-semi-kernel, by Lemma 4 we can consider  $S$  to be a maximal 3-semi-kernel of  $D$ . If  $S$  is 2-absorbent, then  $S$  is the desired 3-kernel. If  $S$  is not 2-absorbent, we can consider  $T \subseteq V(D)$ , the set of vertices not 2-absorbed by  $S$ . It follows from Lemma 30 that there exists of a vertex  $v \in T$  such that if  $u \in T$  and  $(v, u) \in A(D)$ , then  $d(u, v) \leq 2$ . As a consequence of Lemma 26, whenever  $u \in T$  and  $d(v, u) = 2$  it follows that  $(u, v) \in A(D)$ . Besides,  $d(v, S) \geq 3$  and  $d(S, v) \geq 3$ , as a matter of fact, in virtue of Lemma 26 we have that  $d(v, S) = 3 = d(S, v)$ . So  $S \cup \{v\}$  is a 3-independent subset of  $V(D)$ . By the choice of  $v$  and since  $S$  is a 3-semi-kernel,  $S \cup \{v\}$  fulfills the second property of 3-semi-kernel, so it is a 3-semi-kernel properly containing  $S$ , contradicting the maximality of  $S$ . Therefore,  $S$  is 2-absorbent and thus a 3-kernel. ■

**Lemma 32.** *Let  $D$  be a quasi-transitive strong digraph and  $k, l$  be a pair of integers such that  $k \geq 4$ ,  $3 \leq l \leq k - 1$ . If  $D$  has a  $(k, l)$ -semi-kernel, then  $D$  has a  $(k, l)$ -kernel.*

**Proof.** Since  $D$  has a non-empty  $(k, l)$ -semi-kernel, by Lemma 4 we can consider  $S$  to be a maximal  $(k, l)$ -semi-kernel of  $D$ . If  $S$  is  $l$ -absorbent, then  $S$  is the  $(k, l)$ -kernel we have been looking for. If it is not  $l$ -absorbent, then let  $v \in V(D)$  be a vertex not  $l$ -absorbed by  $S$ . Since  $D$  is strong, there must exist a  $vS$ -directed path of length greater than or equal to  $l + 1 \geq 4$ . Let  $s \in S$  be the final vertex in such directed path. In virtue of Lemma 26 we have that  $(s, v) \in A(D)$ , and since  $S$  is a  $(k, l)$ -semi-kernel of  $D$ , a  $vS$ -directed path of length less than or equal to  $l$  must exist in  $D$ , contradicting the choice of  $v$  as a vertex not  $l$ -absorbed by  $S$ . Hence  $S$  is  $l$ -absorbent and it turns out to be the desired  $(k, l)$ -kernel. ■

**Lemma 33.** *Let  $D$  be a quasi-transitive digraph without rays and  $k, l$  be a pair of integers such that either  $k \geq 4$  and  $3 \leq l \leq k - 1$  or  $k = 3$  and  $l = 2$ . Then there is a vertex  $v \in V(D)$  such that  $\{v\}$  is a  $(k, l)$ -semi-kernel of  $D$ .*

**Proof.** It suffices to consider  $S = V(D)$  and an application of Lemma 30 to find a vertex  $v \in V(D)$  such that whenever  $u \in V(D)$  and  $(v, u) \in A(D)$  then  $d(u, v) \leq 2$ . It follows from Lemma 26 that the set  $\{v\}$  is a  $(k, l)$ -semi-kernel. ■

We finish this section with our two principal results.

**Theorem 34.** *If  $D$  is an infinite quasi-transitive strong digraph without rays, then  $D$  has a  $k$ -kernel for every  $k \in \mathbb{N}$ ,  $k \geq 3$ .*

**Proof.** It is an immediate consequence of Lemmas 31, 32 and 33. ■

**Theorem 35.** *If  $D$  is an infinite quasi-transitive digraph without rays, then  $D$  has a  $(k, l)$ -kernel for every pair of integers  $k, l$  such that  $k \geq 4$  and  $3 \leq l \leq k - 1$  or  $k = 3$  and  $l = 2$ .*

**Proof.** In virtue of Theorem 34 every terminal strong component of  $D$  has a  $(k, l)$ -kernel. If  $D$  has  $\{D_i\}_{i \in I}$  as set of strong terminal components (where  $I$  can be an infinite set), it suffices to choose a  $(k, l)$ -kernel  $N_i$  for every  $D_i$ . The union  $N = \bigcup_{i \in I} N_i$  is a  $(k, l)$ -kernel of  $D$ . Since  $N_i \subseteq D_i$  and  $D_i$  is a terminal component for every  $i \in \{1, 2, \dots, t\}$  it is clear that  $N$  is  $k$ -independent. In addition, if  $v \in V(D_i)$ , then  $v$  is  $l$ -absorbed by  $N_i$ ,  $1 \leq i \leq t$ . So it suffices to prove that if  $v$  is in a non-terminal strong component then it is  $l$ -absorbed by some vertex in  $N$ . But by Lemma 27,  $D^*$  is an acyclic transitive digraph. Since  $D$  does not have rays,  $D^*$  does not have rays. Thus, by Lemma 28 every non-terminal strong component of  $D$  is 1-absorbed by a terminal strong component of  $D$  in  $D^*$ . Therefore, by Lemma 27 every vertex in a non-terminal strong component is 1-absorbed by every vertex in at least one terminal strong component of  $D$ , and then is  $l$ -absorbed by  $N$ . So  $N$  is a  $k$ -independent,  $l$ -absorbent set, and thus the desired  $(k, l)$ -kernel. ■

## 8. PRETRANSITIVE DIGRAPHS

Another well-known generalization of transitive digraphs are right and left pretransitive digraphs.

