

THE DEPRESSION OF A GRAPH AND k -KERNELS

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Abstract

An *edge ordering* of a graph G is an injection $f : E(G) \rightarrow \mathbb{R}$, the set of real numbers. A path in G for which the edge ordering f increases along its edge sequence is called an *f -ascent*; an *f -ascent* is *maximal* if it is not contained in a longer *f -ascent*. The *depression* of G is the smallest integer k such that any edge ordering f has a maximal *f -ascent* of length at most k . A *k -kernel* of a graph G is a set of vertices $U \subseteq V(G)$ such that for any edge ordering f of G there exists a maximal *f -ascent* of length at most k which neither starts nor ends in U . Identifying a *k -kernel* of a graph G enables one to construct an infinite family of graphs from G which have depression at most k . We discuss various results related to the concept of *k -kernels*, including an improved upper bound for the depression of trees.

Keywords: edge ordering of a graph, increasing path, monotone path, depression.

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1. INTRODUCTION

An *edge ordering* of a graph G is an injection $f : E(G) \rightarrow \mathbb{R}$, the set of real numbers. Denote the set of all edge orderings of G by $\mathcal{F}(G)$. A path λ in G for which $f \in \mathcal{F}(G)$ increases along its edge sequence is called an *f -ascent*; an *f -ascent* is *maximal* if it is not contained in a longer *f -ascent*. The *flatness* of an edge ordering f , denoted by $h(f)$, is the length of a shortest maximal *f -ascent* of G .

The *depression* of G was defined in [7] as $\varepsilon(G) = \max_{f \in \mathcal{F}(G)} \{h(f)\}$. The interpretation of the depression of a graph G is that any edge ordering f has a

maximal f -ascent of length at most $\varepsilon(G)$, and $\varepsilon(G)$ is the smallest integer for which this statement is true.

Clearly, $\varepsilon(G) = 1$ if and only if K_2 is a component of G . If a connected graph G has a vertex v that is adjacent to u and w , where u and w are pendant vertices or adjacent vertices of degree two, then in any edge ordering f of G , either u, v, w or w, v, u is a maximal $(2, f)$ -ascent, hence $\varepsilon(G) = 2$. In [7] it was shown that the converse of this statement is also true, which gives the following characterization of graphs with depression two.

Theorem 1 [7]. *If G is connected, then $\varepsilon(G) = 2$ if and only if G has a vertex adjacent to two pendant vertices or to two adjacent vertices of degree two.*

Consider two disjoint graphs G_1 and G_2 and vertices $v_i \in V(G_i)$. The *vertex-coalescence* of G_1 and G_2 via v_1 and v_2 is the graph obtained by identifying v_1 and v_2 to form a new vertex v , and is denoted $(G_1 \cdot G_2)(v_1, v_2 : v)$. In forming $G = (G_1 \cdot G_2)(v_1, v_2 : v)$, if v_2 is unimportant we also say we *attach* G_1 to G_2 at v_1 , and if G is the resulting graph, we say that G contains G_1 as an *attachment* at v_1 .

We see from Theorem 1 that if v is the central vertex of P_3 or any vertex of K_3 , and G is any connected graph containing P_3 or K_3 as an attachment at v , then $\varepsilon(G) = 2$.

An interesting question arises from this result.

- What properties should H and $v \in V(H)$ satisfy so that if we attach H to an arbitrary graph at v , the resulting graph has depression at most k ?

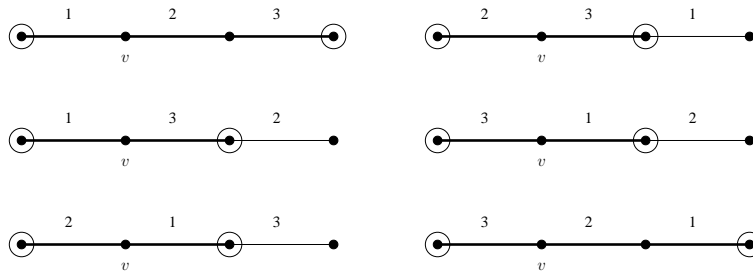


Figure 1. The vertex v is a 3-kernel of G and an ε -kernel of G .

To help answer this question, a k -kernel of a graph G is defined in [14] as a set $U \subseteq V(G)$ such that for any edge ordering f of G there exists a maximal (l, f) -ascent for some $l \leq k$ that neither starts nor ends at a vertex in U , and k is smallest value for which this is true. For example, either vertex of P_4 with degree two forms a 3-kernel of P_4 (see Figure 1). The following theorem relates the concept of k -kernels to the question above.

Theorem 2 [14]. *Let H be an arbitrary graph and let U be a k -kernel of H . Form a graph G by adding any set A of new vertices and arbitrary edges joining vertices in $U \cup A$. Then $\varepsilon(G) \leq k$.*

Therefore, if $\{v\}$ is a k -kernel of H and we attach H to a graph G at $\{v\}$, then by Theorem 2 the resultant graph has depression at most k . In general, identifying a k -kernel of a graph enables one to construct an infinite family of graphs with depression at most k .

