

NOTE

ON THE ERDŐS-GYÁRFÁS CONJECTURE
IN CLAW-FREE GRAPHS

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Abstract

The Erdős-Gyárfás conjecture states that every graph with minimum degree at least three has a cycle whose length is a power of 2. Since this conjecture has proven to be far from reach, Hobbs asked if the Erdős-Gyárfás conjecture holds in claw-free graphs. In this paper, we obtain some results on this question, in particular for cubic claw-free graphs.

Keywords: Erdős-Gyárfás conjecture, claw-free graphs, cycles.

2010 Mathematics Subject Classification: C5038, C5038.

1. INTRODUCTION

All graphs in this paper are assumed to be simple, that is, without any loops and multiple edges. Let us first recall here briefly some notation and terminology we will need in this paper. We denote by $\delta = \delta(G)$ the minimum degree of the the vertices in the graph $G = (V, E)$. A *uv-path* is a path having the vertices u and

v as its ends. The length of a path P (or a cycle C) is denoted by $l(P)$ (resp. $l(C)$). Also, we denote the distance between the vertices u and v by $d(u, v)$, that is the length of a shortest uv -path. A graph that does not contain a particular graph H as an induced subgraph is called H -free. The complete bipartite graph $K_{1,3}$ is referred to as a *claw*; so a graph is called *claw-free* if it does not have $K_{1,3}$ as an induced subgraph. A *triangle* is a cycle of length three. A *chord* of a cycle C is an edge between two vertices of C which are not adjacent in C . By a *hole* we mean a chordless cycle of length at least four. A hole of length n is called an n -hole.

Several questions on cycles in graphs have been posed by Erdős and his colleagues (see, e.g., [1]). In particular, in 1995 Erdős and Gyárfás [4] asked: *If G is a graph with minimum degree at least three, does G have a cycle whose length is a power of 2?*

This is known as the *Erdős-Gyárfás conjecture*. In fact, Erdős and Gyárfás [4] said that “we are convinced now that this is false and no doubt there are graphs for every r every vertex of which has degree $\geq r$ and which contain no cycle of length 2^k , but we never found a counterexample even for $r = 3$ ”.

There seems to be very little published on the Erdős-Gyárfás conjecture. Markström [5] (via computer searches) asserted that any cubic counterexample must have at least 30 vertices. Salehi Nowbandegani and Esfandiari [6] prove that any bipartite counterexample must have at least 32 vertices.

More generally, Erdős asked does there exist an integer sequence a_1, a_2, a_3, \dots with zero density, and a constant c such that every graph with average degree at least c contains a cycle of length a_i for some i . This question is answered affirmatively by Verstraëte [8].

Hobbs asked if the Erdős-Gyárfás conjecture holds in claw-free graphs [3]. Shauger [7] proved the conjecture for $K_{1,m}$ -free graphs having minimum degree at least $m + 1$ or maximum degree at least $2m - 1$. Also, Daniel and Shauger [3] proved it for planar claw-free graphs. In this paper, we investigate claw-free graphs with $\delta \geq 4$ and cubic claw-free graphs.

2. TWO-POWER CYCLE LENGTHS IN CLAW-FREE GRAPHS

Our first theorem concerns claw-free graphs with $\delta \geq 3$.

Theorem 1. *Suppose that G is a claw-free graph with $\delta \geq 3$. Then G has a cycle whose length is 2^k , or $3 \cdot 2^k$, for some positive integer k .*

To prove Theorem 1 we need the following lemma.

Lemma 2. *Let G be a graph with $\delta \geq 3$. If G does not have C_4 as a subgraph, then for some $n \geq 5$ it has an n -hole.*

Proof. It is known that every graph with $\delta \geq 2$ contains a cycle of length at least $\delta + 1$ (see, e.g., [2, Exercise 2.1.5]). Thus G has a cycle D_1 of length $n_1 \geq 5$. If $n = 5$, D_1 must clearly be chordless. If $n > 5$, and D_1 has no chord, we are finished, so suppose D_1 has a chord. The chord separates D_1 into two shorter cycles, none of which have length 4, by assumption. Thus at least one of these two cycles, say D_2 , must have length $5 \leq n_2 < n_1$. Since G is finite, we must by repeating this argument eventually find a chordless cycle D_k of length $n_k \geq 5$. ■

Definition. We call an edge of a graph *triangulated* if it is contained in a triangle. Also if such a triangle is unique, we call the edge *uniquely triangulated*.

Now we are ready to prove Theorem 1.