**Definition.** A digraph  $D$  is *right (left) pretransitive* if  $(u, v), (v, w) \in A(D)$  implies  $(u, w) \in A(D)$  or  $(w, v) \in A(D)$  ( $(v, u) \in A(D)$ ).

This family of digraphs have been previously studied in the context of Kernel Theory. Duchet proved in [9] that finite right/left pretransitive digraphs are kernel perfect. In [15] it is proved that if  $D$  is a finite right/left strong pretransitive digraph such that every directed triangle is symmetric, then  $D$  has a  $k$ -kernel for every integer  $k \geq 2$ . In the finite case, the strong connectivity can be dropped from the hypotheses at least in the right case. For the left case, a similar result is obtained by means of dualization, stating that every left pretransitive digraph such that every directed triangle is symmetric has a  $k$ -solution ( $k$ -independent,  $(k - 1)$  dominant subset of  $V(D)$ ) for every integer  $k \geq 2$ .

The main result of this section is proved with the aid of Lemma 5 in a very similar way that the finite version is proved.

**Lemma 36.** *Let  $k \geq 2$  be an integer. If  $D$  is a right pretransitive strong digraph such that every directed triangle is symmetric, then every vertex of  $D$  is a  $k$ -semi-kernel of  $D$ .*

**Idea of Proof.** Let  $k \geq 2$  be an integer. Let  $v \in V(D)$  be any vertex, consider  $w \in V(D)$  such that there exists a  $vw$ -directed path of length less than or equal to  $k - 1$  and let  $\mathcal{C} = (v = v_0, v_1, \dots, v_n = w)$  be a  $vw$ -directed path of minimum length. For every such  $w$ ,  $\mathcal{C}$  is a directed path of finite length. Also, since  $D$  is strong,  $d(w, v) \in \mathbb{N}$ . So, the same argument as used in the finite version of this theorem (found in [15]) can be used to prove that  $d(w, v) \leq k - 1$ . ■

**Theorem 37.** *If  $D$  is a right pretransitive strong digraph such that every directed triangle is symmetric, then  $D$  has a  $k$ -kernel for every  $k \in \mathbb{N}$ ,  $k \geq 2$ .*

**Proof.** It follows from Lemmas 5 and 36. ■

The analogous results for left pretransitive digraphs can be easily obtained like in finite digraphs by means of dualization.

**Lemma 38.** *If  $D$  is a left pretransitive strong digraph such that every directed triangle is symmetric, then  $\{v\}$  is a  $k$ -semi-kernel of  $D$  for every  $v \in V(D)$ .*

**Proof.** Exactly like the finite case, see [15]. ■

**Theorem 39.** *If  $D$  is a left pretransitive strong digraph such that every directed triangle is symmetric, then  $D$  has a  $k$ -kernel for every  $k \in \mathbb{N}$ ,  $k \geq 2$ .*

**Proof.** The result follows from Lemmas 5 and 38. ■

In [15], it is also proved that if  $D$  is a finite right pretransitive digraph such that every directed triangle is symmetric, then  $D$  has a  $k$ -kernel for every  $k \in \mathbb{N}$ ,  $k \geq 2$ . However, the proof of this fact was done by induction on  $|V(D^*)|$ , which can be uncountable if  $D$  is infinite. So, since we were unable to find a proof for infinite digraphs, we state the following conjecture.

**Conjecture 40.** *If  $D$  is a right pretransitive digraph such that every directed triangle is symmetric and does not contain rays, then  $D$  has a  $k$ -kernel for every  $k \in \mathbb{N}$ ,  $k \geq 2$ .*

Obviously,  $D$  must have a restriction on its rays because, since the digraph  $D_{\mathbb{Z}}$  is transitive, is right/left pretransitive and every directed triangle is symmetric (provided that it has none) but we already have observed that it does not have  $k$ -kernel for any integer  $k \geq 2$ . One remarkable observation is that, in Theorems 37 and 39 this condition is not necessary, the digraphs may contain rays and the results remain valid.



To have an overall state of the current situation with pretransitive digraphs, we would like to point out two problems previously proposed in [15] for finite digraphs.

**Problem 41.** Is it true that every left pretransitive digraph such that every directed triangle is symmetric has a  $k$ -kernel for every integer  $k \geq 2$ ?

**Problem 42.** Are the hypotheses in Theorems 37 and 39 on the directed triangles sharp?

It is worth mention that it can be easily proved that if  $T$  is a directed triangle in a right/left pretransitive digraph, then  $T$  has at least two symmetric arcs. So the restriction in Theorems 37 and 39 is weaker than it seems at a first glance.

### 9. $\kappa$ -STRONGLY CONNECTED DIGRAPHS

The results of this section have somewhat technical proofs and are direct generalizations of the respective finite versions. Once again, the main tool to change from finite to infinite digraphs is Lemma 5.

**Definition.** For a strong digraph  $D$ , a set  $S \subset V(D)$  is a separator (or separating set) if  $D \setminus S$  is not strong. A digraph  $D$  is  $\kappa$ -strongly connected (or  $\kappa$ -strong) if  $|V(D)| \geq \kappa + 1$  and has no separator with less than  $\kappa$  vertices.

As a first observation, let us notice that if  $D$  is a  $\kappa$ -strong digraph with circumference  $l$ , then  $l \geq \kappa + 1$ . To prove this, let  $\mathcal{C}$  be a longest cycle in  $D$ . If  $|V(\mathcal{C})| \leq \kappa$ , then fix an arc  $(x, y)$  in  $\mathcal{C}$  and delete all vertices of  $\mathcal{C} - \{x, y\}$  and the arc  $(x, y)$ . The resulting digraph is strongly connected (since  $|V(\mathcal{C})| \leq \kappa$ ), so there is an  $xy$ -path of length at least 2 in  $D$ . Thus, a cycle longer than  $\mathcal{C}$  can be constructed in  $D$ , contradicting the choice of  $\mathcal{C}$ .