In Section 4.1 we define an ε -kernel of a graph and introduce some terminology to aid us in our discussion of k -kernels. In Section 4.2 we characterize ε -kernels of paths and k -kernels of cycles. We provide a sufficient condition in Section 4.3 for a set of vertices to be an ε -kernel of a spider and use this result to improve the upper bound for the depression of trees given in Theorem 9. In Section 4.4 we consider graphs G for which $\text{diam}(\mathcal{L}(G)) = 2$, where $\mathcal{L}(G)$ denotes the line graph of G . Specifically, we describe a sufficient condition for a vertex of a graph G with $\text{diam}(\mathcal{L}(G)) = 2$ to be a k -kernel of G for $k \in \{2, 3\}$ which in turn identifies a large class of graphs with depression at most three. The paper concludes with a list of some open problems in Section 5.

2. DEFINITIONS AND BACKGROUND

We consider simple, finite graphs $G = (V(G), E(G))$. For basic graph theoretic definitions we refer the reader to the book [4] or any of its predecessors. The *open neighbourhood* of a vertex v of G is the set of all vertices adjacent to v and is denoted by $N_G(v)$, or just $N(v)$, and its *closed neighbourhood* is $N_G[v] = N[v] = N(v) \cup \{v\}$.

A *branch vertex* of a tree is a vertex of degree at least three. Let $L(T)$ and $B(T)$ respectively denote the sets of all leaves and all branch vertices of the tree T , and $\ell(T)$ the minimum length of a path P between two leaves of T such that no two consecutive vertices of P are in $B(T)$. For $v \in V(T)$ and $l \in L(T)$, a (v, l) -*endpath*, or v -*endpath* if the leaf is unimportant, or *endpath* if neither v nor l is important, is a path P from v to l such that each internal vertex of P has degree two in T . A *spider* $S(a_1, a_2, \dots, a_r)$ is a tree with exactly one branch vertex v and v -endpaths (also called *legs*) of lengths $1 \leq a_1 \leq a_2 \leq \dots \leq a_r$.

Given an edge ordering f of the graph G , an f -*ascent* λ is simply called an *ascent* if the ordering is clear, and if λ has length k , it is also called a (k, f) -*ascent*. If the path λ with vertex sequence v_0, v_1, \dots, v_k or edge sequence e_1, e_2, \dots, e_k forms an f -*ascent*, we denote this fact by writing λ as $v_0v_1 \cdots v_k$ or $e_1e_2 \cdots e_k$.

The *height* of an edge ordering f , denoted by $H(f)$, is the length of a longest maximal f -*ascent*. In [2] the *altitude* of G was defined as $\alpha(G) = \min_{f \in \mathcal{F}(G)} \{H(f)\}$. The interpretation of the altitude of a graph G is that any

edge ordering $f \in \mathcal{F}(G)$ has an f -ascent of length at least $\alpha(G)$, and $\alpha(G)$ is the largest integer for which this statement is true.

The study of lengths of increasing paths was initiated by Chvátal and Komlós [5] who posed the problem of determining the altitude of complete graphs. This is a difficult problem and $\alpha(K_n)$ is known only for $1 \leq n \leq 8$ (see [2, 5]). In [7] the authors compare the altitude and depression for various families of graphs. In particular, they show that $\varepsilon(K_n) < \alpha(K_n)$ for $n \geq 4$ while $\varepsilon(P_n) > \alpha(P_n)$ for $n \geq 3$. The altitude of graphs was also investigated in e.g. [1, 3, 6, 8, 10, 12, 13, 15, 17, 18].

3. KNOWN RESULTS

Let $\tau(G)$ denote the length of a longest path in G , called the *detour length* in G . If we assume that G is connected and of size at least two, then

$$2 \leq \varepsilon(G), \alpha(G) \leq \tau(G).$$

By taking the edge ordering f for the path P_n , $n \geq 3$, to increase along its edge sequence we see that $\varepsilon(P_n) = \tau(P_n) = n - 1$. On the other hand, by taking the edge ordering for the path P_n , $n \geq 3$, as $1, n - 1, 2, n - 2, \dots, \lceil \frac{n}{2} \rceil$ along its edge sequence, we see that $\alpha(P_n) = 2$.

It is reasonable to expect a link between the depression of a graph G and the diameter of its line graph $\mathcal{L}(G)$, and indeed the following result appeared in [7].

Theorem 3 [7]. *If $\text{diam}(\mathcal{L}(G)) = 2$, then $\varepsilon(G) \leq 3$.*

The difference $\text{diam}(\mathcal{L}(G)) - \varepsilon(G)$ can be arbitrarily large, a result that easily follows from Theorem 1. Much harder to see is that the difference $\varepsilon(G) - \text{diam}(\mathcal{L}(G))$ can also be arbitrarily large as shown by Gaber-Rosenblum and Roditty in [11].

The depression of complete graphs is a direct result of Theorems 1 and 3.

Corollary 4 [7]. *$\varepsilon(K_n) = 3$ for all $n \geq 4$.*

The depression of cycles is also given in [7].

