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THE NICHE GRAPHS OF INTERVAL ORDERS

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Abstract

The *niche graph* of a digraph D is the (simple undirected) graph which has the same vertex set as D and has an edge between two distinct vertices x and y if and only if $N^+_D(x) \cap N^+_D(y) \neq \emptyset$ or $N^-_D(x) \cap N^-_D(y) \neq \emptyset$, where $N^+_D(x)$ (resp. $N^-_D(x)$) is the set of out-neighbors (resp. in-neighbors) of x in D. A digraph $D = (V, A)$ is called a *semiorder* (or a *unit interval order*) if there exist a real-valued function $f : V \to \mathbb{R}$ on the set V and a positive real number $\delta \in \mathbb{R}$ such that $(x, y) \in A$ if and only if $f(x) > f(y) + \delta$. A digraph $D = (V, A)$ is called an *interval order* if there exists an assignment J of a closed real interval $J(x) \subset \mathbb{R}$ to each vertex $x \in V$ such that $(x, y) \in A$ if and only if $\min J(x) > \max J(y)$.

Kim and Roberts characterized the competition graphs of semiorders and interval orders in 2002, and Sano characterized the competition-common enemy graphs of semiorders and interval orders in 2010. In this note, we give characterizations of the niche graphs of semiorders and interval orders.

Keywords: competition graph, niche graph, semiorder, interval order.

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1. Introduction

Cohen [2] introduced the notion of competition graphs in 1968 in connection with a problem in ecology. The *competition graph* $C(D)$ of a digraph D is the (simple undirected) graph which has the same vertex set as D and has an edge between two distinct vertices x and y if and only if $N^+_D(x) \cap N^+_D(y) \neq \emptyset$, where $N^+_D(x) = \{v \in V(D) \mid (x,v) \in A(D)\}$ is the set of out-neighbors of x in D. (For a digraph D, we denote the vertex set and the arc set of D by $V(D)$ and $A(D)$, respectively.) It has been one of the important research problems in the study of competition graphs to characterize the competition graphs of digraphs satisfying some specified conditions.

A digraph $D = (V, A)$ is called a *semiorder* (or a *unit interval order*) if there exist a real-valued function $f : V \to \mathbb{R}$ on the set V and a positive real number $\delta \in \mathbb{R}$ such that $(x, y) \in A$ if and only if $f(x) > f(y) + \delta$. A digraph $D = (V, A)$ is called an *interval order* if there exists an assignment J of a closed real interval $J(x) \subset \mathbb{R}$ to each vertex $x \in V$ such that $(x, y) \in A$ if and only if $\min J(x) > \max J(y)$. We call J an *interval assignment* of D. (See [3] for details on interval orders.)

A *complete graph* is a graph which has an edge between every pair of vertices. We denote the complete graph with n vertices by K_n. An *edgeless graph* is a graph which has no edges. We denote the edgeless graph with n vertices by I_n. The (*disjoint*) *union* of two graphs G and H is the graph $G \cup H$ whose vertex set is the disjoint union of the vertex sets of G and H and whose edge set is the disjoint union of the edge sets of G and H.

Kim and Roberts characterized the competition graphs of semiorders and interval orders as follows.

Theorem 1 [4]**.** *Let G be a graph. Then the following are equivalent.*

(a) *G is the competition graph of a semiorder,*

(b) *G is the competition graph of an interval order,*

(c) *$G = K_r \cup I_q$ where if $r \geq 2$ then $q \geq 1$.*

Scott [6] introduced the *competition-common enemy graphs* of digraphs in 1987 as a variant of competition graphs. The *competition-common enemy graph* of a digraph D is the graph which has the same vertex set as D and has an edge between two distinct vertices x and y if and only if both $N^+_D(x) \cap N^+_D(y) \neq \emptyset$ and $N^-_D(x) \cap N^-_D(y) \neq \emptyset$ hold, where $N^-_D(x) = \{v \in V(D) \mid (v,x) \in A(D)\}$ is the set of in-neighbors of x in D.

Sano characterized the competition-common enemy graphs of semiorders and interval orders as follows.

Theorem 2 [5]**.** *Let G be a graph. Then the following are equivalent.*

(a) *G is the competition-common enemy graph of a semiorder,*

(b) *G is the competition-common enemy graph of an interval order,*

(c) *$G = K_r \cup I_q$ where if $r \geq 2$ then $q \geq 2$.*

Niche graphs are another variant of competition graphs, which were introduced by Cable, Jones, Lundgren and Seager [1]. The *niche graph* of a digraph D is the graph which has the same vertex set as D and has an edge between two distinct vertices x and y if and only if $N^+_D(x) \cap N^+_D(y) \neq \emptyset$ or $N^-_D(x) \cap N^-_D(y) \neq \emptyset$.