The proofs of some results will be omitted for the sake of brevity since, as many of them are local properties, the proof is just like in finite digraphs.

**Lemma 43.** *Let  $D$  be an  $\kappa$ -strong digraph with circumference  $l$ ,  $k \geq 2$  a fixed integer and  $\mathcal{C} = (x_0, x_1, \dots, x_m)$  a directed path of length  $m$ . If  $m = q\kappa + r$ , where  $q$  and  $r$  are given by the division algorithm, then:*

- (i) *If  $r = 0$ , then  $d(x_m, x_0) \leq (l - \kappa)q$ .*
- (ii) *If  $r > 0$ , then  $d(x_m, x_0) \leq (l - r) + (l - \kappa)\lfloor \frac{m-1}{\kappa} \rfloor$ .*

**Proof.** The proof is by induction on  $q$ , which is finite. So the proof for the finite case remains valid. ■

The proof of the following theorem is also like the one of the finite version, we will reproduce it to emphasize that it is also valid for infinite digraphs.

**Lemma 44.** *Let  $D$  be a  $\kappa$ -strong digraph with circumference  $l$ . Then for every  $v \in V(D)$ ,  $\{v\}$  is a  $(k, (l-1) + (l-\kappa) \lfloor \frac{k-2}{\kappa} \rfloor)$ -semi-kernel for every integer  $k \geq 2$ .*

**Proof.** Let us recall that  $\kappa \leq l-1$ . Let  $k \geq 2$  and  $v \in V(D)$  be fixed and let  $\mathcal{C} = (v = x_0, x_1, \dots, x_m)$  be a  $vx_m$ -directed path of length  $m \leq k-1$ . In virtue of Lemma 43,  $d(x_m, v) \leq (l-1) + (l-\kappa) \lfloor \frac{m-1}{\kappa} \rfloor \leq (l-1) + (l-\kappa) \lfloor \frac{k-2}{\kappa} \rfloor$  and then  $\{v\}$  fulfills the second  $(k, (l-1) + (l-\kappa) \lfloor \frac{k-2}{\kappa} \rfloor)$ -semi-kernel condition. ■

The principal theorem of the section is proved next. It explores what kind of  $(k, l)$ -kernels exists with given values of circumference and strong connectivity.

**Theorem 45.** *Let  $D$  be a  $\sigma$ -strong digraph with circumference  $l$ . Then  $D$  has a  $(k, (l-1) + (l-\sigma) \lfloor \frac{k-2}{\sigma} \rfloor)$ -kernel for every integer  $k \geq 2$ .*

**Proof.** It follows immediately from Lemmas 5 and 44. ■

We would like to point out a special case which give us sufficient conditions for a digraph to have a  $k$ -kernel. This corollary also follows immediately from Theorem 45.

**Corollary 46.** *Let  $D$  be an infinite  $(k-1)$ -strong digraph with circumference  $k$ . Then  $D$  has a  $k$ -kernel.*

## 10. LOCALLY IN/OUT-SEMICOMplete DIGRAPHS

This section is very brief. We only remark which of the existing results for locally in/out-semicomplete finite digraphs remain valid for infinite digraphs. Once again in this section, Lemma 5 makes the proofs for the infinite case possible.

**Definition.** Let  $D$  be a digraph. Then  $D$  is:

- Locally in-semicomplete if  $(v, u), (w, u) \in A(D)$  implies that  $(v, w) \in A(D)$  or  $(w, v) \in A(D)$ .
- Locally out-semicomplete if  $(u, v), (u, w) \in A(D)$  implies that  $(v, w) \in A(D)$  or  $(w, v) \in A(D)$ .
- Locally semicomplete if it is both, locally out-semicomplete and locally in-semicomplete.

The following lemma is a technical one that will be needed to prove the one after it.

**Lemma 47.** *Let  $l \geq 1$  be an integer,  $D$  a locally out-semicomplete digraph and  $(x_0, x_1, \dots, x_n)$  is an  $x_0x_n$ -directed path of length  $n \leq l$ . If  $v_0 \in V(D)$  is such that  $(x_0, v_0) \in A(D)$  and  $(x_n, v_0) \notin A(D)$ , then  $d(v_0, x_n) \leq l$ .*

**Proof.** The proof of the finite case remains valid since we only work with the vertices of the  $x_0x_n$ -directed path. ■

**Lemma 48.** *Let  $D$  be a locally out-semicomplete digraph such that, for a fixed integer  $l \geq 1$ , whenever  $(u, v) \in A(D)$ , then  $d(v, u) \leq l$ . Then  $\{u\}$  is a  $(k, l)$ -semi-kernel for every integer  $k \geq 2$  and every  $u \in V(D)$ .*

**Proof.** Consider a vertex  $w \in V(D)$  such that a  $vw$ -directed path exists in  $D$ . It is proved by induction on  $d(v, w)$  that  $d(w, v) \leq l$ . The proof of the finite case remains valid. ■

**Theorem 49.** *Let  $D$  be a locally out-semicomplete digraph such that, for a fixed integer  $l \geq 1$ , whenever  $(u, v) \in A(D)$ , then  $d(v, u) \leq l$ . Then  $D$  has a  $(k, l)$ -kernel for every integer  $k \geq 2$ .*

**Proof.** Is a direct consequence of Lemmas 5 and 48. ■

The two following corollaries are straightforward from Theorem 49.