Proposition 5 [7]. *$\varepsilon(C_n) = \lceil \frac{n+1}{2} \rceil$ for all $n \geq 3$.*

A lower bound for the depression of trees was given in [9] and it was shown that this bound gives the exact value of $\varepsilon(T)$ in the case where $B(T)$ is independent. The bound requires the following definition. For $v \in B(T)$ with $\deg v = r$, let $e_1(v), e_2(v), \dots, e_r(v)$ be an arrangement of the edges incident with v , and $\ell_i(v)$ the length of a shortest v -endpath that contains $e_i(v)$. We abbreviate $e_i(v)$ and $\ell_i(v)$ to e_i and ℓ_i , if the vertex v is clear from the context. An arrangement

e_1, \dots, e_r is called *suitable* if $\ell_i \leq \ell_j$ whenever $i < j$. From a suitable arrangement e_1, \dots, e_r of the edges incident with v , define

$$\rho(v) = \min\{\ell_1(v) + \ell_2(v), \ell_3(v) + 1\}.$$

Theorem 6 [9]. *For any tree T , $\varepsilon(T) \geq \min_{v \in B(T)} \{\rho(v)\}$. Moreover, if $B(T)$ is independent, then $\varepsilon(T) = \min_{v \in B(T)} \{\rho(v)\}$.*

Two upper bounds for the depression of trees are given in [7]. The first is based on $\ell(T)$.

Theorem 7 [7]. *For any tree T , $\varepsilon(T) \leq \ell(T)$.*

The second is an improvement on Theorem 7 and is a corollary of Theorem 6.

Corollary 8 [7]. $\varepsilon(S(a_1, a_2, \dots, a_r)) = \min\{a_1 + a_2, a_3 + 1\}$.

An upper bound for the depression of trees related to the above result for spiders was determined in [7], which is an obvious improvement on Theorem 7. Those spiders obtained by removing all edges of the tree that are not edges of endpaths are called *hanging spiders* of T . Let $\mathcal{H}(T)$ denote the set of all hanging spiders $H = S(a_1, \dots, a_r)$, $r \geq 3$, of T . If $\mathcal{L}(G) \neq \emptyset$, then define

$$s(T) = \min_{H \in \mathcal{H}(T)} \{a_3 + 1\}.$$

If $\mathcal{L}(G) = \emptyset$, then define $s(T) = \infty$.

Theorem 9 [7]. *For any tree T , $\varepsilon(T) \leq \min\{\ell(T), s(T)\}$.*

This bound is not exact for trees, even in the case where $B(T)$ is independent. An improvement on this bound is given in Section 4.3 which does give the exact value of $\varepsilon(T)$ in the case where $B(T)$ is independent.

As mentioned previously, Theorem 1 characterizes the class of graphs with depression two. The characterization of graphs with depression three remains an open problem, however, trees with depression three were characterized in [14], and graphs with depression three and no adjacent vertices of degree three or higher were characterized in [16]. The concept of k -kernels plays an integral role in establishing the results in [14] and [16].

4. MAIN RESULTS

4.1. An ε -kernel

We define an ε -kernel of a graph G as a set $U \subseteq V(G)$ such that for any edge ordering f of G there exists a maximal f -ascent of length at most $\varepsilon(G)$ that

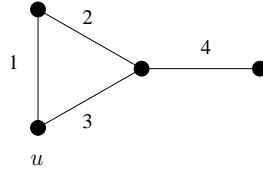


Figure 2. The vertex u is not an ε -kernel but is a 3-kernel.

neither starts nor ends at a vertex in U . That is, a set U is an ε -kernel of G if U is a k -kernel of G and $k = \varepsilon(G)$. For example, as shown in Figure 1, a vertex v of P_4 with degree two is a 3-kernel of P_4 , and since $\varepsilon(P_4) = 3$ we also say that v is an ε -kernel of P_4 .

As illustration of a k -kernel where $k > \varepsilon(G)$, consider the graph G shown in Figure 2. By Theorem 1, $\varepsilon(G) = 2$ and the labelling f in the figure shows that the vertex u is not an ε -kernel of G since the only maximal f -ascent of length two (2 3) ends at u . On the other hand, for any labelling f there exists an f -ascent that does not start or end at u and since the longest possible path in G has length three, we conclude that u is a 3-kernel of G .

For any two adjacent edges of a graph G , say e_1 and e_2 , and an edge ordering f of G , either e_1e_2 or e_2e_1 is an f -ascent of G which is contained in a maximal f -ascent of length at most $\tau(G)$, the length of a longest path in G . Thus, for the vertex v incident with e_1 and e_2 , and any edge ordering f of G , there exists an f -ascent which neither starts nor ends at v . This leads to the following observation.

Observation 10. *Any vertex $v \in V(G)$ with $\deg(v) \geq 2$ is a k -kernel of G for some $\varepsilon(G) \leq k \leq \tau(G)$.*

To aid us in our discussion of k -kernels we introduce the following terminology.

Let f be an edge ordering of a graph G . If an f -ascent λ neither starts nor ends in a set $A \subset V(G)$, we say that λ is an A -avoiding (maximal) f -ascent or an a -avoiding (maximal) f -ascent if $A = \{a\}$ (and λ is not contained in a longer f -ascent).

In order to identify a set $U \subseteq V(G)$ as a k -kernel of a graph G , we must show that for every edge ordering $f \in \mathcal{F}(G)$ there exists a U -avoiding maximal ascent of length at most k .

In the following sections we identify k -kernels for various classes of graphs.

4.2. Paths and cycles

In this section we identify k -kernels of paths and cycles. Since $\varepsilon(P_n) = \tau(P_n)$, it follows that any k -kernel of P_n is necessarily an ε -kernel.