Proof of Theorem 1. If G has a cycle of length four, the theorem holds, with $k = 2$. We may therefore assume that G does not contain any C_4 . Thus, by Lemma 2, for some $n \geq 5$, G has an n -hole. Let $C : a_1a_2 \dots a_s a_1$, $s \geq 5$, be a smallest hole in G . Since $\delta \geq 3$ and C is a hole, each vertex of C has a neighbour in $G - V(C)$. For i , ($1 \leq i \leq s$), suppose that $a_i b_i \in E(G)$, where $a_i \in C$ and $b_i \in V(G) \setminus V(C)$. Then either $a_{i-1} b_i \in E(G)$, or $a_{i+1} b_i \in E(G)$, because G is claw-free. Now we show that $b_i \neq b_j$ if $|j - i| \geq 2$. To get a contradiction, fix i and let a_j be the first vertex of C after a_i such that $b_i = b_j = b$, $|j - i| \geq 2$. If $j - i = 2$, then we get the $C_4 : a_i a_{i+1} a_{i+2} b a_i$, which is absurd. If $|j - i| > 2$, then we get the hole $a_{i+1} \dots a_j b a_{i+1}$ which is certainly smaller than C (note that we do not reject the case that this hole may be a C_4).

Therefore, it follows that every other edge of C is uniquely triangulated; we mark them. Moreover, the third vertices of the corresponding triangles are disjoint. Note also that s is even. Consequently, we find cycles of lengths $s, s + 1, \dots, \frac{3}{2}s$ by traversing C such that as we reach a marked edge, we pass it directly or through the third vertex of its corresponding triangle. Since either there exists a 2^k or a $3 \cdot 2^{k-1}$ between s and $\frac{3}{2}s$, the proof is complete. ■

As mentioned above, Shauger [7] proved the Erdős-Gyárfás conjecture for $K_{1,m}$ -free graphs having minimum degree at least $m + 1$ or maximum degree at least $2m - 1$. Theorem 5 improves on the result of Shauger in claw-free graphs. First we state the following proposition. We omit the easy proof.

Proposition 3. *In a 4-regular claw-free graph which does not contain C_4 , every edge is uniquely triangulated.*

Lemma 4. *Let G be a 4-regular claw-free graph which does not contain C_4 and v be a vertex of G . Let C be a smallest n -hole in G containing v , $n \geq 5$. Then for every edge xy of C , the third vertex $z = z(xy)$ of the corresponding triangle of xy is out of C . Furthermore, if $uw \neq xy$ are two edges of C , then $z(uw) \neq z(xy)$.*

Proof. First note that since C is a hole, for every edge xy in C , $z = z(xy) \notin C$. Let uw and wx be two consecutive edges in C . If $z = z(uw) = z(wx)$, then we get the $C_4 : uwxyz$. Hence $z(uw) \neq z(wx)$. Suppose that uw and xy are two non-consecutive edges in C and suppose C traverses the vertices in order u, w, x, y , and then v . Let Q be the yvu segment of C . Now if $z = z(uw) = z(xy)$, then the cycle $uQyzu$ is a smaller hole containing v ; unless u and y are adjacent in C (and hence v is one of them). But in this case, we see that $uzxyu$ is a C_4 in G . This contradiction shows that $z(uw) = z(xy)$ for $uw \neq xy$ is impossible. ■

Theorem 5. *Let G be a claw-free graph with $\delta \geq 4$, which does not contain C_4 . Then every non-cut vertex of G lies on a cycle whose length is a power of 2.*

Proof. Since $\delta \geq 4$ and G is claw-free, if G has a vertex with degree at least 5, then this vertex lies on a C_4 ; so we can assume that G is 4-regular. Suppose that v is a non-cut vertex of G and let w, x, y , and u be its neighbours. Hence, $G - v$ is connected. In view of G is claw-free, we can assume that $wu, xy \in E(G)$. Let P_1, P_2, P_3 , and P_4 be the shortest wy -path, wx -path, xu -path, and yu -path in $G - v$, respectively. Also, without loss of generality assume that $l(P_1) = \min\{l(P_1), l(P_2), l(P_3), l(P_4)\}$. The path P_1 together with the edges vw and vy make a cycle C . Clearly, $l(P_1) > 1$, otherwise $ywvwy$ will be a C_4 . Therefore, $l(C) = s \geq 5$. Since P_1 was the shortest path among P_1, P_2, P_3 , and P_4 , we see that neither x nor u are in P_1 and, in fact, C is the shortest non-triangle hole containing the vertex v ; for if v lies on another non-triangle shorter hole, then two of its neighbours would have distance less than $l(P_1)$ in $G - v$. By Lemma 4, each edge of C is uniquely triangulated such that the third vertex of its corresponding triangle is not on C and this correspondence is one to one. Since $l(C) = s$, then G contains cycles of lengths $s, s + 1, \dots, 2s$. For, as in the proof of Theorem 1, when we traverse the vertices of C , we can either pass the two ends of every edge directly or through the third vertex of its corresponding triangle.