**Corollary 50.** *Let  $D$  be a locally out-semicomplete digraph such that, for a fixed integer  $l \geq 1$ , whenever  $(u, v) \in A(D)$ , then  $d(v, u) \leq l$ . Then  $D$  has a  $k$ -kernel for every integer  $k \geq l + 1$ .*

**Corollary 51.** *Let  $D$  be a locally out-semicomplete strong digraph with circumference  $l + 1$ , then  $D$  has a  $(k, l)$ -kernel for every integer  $k \geq 2$ .*

The results of Lemma 48 and Theorem 49 can be dualized by means of the next lemma. Its proof is exactly the same as in the finite case.

**Lemma 52.** *A digraph  $D$  is locally in-semicomplete if and only if  $\overleftarrow{D}$  is locally out-semicomplete. As a consequence, a digraph  $D$  is locally semicomplete if and only if  $\overleftarrow{D}$  is locally semicomplete.*

The principal result obtained by means of dualization is stated next.

**Theorem 53.** *Let  $D$  be a locally out-semicomplete digraph such that, for a fixed integer  $l \geq 1$ , whenever  $(u, v) \in A(D)$ , then  $d(v, u) \leq l$ . Then  $D$  has a  $(k, l)$ -kernel for every integer  $k \geq 2$ .*

One of the main results obtained in [16] states that every locally out-semicomplete digraph with circumference  $l + 1$  has a  $(k, l)$ -kernel for every  $k \geq 2$ . This is, in the finite case the condition of strong connectivity can be dropped in Corollary 51.

Nonetheless, the proof of the finite case was done by induction on  $|V(D)|$ , and we were unable to find a proof that works for the infinite case. In the same article, it is conjectured that if  $D$  is a finite digraph with circumference  $l$ , then  $D$  has a  $k$ -kernel for every  $k \leq l$ . In [22] it is proved that if  $D$  is a finite digraph such that every directed cycle has length 3, then  $D$  has a 3-kernel. For the infinite case it seems that there should also be a condition on the rays. Proving the result for some families such as locally out-semicomplete digraphs would be a good start point. Also, a weaker version of the conjecture can be stated.

**Conjecture 54.** *Let  $D$  be a digraph without rays. If every directed cycle of  $D$  has length  $k$ , then  $D$  has a  $k$ -kernel.*

As we have mentioned, Conjecture 54 is true for finite graphs when  $k \in \{2, 3\}$ . It is also true for infinite graphs when  $k = 2$ .

## 11. $k$ -TRANSITIVE AND $k$ -QUASI-TRANSITIVE DIGRAPHS

In [17] three new families of digraphs are introduced, two of them generalizing transitive digraphs and one of them generalizing quasi-transitive digraphs. In this section we will focus on  $k$ -transitive and  $k$ -quasi-transitive digraphs.

**Definition.** A digraph  $D$  is  $k$ -transitive if whenever  $\mathcal{C} = (x_0, x_1, \dots, x_k)$  is a directed path of length  $k$  in  $D$ , then  $(x_0, x_k) \in A(D)$ .

Clearly, a 2-transitive digraph is a transitive digraph in the usual sense. Surprisingly, most of the results stated for transitive or quasi-transitive digraphs have a very good analog for  $k$ -transitive or  $k$ -quasi-transitive digraphs respectively. We begin our results with a simple technical lemma.

**Lemma 55.** *Let  $k \geq 2$  be an integer,  $D$  a  $k$ -transitive infinite digraph and  $u, v \in V(D)$ . If there exists a  $uv$ -directed path in  $D$ , then  $d(u, v) \leq k - 1$ .*

**Proof.** Let  $u, v \in V(D)$  be arbitrary distinct vertices and let  $\mathcal{C} = (u = x_0, x_1, \dots, x_n = v)$  be a  $uv$ -directed path. We will prove by induction on  $n$  that  $d(u, v) \leq k - 1$ . If  $n \leq k - 1$  then we are done. So let us assume that  $n \geq k$ , then, by the  $k$ -transitivity of  $D$ , since  $x_0 \mathcal{C} x_k$  is a directed path of length  $k$  in  $D$ ,  $(x_0, x_k) \in A(D)$ , so  $(x_0, x_k) \cup x_k \mathcal{C} x_n$  is a  $uv$ -directed path of length strictly less than  $n$ . We can derive from the induction hypothesis that  $d(u, v) \leq k - 1$ . The result follows from the principle of mathematical induction. ■

The following theorem generalize the result of Theorem 14.

**Theorem 56.** *Let  $D$  be an infinite  $k$ -transitive digraph such that every ray  $(x_i)_{i \in \mathbb{N}}$  has an arc of the form  $(x_j, x_i)$  with  $i < j$ . Then  $D$  has an  $(n, m)$ -kernel for every*

pair of integers  $n, m$  such that  $n \geq 2, m \geq k - 1$ . Moreover, every  $(n, m)$ -kernel of  $D$  consists in choosing one vertex from every terminal component of  $D$ .

**Proof.** This proof is very similar to the proof of Theorem 14, so we will use the same notation for the relation  $\preceq$ . It suffices to prove that if  $C_0$  is a strong component of  $D$ , then there exists a terminal component  $T$  of  $D$  such that  $C_0 \preceq T$ . This is because, since  $D$  is a  $k$ -transitive digraph, by Lemma 55 every vertex in  $C_0$  will be  $(k - 1)$ -absorbed by every vertex in  $T$ . Also, if we choose one vertex in every terminal strong component of  $D$ , the set of the chosen vertices will be  $k$ -independent for every integer  $k \geq 2$ , because every vertex is in a distinct terminal component. So, every set consisting of one vertex from every terminal component of  $D$  will be  $n$ -independent and  $(k - 1)$ -absorbent, and thus,  $m$ -absorbent, for every pair of integers  $n, m$  such that  $n \geq 2, m \geq k - 1$ .