Proposition 11. *Let $U \subseteq V(P_n)$ where $n \geq 3$. Then U is an ε -kernel of P_n if and only if U is an independent set and for each $u \in U$, $\deg(u) = 2$.*

Proof. Suppose to the contrary that U is an independent set, for each $u \in U$, $\deg(u) = 2$, and U is not an ε -kernel of $P_n = v_1, v_2, \dots, v_n$. Then there exists an edge ordering of P_n , say f , such that every maximal f -ascent of length at most $\varepsilon(P_n)$ either starts or ends in U . Since $\varepsilon(P_n) = n - 1$, this means that every maximal f -ascent of P_n starts or ends at a vertex in U . Necessarily, for some $3 \leq k \leq n$, either $v_1v_2 \cdots v_k$ or $v_kv_{k-1} \cdots v_1$ is a maximal f -ascent. Without loss of generality we assume the former. Since $v_1 \notin U$, it follows that $v_k \in U$ and $k < n$. Since $v_1v_2 \cdots v_k$ is a maximal f -ascent, $f(v_{k-1}v_k) > f(v_kv_{k+1})$, which means $v_{k+1}v_kv_{k-1}$ is an f -ascent that ends at v_{k-1} . Since U is an independent set, v_{k-1} and v_{k+1} are not in U . This implies $v_{k+1}v_kv_{k-1}$ is contained in a longer f -ascent λ . Since λ starts or ends in U , the initial vertex, say $v_{k'}$, is in U and $k' > k + 1$. By a similar argument, $v_{k'-1}v_{k'}v_{k'+1}$ is an f -ascent contained in a longer f -ascent, say λ' , and the end vertex k'' of λ' is in U , where $k'' > k' - 1$. Since P_n is of finite length, eventually we obtain a maximal f -ascent which neither starts nor ends in U , a contradiction.

Conversely, suppose U is an ε -kernel of P_n . Then every edge ordering f contains a U -avoiding maximal f -ascent. Suppose that U is not an independent set. Let $v_i, v_{i+1} \in U$ for some $2 \leq i \leq n - 1$. Let f be the edge ordering defined as $f(v_iv_{i+1}) = 1$, $f(v_jv_{j+1}) = j + 1 - i$ for each $j > i$, and $f(v_jv_{j+1}) = i - j + f(v_{n-1}v_n)$ for each $j < i$. Thus any maximal f -ascent of P_n starts at either v_i or v_{i+1} , a contradiction. Suppose that U contains an end vertex of P_n . Consider the edge ordering f defined by $f(v_iv_{i+1}) = i$ for all $1 \leq i \leq n - 1$. Clearly, $v_1v_2 \cdots v_n$ is the only maximal f -ascent and by our assumption it starts or ends in U , a contradiction. ■

Note that Theorem 7 is a corollary of Theorem 2 and Proposition 11. Furthermore, if we define $\ell(G)$ as the minimum length of a path between two end-vertices of G which contains no adjacent vertices of degree three or more, and define $\ell(G) = \tau(G)$ if no such path exists, then we obtain a bound similar to Theorem 7 which applies to graphs in general.

Corollary 12. For any graph G , $\varepsilon(G) \leq \ell(G)$.

Proposition 13. Let $U \subseteq V(C_n)$ where $n \geq 3$. If U is a k -kernel of C_n , then $k = n - 1$. Furthermore, U is an $(n - 1)$ -kernel of C_n if and only if $|U| = 1$.

Proof. By Observation 10 any single vertex v is a k -kernel of C_n for some $\varepsilon(C_n) \leq k \leq n - 1$. Consider a cycle $C_n = v_1, v_2, \dots, v_n$ and the edge ordering f given by $f(v_iv_{i+1}) = i$ for $1 \leq i \leq n - 1$ and $f(v_nv_1) = n$. The only v_2 -avoiding maximal f -ascent has length $n - 1$. Hence v_2 is an $(n - 1)$ -kernel of C_n . Since C_n is vertex transitive, any $U \subseteq V(C_n)$ with $|U| = 1$ is an $(n - 1)$ -kernel of C_n .

Suppose that $U \subseteq V(G)$ and $|U| \geq 2$. Let $u, v \in U$, and say $u = v_1$ and $v = v_k$ where $2 \leq k \leq n$. Let $f : E(C_n) \rightarrow \{1, \dots, n\}$ such that $f(v_1v_2) = 1$,

$f(v_k v_{k+1}) = n$ (or $f(v_k v_1) = n$ if $k = n$), and the remaining edges are labelled so that there are exactly two maximal f -ascents in G , both of which start with the edge labelled 1 and end with the edge labelled n (one in each direction around the cycle). One of the ascents starts at u and the other ends at v , which implies that U is not a kernel of C_n . Therefore, if U is a k -kernel of C_n , then $|U| = 1$ and $k = n - 1$. ■

4.3. Spiders

In this section we identify sets which are an ε -kernel of a spider $S(a_1, a_2, \dots, a_r)$ and use this result to determine a new upper bound for the depression of trees. Recall that for a tree T and a vertex $v \in B(T)$ with $\deg v = r$, an arrangement of the edges e_1, \dots, e_r incident with v is called suitable if $\ell_i(v) \leq \ell_j(v)$ whenever $i < j$, where $\ell_i(v)$ is the length of shortest v -endpath containing e_i .