In this note, we characterize the niche graphs of semiorders and interval orders. As a consequence, it turns out that the class of the niche graphs of interval orders is larger than the class of the niche graphs of semiorders. In fact, the graph $P_3 \cup I_1$ (the union of a path with three vertices and an isolated vertex) is the niche graph of an interval order, but $P_3 \cup I_1$ is not the niche graph of a semiorder.

2. Main Results

To state our main results, we first recall basic terminology in graph theory. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The *complement* of a graph G is the graph $\overline{G}$ defined by $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{vv' \mid v, v' \in V(G), v \neq v', vv' \notin E(G)\}$. For positive integers m and n, the *complete bipartite graph* $K_{m,n}$ is the graph defined by $V(K_{m,n}) = X \cup Y$, where $|X| = m$ and $|Y| = n$, and $E(K_{m,n}) = \{xy \mid x \in X, y \in Y\}$. We can observe that $\overline{K_{m,n}} = K_m \cup K_n$.

The niche graphs of semiorders are characterized as follows.

Theorem 3. *A graph G is the niche graph of a semiorder if and only if G is one of the following graphs.*

(i) *an edgeless graph I_q,*

(ii) *the union of two complete graphs $K_m \cup K_n$,*

(iii) *the union of two complete graphs and an edgeless graph $K_m \cup K_n \cup I_q$,*

(iv) *the complement of the union of a complete bipartite graph and an edgeless graph $\overline{K_{m,n} \cup I_q}$,*

where m, n, and q are positive integers.

Proof. First, we show the "only if" part. Let G be the niche graph of a semiorder D. Then there exist a function $f : V(D) \to \mathbb{R}$ and a positive real number $\delta \in \mathbb{R}_{>0}$ such that $A(D) = \{(x, y) \mid x, y \in V(D), f(x) > f(y) + \delta\}$. Let r_1 and r_2 be real numbers defined by

$$r_1 = \min_{x \in V(D)} f(x) \qquad \text{and} \qquad r_2 = \max_{x \in V(D)} f(x).$$

We consider the following three cases: *Case* 1. $r_1+\delta \geq r_2$, *Case* 2. $r_1+\delta < r_2 \leq r_1+2\delta$, *Case* 3. $r_1+2\delta < r_2$.

Case 1. Consider the case where $r_1+\delta \geq r_2$. In this case, we can observe that D has no arcs. Therefore G is an edgeless graph.

Case 2. Consider the case where $r_1+\delta < r_2 \leq r_1+2\delta$. Note that $r_1 < r_2-\delta \leq r_1+\delta < r_2$. Let V_1, V_2, and V_3 be subsets of $V(D)$ defined by

$$\begin{aligned} V_1 &= \{v \in V(D) \mid r_1 \leq f(v) < r_2-\delta\},\\ V_2 &= \{v \in V(D) \mid r_2-\delta \leq f(v) \leq r_1+\delta\},\\ V_3 &= \{v \in V(D) \mid r_1+\delta < f(v) \leq r_2\}. \end{aligned}$$

Then it follows that $V(G) = V_1 \cup V_2 \cup V_3$ and $V_i \cap V_j = \emptyset$ if $i \neq j$. Note that $V_1 \neq \emptyset$ since there exists a vertex $x \in V(D)$ such that $f(x) = r_1$, and that $V_3 \neq \emptyset$ since there exists a vertex $x \in V(D)$ such that $f(x) = r_2$. The set V_1 forms a clique in G since any vertex in V_1 has a common in-neighbor which belongs to V_3 in D. The set V_3 forms a clique in G since any vertex in V_3 has a common out-neighbor which belongs to V_1 in D. Any vertex in V_1 and any vertex in V_3 are not adjacent in G since any vertex in V_1 has no out-neighbor in D and any vertex in V_3 has no in-neighbor in D. Furthermore, any vertex in the set V_2 is an isolated vertex in G since it has neither an in-neighbor nor an out-neighbor in D. That is, the set V_2 induces an edgeless graph if $V_2 \neq \emptyset$. Thus, G is the union of two complete graphs, or G is the union of two complete graphs and an edgeless graph.