We will proceed by contradiction. Assume that for every  $C \in \mathfrak{C}$  such that  $C_0 \preceq C$  there exists  $C' \in \mathfrak{C}$  such that  $C' \neq C$  and  $C \preceq C'$ . In virtue of the Axiom of Choice we can build a sequence  $\{C_i\}_{i \in \mathbb{N}}$  satisfying  $C_0 \preceq C_1$  and, for every  $i < j$ ,  $C_i \neq C_{i+1}$  and  $C_i \preceq C_j$ . As in the proof of Theorem 14, we want to obtain an infinite outward in  $D$  from this sequence. Let us choose a vertex  $x_1 \in V(C_1)$ . Since  $D$  is  $k$ -transitive, using the Axiom of Choice we can recursively construct a ray in  $D$  in the following way: The first vertex is  $x_1$ ; If  $x_n$  has been chosen in  $V(C_i)$  for some  $i \in \mathbb{N}$ , choose  $x_{n+1}$  as any vertex such that  $(x_n, x_{n+1}) \in A(D)$  and  $x_{n+1} \in V(C_j)$  with  $i < j$ . We affirm that such vertex exists because  $D$  is a  $k$ -transitive digraph and  $\{C_i\}_{i \in \mathbb{N}}$  is an infinite chain in the partial order  $\preceq$ . Clearly,  $x_n$  can reach (at a finite distance)  $C_{i+r}$  for every  $r \in \mathbb{N}$ , so we can choose a vertex  $x_{n+1} \in V(C_j)$  for  $i < j$  such that a  $x_n x_{n+1}$ -directed path of length  $\equiv 0 \pmod{k^2}$  exists. Thus, the  $k$ -transitivity of  $D$  implies that  $(x_n, x_{n+1}) \in A(D)$ .

Now, we have a ray  $\{x_i\}_{i \in \mathbb{N}}$ . Moreover, if  $i < j$ , then  $(x_j, x_i) \notin A(D)$ . This is because, by the construction of  $\{x_i\}_{i \in \mathbb{N}}$ , if  $i < j$ , then  $x_i \in V(C_r)$  and  $x_j \in V(C_s)$  for some  $r < s$ . Then we would have that  $C_s \preceq C_r$  for some  $r < s$  and it would follow from the antisymmetry of  $\preceq$  that  $C_r = C_s$ . By the construction of  $\{C_i\}_{i \in \mathbb{N}}$ , we know that  $s \neq r + 1$ , but this implies the existence of a directed cycle  $(C_r, C_{r+1}, \dots, C_s, C_r)$  in  $D^*$ , contradicting that  $D^*$  is acyclic. Therefore,  $(x_i)_{i \in \mathbb{N}}$  is a ray in  $D$  such that  $(x_j, x_i) \notin A(D)$  for each  $i < j$ , a contradiction. From this point we can conclude as in the proof of Theorem 14. ■

Now we introduce a definition generalizing quasi-transitive digraphs.

**Definition.** A digraph  $D$  is called  $k$ -quasi-transitive if, whenever  $(x_0, x_1, \dots, x_k)$  is a directed path of length  $k$ , then  $(x_0, x_k) \in A(D)$  or  $(x_k, x_0) \in A(D)$ .

Once again, 2-quasi-transitive digraphs are quasi-transitive digraphs in the usual sense. We use a very similar technique that the one we used to work with quasi-transitive digraphs, using local properties, to prove the existence of  $(n, m)$ -semi-

kernels in strong  $k$ -quasi-transitive digraphs. Then we prove that such  $(n, m)$ -semi-kernels are indeed  $(n, m)$ -kernels. The three following lemmas originally stated in [17] for finite digraphs, trivially remain valid for the infinite case. Although they are only a tool for our immediate concern of finding  $(n, m)$ -kernels, we think that they are very interesting on their own.

**Lemma 57.** *Let  $k \in \mathbb{N}$  be an even natural number,  $D$  a  $k$ -quasi-transitive infinite digraph and  $u, v \in V(D)$  such that a  $uv$ -directed path exists. Then:*

- (i) *If  $d(u, v) = k$ , then  $d(v, u) = 1$ .*
- (ii) *If  $d(u, v) = k + 1$ , then  $d(v, u) \leq k + 1$ .*
- (iii) *If  $d(u, v) \geq k + 2$ , then  $d(v, u) = 1$ .*

**Lemma 58.** *Let  $k \in \mathbb{N}$  be an odd natural number,  $D$  a  $k$ -quasi-transitive infinite digraph and  $u, v \in V(D)$  such that a  $uv$ -directed path exists. Then:*

- (i) *If  $d(u, v) = k$ , then  $d(v, u) = 1$ .*
- (ii) *If  $d(u, v) = k + 1$ , then  $d(v, u) \leq k + 1$ .*
- (iii) *If  $d(u, v) = n \geq k + 2$  with  $n$  odd, then  $d(v, u) = 1$ .*
- (iv) *If  $d(u, v) = n \geq k + 3$  with  $n$  even, then  $d(v, u) \leq 2$ .*

It can be observed that  $k$ -quasi-transitive digraphs have a “better” behavior when  $k$  is an even integer. This fact will have important consequences to our concern.