Proposition 14. *Let $T = S(a_1, a_2, \dots, a_r)$. If $U \subseteq V(T) - L(T)$ and $U \cup B(T)$ is independent, then U is an ε -kernel of T .*

Proof. Let $B(T) = \{v\}$ and $U \subseteq V(T) - L(T)$ such that $U \cup \{v\}$ is independent. By Corollary 8, $\varepsilon(S(a_1, a_2, \dots, a_r)) = \min\{a_1 + a_2, a_3 + 1\}$. Hence, to prove the result we must show that for any edge ordering f of T there exists a U -avoiding maximal f -ascent of length at most $\min\{a_1 + a_2, a_3 + 1\}$.

Let e_1, e_2, \dots, e_r be a suitable arrangement of the edges incident with the branch vertex v . For $1 \leq i < j \leq 3$, let $P_{i,j}$ be the path of length $a_i + a_j$ which contains e_i and e_j . From Proposition 11, for $G = P_{1,2}$, any independent set of $V(G)$ forms an ε -kernel of G , where $\varepsilon(G) = a_1 + a_2$. This implies that for any edge ordering f of T , there exists a $(U \cup \{v\})$ -avoiding maximal f -ascent of length at most $a_1 + a_2$ which is contained in $P_{1,2}$. Similarly, there exist $(U \cup \{v\})$ -avoiding maximal f -ascents of lengths at most $a_1 + a_3$ and $a_2 + a_3$ contained in $P_{1,3}$ and $P_{2,3}$ respectively. Let $\lambda_{i,j}$ be a $(U \cup \{v\})$ -avoiding maximal f -ascent contained in the path $P_{i,j}$ where $1 \leq i < j \leq 3$.

Let f be an edge ordering of T . If $a_1 + a_2 \leq a_3 + 1$, then we are done. Hence we assume $a_1 + a_2 > a_3 + 1$. Suppose to the contrary that there does not exist a U -avoiding maximal f -ascent of length at most $a_3 + 1$ in T . Then each $\lambda_{i,j}$ has length at least $a_3 + 2 \geq a_2 + 2 \geq a_1 + 2$ which implies the edges e_i and e_j are contained in $\lambda_{i,j}$.

Without loss of generality assume that $f(e_1) < f(e_2)$. For $1 \leq i \leq 3$, let e'_i be the edge adjacent to e_i and not incident with v . Then, since the length of $\lambda_{1,2}$ is at least $a_2 + 2$, $f(e'_1) < f(e_1)$ and $f(e_2) < f(e'_2)$. This implies that $f(e'_3) < f(e_3) < f(e_2)$ or else the length of $\lambda_{2,3}$ is at most $a_3 + 1$, which is a contradiction. But then either $\lambda_{1,3}$ has length at most $a_1 + 1$ (if $f(e_3) > f(e_1)$), or $\lambda_{1,3}$ has length at most $a_3 + 1$ (if $f(e_3) < f(e_1)$), which again is a contradiction. ■

We use Proposition 14 to establish an upper bound for the depression of a tree. The bound requires the following definition. An *embedded spider* of a tree T is a subgraph $H = S(a_1, a_2, \dots, a_r)$ of T which is a spider, no endpath of H contains consecutive vertices in $B(T)$, and leaves of H are also leaves of T . Let $\mathcal{H}_{es}(T)$ denote the set of all embedded spiders $H = S(a_1, a_2, \dots, a_r)$ of T where $r \geq 3$. If $\mathcal{H}_{es}(T) \neq \emptyset$, define

$$\sigma(T) = \min_{H \in \mathcal{H}_{es}(T)} \{a_3 + 1\}.$$

If $\mathcal{H}_{es}(T) = \emptyset$, then define $\sigma(T) = \infty$.

Note that $\sigma(T) \leq s(T)$, where $s(T) = \min_{H \in \mathcal{H}(T)} \{a_3 + 1\}$ and $\mathcal{H}(T)$ is the set of all hanging spiders of T with at least three leaves.

Recall that $\ell(T)$ is the minimum length of a path P between two leaves of T such that P contains no two consecutive branch vertices.

Theorem 15. *For any tree T , $\varepsilon(T) \leq \min\{\ell(T), \sigma(T)\}$.*

Proof. If $\min\{\ell(T), \sigma(T)\} = \ell(T)$, then the result follows from Theorem 7. Suppose then that $\ell(T) > \sigma(T)$. Let $H = S(a_1, a_2, \dots, a_r)$ be an embedded spider of T such that $a_3 + 1 = \sigma(T)$. Let U be the set of vertices of H that are adjacent to vertices of $T - H$. Since H is an embedded spider, $U \cup B(H)$ is independent. By Proposition 14, U is an ε -kernel of H . By Theorem 2, $\varepsilon(T) \leq \min\{a_1 + a_2, a_3 + 1\}$, and since $\ell(T) > \sigma(T)$, $a_1 + a_2 > a_3 + 1$ and the bound is established. ■

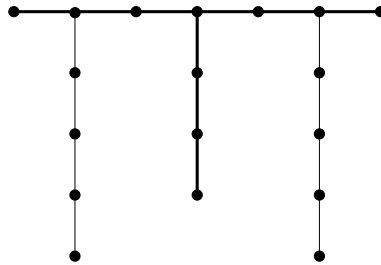


Figure 3. A tree T with $\varepsilon(T) = 4$.