Case 3. Consider the case where $r_1+2\delta < r_2$. Note that $r_1 < r_1+\delta < r_2-\delta < r_2$. Let V_1, V_2, and V_3 be subsets of $V(D)$ defined by

$$\begin{aligned} V_1 &= \{v \in V(D) \mid r_1 \leq f(v) \leq r_1+\delta\},\\ V_2 &= \{v \in V(D) \mid r_1+\delta < f(v) < r_2-\delta\},\\ V_3 &= \{v \in V(D) \mid r_2-\delta \leq f(v) \leq r_2\}. \end{aligned}$$

Then it follows that $V(G) = V_1 \cup V_2 \cup V_3$ and $V_i \cap V_j = \emptyset$ if $i \neq j$. Note that $V_1 \neq \emptyset$ and $V_3 \neq \emptyset$. The set $V_2 \cup V_3$ forms a clique in G since any vertex in $V_2 \cup V_3$ has a common out-neighbor which belongs to V_1 in D. The set $V_1 \cup V_2$ forms a clique in G since any vertex in $V_1 \cup V_2$ has a common in-neighbor which belongs to V_3 in D. Any vertex in V_1 and any vertex in V_3 are not adjacent in G since any vertex in V_1 has no out-neighbor in D and any vertex in V_3 has no in-neighbor in D. Therefore, $G = \overline{K_{m,n} \cup I_q}$ where $m = |V_1|$, $n = |V_3|$, and $q = |V_2|$. Thus, G is the union of two complete graphs if $V_2 = \emptyset$, and G is the complement of the union of a complete bipartite graph and an edgeless graph if $V_2 \neq \emptyset$.

Second, we show the "if" part.

Case (i). Let G be an edgeless graph. We define a function $f : V(G) \to \mathbb{R}$ by $f(x) = 1$ for all $x \in V(G)$, and let $\delta = 1$. Then f and δ gives a semiorder $D = (V, A)$ where $V = V(G)$ and $A = \emptyset$, and the niche graph of the semiorder D is the graph G.

Cases (ii) and (iii). Let G be the union of two complete graphs K and K' and an edgeless graph I, where I may possibly be the graph with no vertices. We define a function $f : V(G) \to \mathbb{R}$ by $f(x) = 1$ if $x \in V(K)$, $f(x) = 4$ if $x \in V(K')$, $f(x) = 2$ if $x \in V(I)$, and let $\delta = 2$. Then f and δ gives a semiorder $D = (V, A)$ where $V = V(G)$ and $A = \{(x, y) \mid x \in V(K'), y \in V(K)\}$, and the niche graph of the semiorder D is the graph G.

Case (iv). Let $G = \overline{K_{m,n} \cup I_q}$. Let X and Y be the partite sets of the complete bipartite graph $K_{m,n}$ and let Z be the vertex set of the edgeless graph I_q. Then (X, Y, Z) is a tripartition of the vertex set of G and $E(G) = \{vv' \mid v, v' \in V(G), v \neq v'\} \setminus \{xy \mid x \in X, y \in Y\}$. Now, we define a function $f : V(G) \to \mathbb{R}$ by $f(x) = 1$ if $x \in X$, $f(z) = 3$ if $z \in Z$, $f(y) = 5$ if $y \in Y$, and let $\delta = 1$. Then f and δ gives a semiorder $D = (V, A)$ where $V = V(G)$ and $A = \{(y, x) \mid x \in X, y \in Y\} \cup \{(z, x) \mid x \in X, z \in Z\} \cup \{(y, z) \mid z \in Z, y \in Y\}$, and the niche graph of the semiorder D is the graph G. Hence the theorem holds. ∎

The next theorem characterizes the niche graphs of interval orders.

Theorem 4. *A graph G is the niche graph of an interval order if and only if G is one of the following graphs:*

(i) *an edgeless graph I_q,*

(ii) *the union of two complete graphs $K_m \cup K_n$,*

(iii) *the union of two complete graphs and an edgeless graph $K_m \cup K_n \cup I_r$,*

(iv) *the complement of the union of a complete bipartite graph and an edgeless graph $\overline{K_{m,n} \cup I_q}$,*

(v) *the union of an edgeless graph and the complement of the union of a complete bipartite graph and an edge less graph $I_r \cup \overline{K_{m,n} \cup I_q}$,*

where m, n, q, and r are positive integers.

Proof. For positive integers m and n and non-negative integers q and r, let

$$\Gamma(m, n, q, r) = \overline{K_{m,n} \cup I_q} \cup I_r.$$

We remark that $\Gamma(m, n, 0, 0) = K_m \cup K_n$, $\Gamma(m, n, 0, r) = K_m \cup K_n \cup I_r$, and $\Gamma(m, n, q, 0) = \overline{K_{m,n} \cup I_q}$.