**Lemma 59.** *Let  $D$  be a  $k$ -quasi-transitive digraph. If  $A \neq B$  are strong components of  $D$  such that there exists an  $AB$ -directed path in  $D$ , then  $A \xrightarrow{k-1} B$ .*

The following lemma resembles Lemma 29.

**Lemma 60.** *Let  $D$  be a  $k$ -quasi-transitive digraph. Then, for every directed cycle  $\mathcal{C}$  of  $D$ , there are at least  $r$  arcs of  $\mathcal{C}$ , say  $(u_i, v_i) \in A(\mathcal{C})$ , such that  $d(v_i, u_i) \leq k$ ,  $i \in \{1, 2, \dots, r\}$ , where  $r = \min\{k, \ell(\mathcal{C})\}$ .*

**Proof.** By induction on  $\ell(\mathcal{C})$ . If  $\ell(\mathcal{C}) \leq k + 1$ , the result is clear. Let  $\mathcal{C} = (x_0, x_1, \dots, x_n = x_0)$  be a directed cycle of length  $n \geq k + 2$  in  $D$ . Since  $D$  is  $k$ -quasi-transitive and  $(x_0, x_1, \dots, x_k) \in A(D)$ , then  $(x_0, x_k) \in A(D)$  or  $(x_k, x_0) \in A(D)$ . In the latter case it is clear that  $d(x_i, x_{i-1}) \leq k$  for  $1 \leq i \leq k$  and we are done. In the former case, let us apply the induction hypothesis to the cycle  $\mathcal{C}' = (x_0, x_k) \cup (x_k \mathcal{C} x_0)$ , which has length  $n - k + 1 < n$ , to obtain  $k$  arcs with the desired condition in  $A(\mathcal{C}')$ . Since  $A(\mathcal{C}' - (x_0, x_k)) \subset A(\mathcal{C})$ , if the  $k$  arcs obtained from the induction hypothesis are different from  $(x_0, x_k)$ , then we are done. Let us assume that one of the arcs is  $(x_0, x_k)$ . Hence,  $d(x_k, x_0) \leq k$ . If  $d(x_k, x_0) = 1$ , it is the case we have already analyzed. So  $d(x_k, x_0) > 1$ . Let  $\mathcal{D} = (x_k = y_0, y_1, \dots, y_s = x_0)$  be a  $x_k x_0$ -directed path with  $s \leq k$ . If  $y_1 = x_1$ , then  $d(x_k, x_{k-1}) \leq k$  and we are done. Let us assume that  $y_1 \neq x_1$ . Since

$D$  is  $k$ -quasi-transitive and  $x_1 \mathcal{C} x_k \cup (x_k, y_1)$  is an  $x_1 y_1$ -directed path of length  $k$ , it follows that  $(y_1, x_1) \in A(D)$ , which implies that  $d(x_k, x_{k-1}) \leq k$  because  $(x_k, y_1, x_1) \cup (x_1 \mathcal{C} x_{k-1})$  is an  $x_k x_{k-1}$  directed path of length  $k$ ; or  $(x_1, y_1) \in A(D)$ , which implies that  $d(x_1, x_0) \leq k$  because  $(x_1, y_1) \cup (y_1 \mathcal{D} x_0)$  is an  $x_k x_{k-1}$  directed path of length less than or equal to  $k$ . In either case we reach the desired conclusion. ■

**Lemma 61.** *Let  $k \geq 2$  be an integer and  $D$  be an infinite  $k$ -quasi-transitive digraph such that for every ray  $\{x_i\}_{i \in \mathbb{N}}$  there exists an arc  $(x_j, x_i)$  with  $i < j$ . Then there exists a vertex  $v \in V(D)$  such that whenever  $(v, u) \in A(D)$ , then  $d(u, v) \leq k$ .*

**Proof.** We will proceed by contradiction. Let us assume that for every vertex  $v \in V(D)$  there exists an arc  $(v, u) \in V(D)$  such that  $d(u, v) \geq k + 1$ . Then, since the subdigraph  $H$  of  $D$  induced by these arcs has  $\delta^+(H) \geq 1$ , we have two possibilities. There exists a directed cycle  $\mathcal{C}$  in  $D$  such that for every arc  $(v, u) \in A(\mathcal{C})$ ,  $d(u, v) \geq k + 1$ , which clearly results in a contradiction by Lemma 60. Or there exists a ray  $\mathcal{C} = \{x_i\}_{i \in \mathbb{N}}$  such that for every arc  $(x_i, x_{i+1}) \in A(\mathcal{C})$ ,  $d(x_{i+1}, x_i) \geq k + 1$ . But by hypothesis there is an arc  $(x_j, x_i) \in A(D)$  with  $i < j$ . So,  $(x_i \mathcal{C} x_j) \cup (x_j, x_i)$  is a directed cycle in  $D$ . By Lemma 60, at least one arc  $(x_i, x_{i+1}) \in A(\mathcal{C})$  is such that  $d(x_{i+1}, x_i) \leq k$ , a contradiction. ■

**Lemma 62.** *Let  $k \geq 2$  be an even integer and let  $D$  be a  $k$ -quasi-transitive digraph such that for every ray  $\{x_i\}_{i \in \mathbb{N}}$  there exists an arc  $(x_j, x_i)$  with  $i < j$ . Then  $D$  has a  $(k + 2)$ -semi-kernel consisting in a single vertex.*

**Proof.** By Lemma 61 we can choose a vertex  $v \in V(D)$  such that for every arc  $(v, u) \in A(D)$ ,  $d(u, v) \leq k$ . So let  $u \in V(D)$  be a vertex such that  $2 \leq d(v, u) \leq k + 1$ . It cannot happen that  $d(u, v) \geq k + 2$ , because this would imply by Lemma 57 that  $d(v, u) = 1$ , but  $2 \leq d(v, u)$ , so  $d(u, v) \leq k + 1$  and thus  $\{v\}$  is a  $(k + 2)$ -semi-kernel of  $D$ . ■