The bound in Theorem 15 is an improvement on the bound in Theorem 9. For example, consider the tree T shown in Figure 3. We note that $\ell(T) = 5$ and that T does not contain any hanging spiders with at least three leaves, thus from Theorem 9 it follows that $\varepsilon(T) \leq 5$. On the other hand, for the embedded spider $S(3, 3, 3)$ indicated by the emphasized edges, $a_3 + 1 = 4$ which implies $\sigma(T) \leq 4$. Hence by Theorem 15, $\varepsilon(T) \leq 4$.

For the tree shown in Figure 3 the lower bound in Theorem 6 gives $\varepsilon(T) \geq 4$ and it was shown in [9] that this bound is tight for trees with no adjacent branch vertices. Hence, for this example, the bound from Theorem 15 is best possible.

Next we show that in general the bound in Theorem 15 gives the exact value of $\varepsilon(T)$ whenever $B(T)$ is independent.

Recall that for $v \in B(T)$ with $\deg v = r$, from a suitable arrangement e_1, \dots, e_r of the edges incident with v ,

$$\rho(v) = \min\{\ell_1(v) + \ell_2(v), \ell_3(v) + 1\}.$$

Theorem 16. *If $B(T)$ is independent, then $\varepsilon(T) = \min\{\ell(T), \sigma(T)\}$.*

Proof. If T is a path, then the result is obvious. We consider then only trees for which $B(T) \neq \emptyset$. To prove the result we show that if $B(T)$ is independent, then the lower bound in Theorem 6 is equivalent to the upper bound in Theorem 15, that is, $\min\{\ell(T), \sigma(T)\} = \min_{v \in B(T)}\{\rho(v)\}$. Since for any tree T , $\min_{v \in B(T)}\{\rho(v)\} \leq \varepsilon(T) \leq \min\{\ell(T), \sigma(T)\}$, it is enough to show that $\min\{\ell(T), \sigma(T)\} \leq \min_{v \in B(T)}\{\rho(v)\}$.

Let T be a tree with $B(T)$ independent, and v a vertex in $B(T)$ such that $\rho(v) = \min_{w \in B(T)}\{\rho(w)\} = k$. Necessarily, v is the branch vertex of an embedded spider of T , say $S(a_1, a_2, \dots, a_r)$ where $r \geq 3$. By definition $\rho(v) = \min\{a_1 + a_2, a_3 + 1\}$. Moreover, $\ell(T) \leq a_1 + a_2$, and $\sigma(T) \leq a_3 + 1$. Hence, $\min\{\ell(T), \sigma(T)\} \leq \rho(v)$ and the result follows. ■

4.4. Graphs whose line graph has diameter two

Recall that if $\text{diam}(\mathcal{L}(G)) = 2$, then $\varepsilon(G) \leq 3$. In this section we describe a sufficient condition for a vertex of a graph G with $\text{diam}(\mathcal{L}(G)) = 2$ to be a k -kernel of G for $k \in \{2, 3\}$.

We introduce the following notation which we utilize in this section. For a graph G and sets $A, B \subseteq V(G)$, define $E(A, B)$ as the set of all edges $ab \in E(G)$ such that $a \in A$ and $b \in B$.

Theorem 17. *Let G be a graph with $\text{diam}(\mathcal{L}(G)) = 2$. If v is a vertex such that $N[v]$ is a vertex cover of G , then v is a k -kernel of G , where $k \in \{2, 3\}$.*

Proof. Let $v \in V(G)$ be a vertex such that $N[v]$ is a vertex cover of G . It suffices to show that for any edge ordering f there exists a v -avoiding maximal f -ascent of length at most three. Suppose $|E(G)| = n$ and let $f : E(G) \rightarrow \{1, \dots, n\}$ be an edge ordering of G . Let uw and xy be the edges with $f(uw) = 1$ and $f(xy) = n$. Since $\text{diam}(\mathcal{L}(G)) = 2$, uw and xy lie on a common P_4 . If $v \in \{u, w\} \cap \{x, y\}$, say $v = w = y$, then uvx is a v -avoiding maximal f -ascent of length at most three. Similarly, if (say) $w = y$ and $v \notin \{u, w, x, y\}$, then uvw is a v -avoiding maximal ascent. If $\{u, w\} \cap \{x, y\} = \emptyset$ and $v \notin \{u, w, x, y\}$, then $E(\{u, w\}, \{x, y\}) \neq \emptyset$ since uw and xy lie on a common P_4 . Any $e \in E(\{u, w\}, \{x, y\})$ has label k with $1 < k < n$, thus one of $uwxy$, $uwyx$, $wuxy$ and $wuyx$ is a maximal v -avoiding ascent. We may therefore assume that $v \in \{u, w, x, y\}$ and $v \notin \{u, w\} \cap \{x, y\}$.

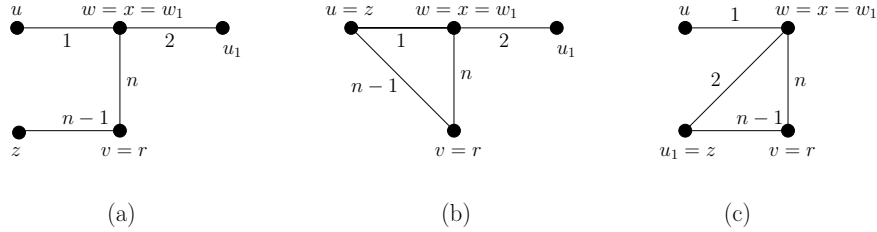


Figure 4. In each case the edge labelled 2 is incident with w .