First, we show the "only if" part. Let G be the niche graph of an interval order D. Then there exists an interval assignment J of D. Let r_1 and r_2 be real numbers defined by

$$r_1 = \min_{x \in V(D)} \max J(x) \qquad \text{and} \qquad r_2 = \max_{x \in V(D)} \min J(x).$$

If $r_1 \geq r_2$, then we can observe that D has no arcs and therefore G is an edgeless graph. Now, we consider the case where $r_1 < r_2$. Note that $|V(G)| \geq 2$ since r_1 and r_2 are attained by different vertices. Let V_1, V_2, V_3, and V_4 be subsets of $V(D)$ defined by

$$\begin{aligned} V_1 &= \{v \in V(D) \mid \min J(v) \leq r_1 \leq \max J(v) < r_2\}, \\ V_2 &= \{v \in V(D) \mid r_1 < \min J(v), \max J(v) < r_2\}, \\ V_3 &= \{v \in V(D) \mid r_1 < \min J(v) \leq r_2 \leq \max J(v)\}, \\ V_4 &= \{v \in V(D) \mid \min J(v) \leq r_1, r_2 \leq \max J(v)\}. \end{aligned}$$

Then it follows that $V(G) = V_1 \cup V_2 \cup V_3 \cup V_4$ and $V_i \cap V_j = \emptyset$ if $i \neq j$. Note that $V_1 \neq \emptyset$ since there exists a vertex $x \in V(D)$ such that $\max J(x) = r_1$, and that $V_3 \neq \emptyset$ since there exists a vertex $x \in V(D)$ such that $\min J(x) = r_2$. The set $V_2 \cup V_3$ forms a clique in G since any vertex in $V_2 \cup V_3$ has a common out-neighbor which belongs to V_1 in D. The set $V_1 \cup V_2$ forms a clique in G since any vertex in $V_1 \cup V_2$ has a common in-neighbor which belongs to V_3 in D. Any vertex in V_1 and any vertex in V_3 are not adjacent in G since any vertex in V_1 has no out-neighbor in D and any vertex in V_3 has no in-neighbor in D. Therefore, the set $V_1 \cup V_2 \cup V_3$ induces the graph $\overline{K_{m,n} \cup I_q}$ where $m = |V_1|$, $n = |V_3|$, and $q = |V_2|$. Furthermore, any vertex in the set V_4 is an isolated vertex in G since it has neither an in-neighbor nor an out-neighbor in D. That is, the set V_4 induces an edgeless graph. Thus G is the graph $\Gamma(m, n, q, r)$ with $m = |V_1|$, $n = |V_3|$, $q = |V_2|$, and $r = |V_4|$.

Second, we show the "if" part.

Case (i). Let G be an edgeless graph. We define an interval assignment J by $J(x) = [1, 2]$ for all $x \in V(G)$, where $[a, b]$ denotes the closed real interval $\{r \in \mathbb{R} \mid a \leq r \leq b\}$. Then J gives an interval order $D = (V, A)$ where $V = V(G)$ and $A = \emptyset$, and the niche graph of the semiorder D is the graph G.

Cases (ii)–(v). Let G be the graph $\Gamma(m, n, q, r)$ for some positive integers m and n and non-negative integers q and r. Then, there exists a partition (U_1, U_2, U_3, U_4) of the vertex set of G such that $E(G) = \{vv' \mid v, v' \in U_1 \cup U_2 \cup U_3, v \neq v'\} \setminus \{u_1u_3 \mid u_1 \in U_1, u_3 \in U_3\}$. Note that $\{|U_1|, |U_3|\} = \{m, n\}$, $|U_2| = q$, and $|U_4| = r$. Now, we define an interval assignment J as follows: $J(x) = [1, 2]$ if $x \in U_1$; $J(x) = [3, 4]$ if $x \in U_2$; $J(x) = [5, 6]$ if $x \in U_3$; $J(x) = [1, 6]$ if $x \in U_4$. Then J gives an interval order $D = (V, A)$ where $V = V(G)$ and $A = \{(x, y) \mid x \in U_i, y \in U_j, (i, j) \in \{(3, 2), (3, 1), (2, 1)\}\}$, and the niche graph of the interval order D is the graph G. Hence the theorem holds. ■

References

[1] C. Cable, K.F. Jones, J.R. Lundgren and S. Seager, *Niche graphs*, Discrete Appl. Math. **23** (1989) 231–241.
doi:10.1016/0166-218X(89)90015-2

[2] J.E. Cohen, *Interval graphs and food webs. A finding and a problem*, RAND Corporation, Document 17696-PR, Santa Monica, California (1968).

[3] P.C. Fishburn, Interval Orders and Interval Graphs: A Study of Partially Ordered Sets, Wiley-Interscience Series in Discrete Mathematics, A Wiley-Interscience Publication (John Wiley & Sons Ltd., Chichester, 1985).

[4] S.-R. Kim and F.S. Roberts, *Competition graphs of semiorders and Conditions $C(p)$ and $C^*(p)$*, Ars Combin. **63** (2002) 161–173.

[5] Y. Sano, *The competition-common enemy graphs of digraphs satisfying conditions $C(p)$ and $C'(p)$*, Congr. Numer. **202** (2010) 187–194.

[6] D.D. Scott, *The competition-common enemy graph of a digraph*, Discrete Appl. Math. **17** (1987) 269–280.
doi:10.1016/0166-218X(87)90030-8

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