**Lemma 63.** *Let  $k \geq 3$  be an odd integer and  $D$  be a  $k$ -quasi-transitive digraph such that for every ray  $\{x_i\}_{i \in \mathbb{N}}$  there exists an arc  $(x_j, x_i)$  with  $i < j$  and such that at least one vertex  $v \in S = \{u \in V(D) : (u, w) \in A(D) \text{ implies that } d(w, u) \leq k + 1\}$  is such that whenever  $d(v, x) = 2$ , then  $d(x, v) \leq k + 1$ . Then  $\{v\}$  is a  $(k + 2)$ -semi-kernel for  $D$ .*

**Proof.** By Lemma 61 the set  $S$  is non-empty and also there is a vertex  $v \in S$  such that whenever  $d(v, x) = 2$ , then  $d(x, v) \leq k + 1$ . So let  $u \in V(D)$  be a vertex such that  $3 \leq d(v, u) \leq k + 1$ . It can not happen that  $d(u, v) \geq k + 2$ , because this would imply, by Lemma 57, that  $d(v, u) \leq 2$ , but  $3 \leq d(v, u)$ , so  $d(u, v) \leq k + 1$  and thus  $\{v\}$  is a  $(k + 2)$ -semi-kernel of  $D$ . ■

**Lemma 64.** *Let  $D$  be an infinite  $k$ -quasi-transitive strong digraph. If  $D$  has a non-empty  $(k + 2)$ -semi-kernel  $S$ , then  $S$  is a  $(k + 2)$ -kernel of  $D$ .*

**Proof.** Let  $S \subseteq V(D)$  be a  $(k + 2)$ -semi-kernel for  $D$  and  $N^{-(k+1)}(S)$  the set of all vertices in  $D$  which are  $(k + 1)$ -absorbed by  $S$ . Define  $T := V(D) \setminus (S \cup N^{-(k+1)}(S))$ . If  $T = \emptyset$ , then  $S$  is a  $(k + 2)$ -kernel of  $D$ . If  $T \neq \emptyset$ , then we can consider a vertex  $v \in T$  which, by the definition of  $T$ , is not  $(k + 1)$ -absorbed by  $S$ , but since  $D$  is strong, there exists a  $vS$ -directed path. Let  $u \in S$  be a vertex such that  $d(v, u) = d(v, S)$ . Then  $d(v, u) \geq k + 2$  because  $v \notin N^{-(k+1)}(S)$ , but from Lemmas 57 and 58 it can be derived that  $d(u, v) \leq 2$ . This fact, altogether with the second  $(k + 2)$ -semi-kernel condition implies that  $v \in N^{-(k+1)}(S)$ , a contradiction. Since the contradiction arises from assuming that  $T \neq \emptyset$ , we can conclude that  $T = \emptyset$  and then  $S$  is a  $(k + 2)$ -kernel for  $D$ . ■

**Theorem 65.** *Let  $k \geq 2$  be an even integer and let  $D$  be a  $k$ -quasi-transitive strong digraph such that for every ray  $\{x_i\}_{i \in \mathbb{N}}$  there exists an arc  $(x_j, x_i)$  with  $i < j$ . Then  $D$  has an  $(n, m)$ -kernel for every pair of integers  $n, m$  such that  $n \geq 2$ ,  $m \geq k + 1$ .*

**Proof.** By Lemma 62,  $D$  has a  $(k + 2)$ -semi-kernel  $N$  consisting in a single vertex, but by Lemma 64,  $N$  is indeed a  $(k + 2)$ -kernel of  $D$ . But since  $N$  has only one vertex, then  $N$  is  $n$ -independent for every  $n \geq 2$ , and since it is  $(k + 1)$ -absorbent, then it is  $m$ -absorbent for every  $m \geq k + 1$ , so  $N$  is an  $(n, m)$ -kernel for every pair of integers  $n, m$  such that  $n \geq 2$ , and  $m \geq k + 1$ . ■

**Theorem 66.** *Let  $k \geq 2$  be an even integer and let  $D$  be a  $k$ -quasi-transitive digraph such that for every ray  $\{x_i\}_{i \in \mathbb{N}}$  there exists an arc  $(x_j, x_i)$  with  $i < j$ . Then  $D$  has an  $(n, m)$ -kernel for every pair of integers  $n, m$  such that  $n \geq 2$ ,  $m \geq k + 1$ .*

**Proof.** Once again it suffices to prove that if  $C_0$  is a strong component of  $D$ , then there exists a terminal component  $T$  of  $D$  such that  $C_0 \preceq T$ . This is because, in virtue of Lemmas 59 and 65, if we choose a subset  $N \subseteq V(D)$  consisting in an  $(n, m)$ -kernel for every terminal component of  $D$ , then this set will be  $n$ -independent for every  $n \in \mathbb{Z}^+$  because every such  $(n, m)$ -kernel consist in a single vertex and terminal components are path-independent. Also  $N$  will be  $(k + 1)$ -absorbent because every  $(n, m)$ -kernel is inside its component and every vertex of  $D$  not in a terminal component is  $(k - 1)$ -absorbed by every vertex in some terminal component.

Let us observe that the same proof as of Theorem 56 works for  $k$ -quasi-transitive digraphs. The only part of the proof where the hypothesis of being  $k$ -transitive is used is when we recursively construct a ray in  $D$  (using the Axiom of Choice). When  $x_n \in V(C_i)$  has been chosen and we choose  $x_{n+1}$  as any vertex such that  $(x_n, x_{n+1}) \in A(D)$  and  $x_{n+1} \in V(C_j)$  with  $i < j$ .