Without loss of generality suppose $v = y$. We consider two cases.

Case 1. $\{u, w\} \cap \{x, v\} = \emptyset$. Since $N[v]$ is a vertex cover, v is joined to u or w with an edge labelled k , where $1 < k < n$. In the former case $wuvx$ is a maximal v -avoiding f -ascent, and in the latter case $uwvx$ is such an ascent.

Case 2. $\{u, w\} \cap \{x, v\} \neq \emptyset$. By our assumption $v \notin \{u, w\}$ and we may assume without loss of generality that $x = w$. Let $f(zr) = n - 1$ and suppose $v \notin \{z, r\}$. If $r \in \{u, w\}$, then uwz or wuz is a maximal v -avoiding ascent. Hence we may assume $r \notin \{u, w\}$ and similarly $z \notin \{u, w\}$. But zr and uw lie on a common P_4 , hence there exists an edge $e \in E(\{z, r\}, \{u, w\})$ and this edge has label k with $1 < k < n - 1$, thus forming a v -avoiding maximal $(3, f)$ -ascent. Therefore $v \in \{z, r\}$; say $v = r$. (Note that possibly $z = u$.)

Let $f(u_1, w_1) = 2$ and suppose $\{u_1, w_1\} \cap \{u, v, w, z\} = \emptyset$. Since $N[v]$ is a vertex cover, v is adjacent to u_1 or w_1 and this edge has label k with $2 < k < n - 1$. Since $u_1 \notin \{u, w\}$, u_1w_1 is not adjacent to an edge with a smaller label. Thus u_1w_1vw or w_1u_1vw is a v -avoiding maximal ascent. It follows that $\{u_1, w_1\} \cap \{u, v, w, z\} \neq \emptyset$. We show that the edge labelled 2 is incident with w .

Now suppose that $|\{u_1, w_1\} \cap \{u, v, w, z\}| = 1$ and without loss of generality $w_1 \in \{u, v, w, z\}$. If $w_1 = z$ then u_1zvw is a maximal ascent and if $w_1 = v$ then u_1vw is a maximal ascent. Suppose then that $w_1 = u$. Then u_1 is not incident with an edge with a smaller label. Since $N[v]$ is a vertex cover, v is joined to u_1 or u by an edge with label k , $2 < k < n$ (possibly $k = n - 1$ if $u = z$), so u_1uvw or uu_1vw is a v -avoiding maximal ascent. Hence $w_1 = w$ (see Figure 4(a) and 4(b)). If $|E(G)| = 4$, then zvw is a v -avoiding maximal f -ascent.

Suppose next that $|\{u_1, w_1\} \cap \{u, v, w, z\}| = 2$. If $u_1w_1 = zu$, then $z \neq u$ and $uzvw$ is a v -avoiding maximal f -ascent. If $u_1w_1 = vu$, then $z \neq u$ and uvw is such an ascent. Hence $u_1w_1 = zw$; say $u_1 = z$ and $w_1 = w$ and note that $z \neq u$. Thus we see that in each case the edge labelled 2 is incident with w (see Figure 4(c)). If $|E(G)| = 4$, then zvw is a v -avoiding maximal f -ascent.

Assume $|E(G)| \geq 5$ and let $f(u_2w_2) = 3$. Suppose $u_2w_2 = uu_1$. If $z \notin \{u, u_1\}$, then since $N[v]$ is a vertex cover, v is joined to u or u_1 by an edge with label k , $3 \leq k \leq n - 1$. Suppose $uv \in E(G)$ and consider the ascent u_1uvw . Since u_1

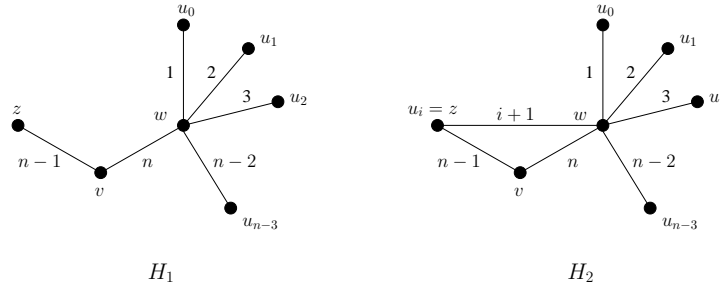


Figure 5. The ascent zvw in H_1 and H_2 is a v -avoiding maximal ascent.

is not incident with the edge labelled 1, and the addition of the edge u_1w with label 2 forms a 4-cycle, u_1uvw is a v -avoiding maximal f -ascent. Similarly, if $u_1v \in E(H)$, then wu_1vw is a v -avoiding maximal f -ascent. If $z = u$, u_1uvw is a v -avoiding maximal ascent, and if $z = u_1$, then wu_1vw is such an ascent. For all other possibilities similar arguments as for u_1w_1 show that $u_2w_2 = zw$ (if $f(zw) \notin \{1, 2\}$) or, without loss of generality, $w_2 = w$ and no edge incident with u_2 has label 1, 2, $n - 1, n$. Let $u_0 = u$. By repeating the above argument we see that each edge u_iw_i with $f(u_iw_i) = i + 1, i = 0, 1, \dots, n - 3$, is incident with w , say $w_i = w$, and possibly $u_i = z$ for one $i = 0, 1, \dots, n - 3$. Therefore, the graph H is either the graph H_1 or H_2 in Figure 5. But in either graph the ascent zvw is a v -avoiding maximal f -ascent and the proof of Case 2 is complete. ■