This implies that G has a cycle containing v whose length is 2^k , for some $k \geq 3$. ■

3. THE ERDŐS-GYÁRFÁS CONJECTURE IN CUBIC CLAW-FREE GRAPHS

In this section, we investigate the Erdős-Gyárfás conjecture in cubic claw-free graphs. Indeed, we discuss on the cubic claw-free graphs for which the Erdős-Gyárfás conjecture possibly does not hold.

Suppose that G is a cubic claw-free graph that does not contain C_4 . Let v be an arbitrary vertex of G , and let its neighbours be x, y , and z . Since G is claw-free, so we can assume that $xy \in E(G)$. Thus, $xz, yz \notin E(G)$; otherwise a C_4 appears. Let x_1 and y_1 be respectively the other neighbours of x and y . Easily we see that

$x_1 \neq y_1$. Therefore, for every vertex there exists a unique triangle containing it, such that the other neighbours of its vertices are distinct. Hence G consists of some vertex-disjoint triangles which are connected by a perfect matching of G . Furthermore, if two vertices from two triangles are matched, then there is no more link between these two triangles, again because we have no C_4 in G . This means if we look locally at the graph, we see a triangle together with three appended edges, such that these edges connect to three disjoint triangles. Now define \hat{G} to be the graph whose vertices are triangles of G and two vertices are adjacent in \hat{G} whenever their corresponding triangles in G are linked by an edge. The graph \hat{G} is then a simple cubic graph. We can imagine \hat{G} as a graph obtained from G by shrinking each triangle to a vertex.

Conversely, we can start from a simple cubic graph \hat{G} and replacing each vertex v with a triangle T ; linking the three vertices of T to the three triangles corresponding to the three neighbours of v . This procedure results in a cubic claw-free graph G without C_4 . To sum up, we have the following proposition.

Proposition 6. *The mapping $G \leftrightarrow \hat{G}$ is a one to one correspondence between simple cubic graphs and simple cubic claw-free graphs without C_4 .*

Corollary 7. *If \hat{G} contains a cycle of length k , then this cycle provides cycles of lengths $2k, 2k + 1, \dots, 3k$ in G .*

Proof. Consider a cycle \hat{C} of length k in \hat{G} . The subgraph S of G corresponding to \hat{C} consists of a cycle of length $2k$ such that every other edge of it is triangulated. Hence we can find cycles of lengths $2k, 2k + 1, \dots, 3k$ in S . ■

Based on Proposition 6 and Corollary 7, we think the following conjecture is true.

Conjecture 8. *Every cubic graph contains a cycle of length l such that $2l \leq 2^k < 3l$, for some positive integer k .*

If this conjecture holds, it will lead to a proof of the Erdős-Gyárfás conjecture in cubic claw-free graphs. Also note that this conjecture can be easily deduced from the Erdős-Gyárfás conjecture. But for simplicity, we restrict ourselves to cubic graphs, and the length of the desired cycle has a very wide range.

At the end, we investigate minimal cubic claw-free graphs which possibly have no cycle with length a power of 2.

Theorem 9. *Any counterexample to the Erdős-Gyárfás conjecture in cubic claw-free graphs must have at least 114 vertices.*

Proof. Let G be a claw-free cubic graph of order $3n$. Then \hat{G} (defined in Proposition 6) is a cubic graph of order n . By Corollary 7, if \hat{G} contains a cycle of length l , where $l \in \{2, 3, 4, 6, 7, 8\}$, then the Erdős-Gyárfás conjecture holds for

G . So let us assume that \hat{G} does not contain such cycles. Let v_0 be a vertex of \hat{G} . We consider $\{v_0\}$ as level 0, and define level i , $i \geq 1$, as the set

$$L_i = \{v \in V(\hat{G}) : d(v, v_0) = i\}.$$

Clearly, L_1 is an independent set. It is easy to see that the subgraph induced by L_2 has at most one edge. One can check that if the subgraph induced by L_2 has no edge, then the subgraph induced by L_3 has at most three edges, and if the subgraph induced by L_2 has one edge, then the subgraph induced by L_3 has at most one edge. No two elements of L_3 have common neighbours in L_4 , because otherwise, \hat{G} contains the cycles of lengths 2, 4, 6, or 8. An easy calculation shows that \hat{G} has at least 38 vertices. Consequently, any counterexample for the Erdős-Gyárfás conjecture must have at least $3 \times 38 = 114$ vertices. ■

Acknowledgments

The authors would like to thank anonymous referees for helpful mathematical and grammatical comments.

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doi:10.1002/jgt.20072

Received 29 August 2012

Revised 6 February 2013

Accepted 6 February 2013