Using  $k$ -quasi-transitivity instead of  $k$ -transitivity, we affirm that such vertex exists because  $\{C_i\}_{i \in \mathbb{N}}$  is an infinite chain in the partial order  $\preceq$ . Clearly,  $x_n$  can reach (at a finite distance)  $C_{i+r}$  for every  $r \in \mathbb{N}$ . Moreover, for every  $C_j$  such that  $i < j$  and  $x \in V(C_j)$  we have that  $d(x_n, x) \leq k - 1$ . Otherwise, by Lemmas 57 and 58 we would have an  $xx_n$ -directed path, but there are not  $C_j C_i$ -directed paths for  $i < j$  because we would have a cycle in  $D^*$ . Now, let  $l$  be an integer greater than  $i$  and  $y \in V(C_l)$  an arbitrary vertex. Then  $d(x_n, y) \leq k - 1$ . If  $d(x_n, y) = 1$ , then  $y = x_{n+1}$ . If not, let  $\mathcal{C}$  be an  $x_n y$ -directed path of minimum length and consider an arbitrary vertex  $z \in V(C_j)$  for some  $j \geq l$  such that a  $yz$ -directed path  $\mathcal{D}$ , internally disjoint with  $\mathcal{C}$  and of length  $k - d(x_n, y)$ , exists (such vertex always exists since  $\{C_n\}_{n \in \mathbb{N}}$  is an infinite sequence). Clearly,  $\mathcal{C} \cup \mathcal{D}$  is an  $x_n z$ -directed path of length  $k$ . By the  $k$ -quasi-transitivity of  $D$ ,  $(x_n, z) \in A(D)$  or  $(z, x_n) \in A(D)$ , but since  $z \in V(C_j)$  with  $i < j$ , it follows that  $(x_n, z) \in A(D)$ , and thus we can choose  $x_{n+1} = z$ . So, the ray can be constructed and the rest of the proof is just like the aforementioned proof. ■

**Theorem 67.** *Let  $k \geq 3$  be an odd integer and let  $D$  be a  $k$ -quasi-transitive strong digraph such that for every ray  $\{x_i\}_{i \in \mathbb{N}}$  there exists an arc  $(x_j, x_i)$  with  $i < j$  and such that at least one vertex  $v \in S = \{u \in V(D) : (u, w) \in A(D) \text{ implies that } d(w, u) \leq k + 1\}$  is such that whenever  $d(v, x) = 2$ , then  $d(x, v) \leq k + 1$ . Then  $D$  has an  $(n, m)$ -kernel for every pair of integers  $n, m$  such that  $n \geq 2, m \geq k + 1$ .*

**Proof.** It is analogous to the proof of Theorem 65. ■

**Theorem 68.** *Let  $k \geq 3$  be an odd integer and let  $D$  be an infinite  $k$ -quasi-transitive digraph such that for every ray  $\{x_i\}_{i \in \mathbb{N}}$  there exists an arc  $(x_j, x_i)$  with  $i < j$  and such that at least one vertex  $v \in S = \{u \in V(D) : (u, w) \in A(D) \text{ implies that } d(w, u) \leq k + 1\}$  is such that whenever  $d(v, x) = 2$ , then  $d(x, v) \leq k + 1$ . Then  $D$  has an  $n$ -kernel for every  $n \geq k + 2$ .*

**Proof.** It is analogous to the proof of Theorem 66. ■

As a final comment, we would like to point out again that, in virtue of Lemma 61, the set  $S = \{u \in V(D) : (u, w) \in A(D) \text{ implies that } d(w, u) \leq k + 1\}$  in Lemma 63 is always non-empty. So, it would suffice to prove that there is a vertex  $v \in S$  such that whenever  $d(v, x) = 2$ , then  $d(x, v) \leq k + 1$  for every  $x \in V(D)$  to have as a consequence that, for every integer  $k \geq 2$ , every  $k$ -quasi-transitive digraph has an  $(n, m)$ -kernel for every pair of integers  $n, m$  such that  $n \geq 2, m \geq k + 1$ .

From the various properties that  $k$ -quasi-transitive digraphs have proved to have, we state the following conjecture.

**Conjecture 69.** *If  $k \geq 3$  is an odd integer and  $D$  is a  $k$ -quasi-transitive strong digraph, then  $D$  has a non-empty  $(k + 2)$ -kernel.*

In the finite case it is proved in [17], by means of a structural characterization of 3-quasi-transitive digraphs, that every 3-quasi-transitive digraph has a 4-kernel. Since the structural characterization works only for finite digraphs, an analogous for infinite digraphs could not be obtained. Nonetheless, we believe that the existence of a  $(k+2)$ -kernel can be replaced by the existence of a  $(k+1)$ -kernel in Conjecture 69. Once again, a good starting point would be to prove the result for infinite 3-quasi-transitive digraphs. Also, recently, it has been proved in [19] that Conjecture 69 is true for finite digraphs. Nonetheless, the proof relies heavily in the fact that finite digraphs have bounded out-degree, and hence, it is not valid for infinite digraphs.

### Acknowledgements

The authors would like to express their deepest gratitude to Professor Lajos Soukup, whose remarks and suggestions helped to greatly improve the final quality and presentation of this work. Specially, Section 2 would not have been possible without him introducing us to the wonderful technique using the Ultrafilter Lemma.

Also, the authors would like to thank to an Anonymous Referee whose comments helped us to have a better organization of the present paper.

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Received 8 December 2010

Revised 16 May 2013

Accepted 16 May 2013