Corollary 18. *Let G be a graph with $\text{diam}(\mathcal{L}(G)) = 2$ and $\varepsilon(G) = 3$. If v is a vertex such that $N[v]$ is a vertex cover of G , then v is an ε -kernel of G .*

To illustrate the above corollary, note that for $n \geq 4, \text{diam}(\mathcal{L}(K_n)) = 2, \varepsilon(K_n) = 3$, and for any vertex $v \in V(K_n), N[v]$ is a vertex cover of K_n . Therefore, by Corollary 18 we see that for any $v \in V(K_n), v$ is an ε -kernel of $K_n, n \geq 4$.

Theorem 2 and Theorem 17 allow us to identify a large class of graphs with depression at most three. We state this result in the following corollary.

Corollary 19. *Let G be a graph with an end-block B such that $\text{diam}(\mathcal{L}(B)) \leq 2$, and v the cut vertex of G contained in B . If $N[v]$ is a vertex cover of B , then $\varepsilon(G) \leq 3$.*

Next we show that the converse of Theorem 17 is false. As a counterexample consider the vertex v of the graph G shown in Figure 6. Clearly, $\text{diam}(\mathcal{L}(G)) = 2$ and $N[v]$ is not a vertex cover of G . In order to show that v is a k -kernel where $k \in \{2, 3\}$ we must show that for every edge ordering f of G there exists a v -avoiding maximal f -ascent of length at most three.

Suppose to the contrary that $f : E(G) \rightarrow \{1, 2, \dots, 8\}$ is an edge ordering of G for which there does not exist a v -avoiding maximal f -ascent of length at

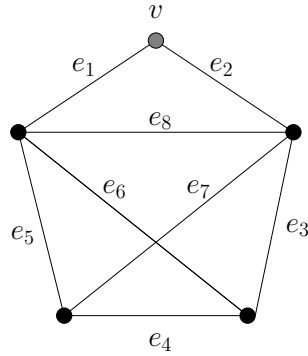


Figure 6. A graph G with $\text{diam}(\mathcal{L}(G)) = 2$ and a vertex v such that $N[v]$ is not a vertex cover of G .

most three. If $\{f^{-1}(1), f^{-1}(8)\} \subseteq \{e_3, e_4, e_5, e_6, e_7, e_8\}$, then there clearly exists a v -avoiding maximal ascent of length at most three. Thus for some $e \in \{e_1, e_2\}$, $f(e) \in \{1, 8\}$, say $f(e_1) = 1$. Then e_1e_2 is contained in a v -avoiding maximal f -ascent λ , and by our assumption λ has length four. Thus either $\lambda = e_1e_2e_3e_4$ or $\lambda = e_1e_2e_7e_4$ and without loss of generality we assume the former. To complete the proof we consider the following cases.

Let $k = \min(\{f(e_3), f(e_5), f(e_6), f(e_7)\})$. If $f(e_3) = k$, then either e_3e_7 or $e_3e_7e_5$ is a v -avoiding maximal f -ascent. Similarly, if $f(e_7) = k$, then e_7e_3 or $e_7e_3e_6$ is a v -avoiding maximal f -ascent, and if $f(e_5) = k$, then e_5e_6 or $e_5e_6e_3$ is a v -avoiding maximal f -ascent.

We assume then that $f(e_6) = \min(\{f(e_3), f(e_5), f(e_6), f(e_7)\})$. Then e_6e_5 are the first two edges of a maximal ascent, and since G does not contain a v -avoiding maximal f -ascent of length at most three, it follows that $f(e_5) < f(e_7) < f(e_2)$. Now if $f(e_8) > f(e_2)$, then e_6e_8 is a v -avoiding maximal f -ascent. Assume then that $f(e_8) < f(e_2)$. Since $e_1e_2e_3e_4$ is an f -ascent, it follows that $f(e_8) < f(e_3)$. If $f(e_8) < f(e_5)$, then $e_8e_5e_4$ is a v -avoiding maximal f -ascent. Finally, if $f(e_8) > f(e_5)$, then $e_5e_8e_3$ is a v -avoiding maximal f -ascent.

This covers all cases and establishes the proof of the counterexample to the converse of Theorem 17.

5. OPEN PROBLEMS

1. In Section 4.3 we identified ε -kernels for the class of trees known as spiders and this result was used to determine an upper bound for the depression of trees. A *double spider* is a tree with exactly two branch vertices, these two vertices being adjacent. It may be possible to improve on the upper bound for trees by identifying ε -kernels for other classes of trees such as double spiders.
2. Let G be a graph with $\text{diam}(\mathcal{L}(G)) = 2$. Theorem 17 identifies a sufficient

condition for a vertex $v \in V(G)$ to be a k -kernel of G where $k \in \{2, 3\}$. Determine a necessary condition for such a vertex.

3. Obtain a similar result to Theorem 17 for graphs with $\text{diam}(\mathcal{L}(G)) \geq 3$.

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