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THE RAMSEY NUMBER FOR THETA GRAPH VERSUS A CLIQUE OF ORDER THREE AND FOUR

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Abstract

For any two graphs F_1 and F_2, the graph Ramsey number $r(F_1, F_2)$ is the smallest positive integer N with the property that every graph on at least N vertices contains F_1 or its complement contains F_2 as a subgraph. In this paper, we consider the Ramsey numbers for theta-complete graphs. We determine $r(\theta_n, K_m)$ for $m = 2, 3, 4$ and $n > m$. More specifically, we establish that $r(\theta_n, K_m) = (n-1)(m-1)+1$ for $m = 3, 4$ and $n > m$.

Keywords: Ramsey number, independent set, theta graph, complete graph.

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1. Introduction

Graphs considered in this paper are finite, undirected and have no loops or multiple edges. For a given graph G, we use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ to denote the vertex set, edge set, maximum degree and minimum degree of G, respectively. An independent set of vertices of a graph G is a subset of $V(G)$ in which no two vertices are adjacent. The independence number of a graph $G, \alpha(G)$, is the size of the largest independent set. The degree of a vertex u in G, denoted by $d_G(u)$, is the number of edges of G incident with u. The neighbor of a vertex $u \in V(G)$, denoted by $N(u)$, is the set of all vertices that are adjacent to u. For a nonempty set $V_1 \subseteq V(G)$, the induced subgraph of G induced by V_1, denoted by $\langle V_1 \rangle$, is the subgraph of G with vertex set V_1 and those edges of G that have both ends in V_1. A theta graph θ_n on n vertices is a cycle C_n with a new edge joining two non-adjacent vertices of C_n.

The graph Ramsey number $r(F_1, F_2)$ is the smallest positive integer N with the property that every graph on at least N vertices contains F_1 or its complement contains F_2 as a subgraph. It is well known that the problem of determining the Ramsey numbers for complete graphs is very difficult and it is easier to deal with sparse graphs instead of complete graphs.

Ramsey numbers for theta graphs were investigated by Jaradat *et al.* [5], in fact they determined $r(\theta_4, \theta_k), r(\theta_5, \theta_k)$ for $k \geq 4$. More specifically, they established that $r(\theta_4, \theta_k) = r(\theta_5, \theta_k) = 2k - 1$ for $k \geq 5$. Furthermore, they determined $r(\theta_n, \theta_n)$ by proving that for $n \geq 5, R(\theta_n, \theta_n) = (3n/2) - 1$ if n is even and $2n - 1$ if n is odd. The Ramsey number of theta graphs versus complete graphs dropping an edge and also theta-complete graph were studied by several authors. Chvátal and Harary [1], proved that $r(\theta_4, K_4) = 11$. Bolze and Harborth [2] and Faudree *et al.* [4] showed that $r(\theta_4, K_5) = 16$ and $r(\theta_4, K_5 - e) = 13$, respectively. McNamara [6] proved that $r(\theta_4, K_6) = 21$ and McNamara and Radziszowski [7] gave the following two results: $r(\theta_4, K_6 - e) = 17$ and $r(\theta_4, K_7 - e) = 28$. An upper bound for $r(\theta_4, K_7)$ and the exact number for $r(\theta_4, K_8)$ were established by Boza [3], in fact, he proved that $r(\theta_4, K_7) \leq 31$ and $r(\theta_4, K_8) = 42$. For more results concerning Ramsey numbers of graphs, we refer the reader to the updated bibliography by Radziszowski [8].

In this paper, we continue studying the theta-complete Ramsey number by extending the above special results to a more general results.

2. Main Results

In this section, we determine the Ramsey number of theta graphs versus complete graphs of order 3 and 4. By taking $G = (m - 1)K_{n-1}$, one can notice that G contains neither θ_n nor m-element independent set. Thus, we establish

$r(\theta_n, K_m) \geq (n-1)(m-1)+1$. This lower bound is certainly the case for all results of this paper, therefore we shall always prove just the claimed upper bounds for the Ramsey numbers. The following result is a straightforward from the above inequality and the fact that if G is a graph with n vertices and $\alpha(G) = 1$, then $G = K_n$.

Theorem 1. *For all* $n \geq 2, r(\theta_n, K_2) = n$.

Theorem 2. *For all* $n \geq 4, r(\theta_n, K_3) = 2n-1$.

Proof. It is sufficient to prove that for $n \geq 4, r(\theta_n, K_3) \leq 2n-1$. We prove it by induction on n. Let $n = 4$ and G be a graph with order 7 that contains neither θ_4 nor 3-element independent set. Since $r(C_3, K_3) = 6$, G contains a cycle C of length 3, say $C = v_1v_2v_3v_1$. Since $r(\theta_4, K_2) = 4$ and $|G-C| = 4$, $G-C$ contains 2-element independent set $X = \{x_1, x_2\}$. Since G has no 3-element independent set, each vertex of C is adjacent to at least one vertex in X. Moreover, no vertex in X is adjacent to two vertices of the cycle, otherwise θ_4 is produced. Let $x_1v_1, x_2v_2 \in E(G)$. Then $\{x_1, x_2, v_3\}$ is an independent set, a contradiction.

Now, assume that G is a graph of order $2n-1$ that contains neither θ_n nor a 3-element independent set. Since $r(\theta_{n-1}, K_3) = 2n-3$ by induction, G contains θ_{n-1}, say $\theta_{n-1} = v_1v_2\cdots v_{n-1}v_1v_j$ for some $3 \leq j \leq n-2$, and since $r(\theta_n, K_2) = n$ and $|G-\theta_{n-1}| = n$, $G-\theta_{n-1}$ contains 2-element independent set $X = \{x_1, x_2\}$. Since G has no 3-element independent set, each vertex of θ_{n-1} is adjacent to at least one vertex of X. No vertex in X is adjacent to two consecutive vertices of θ_{n-1}, since otherwise θ_n is produced. Suppose x_1 is adjacent to v_1. Then x_1 cannot be adjacent to v_2 and so v_2 must be adjacent to x_2, otherwise $\{x_1, x_2, v_2\}$ is a 3-element independent set. x_2 cannot be adjacent to v_3 so v_3 must be adjacent to x_1, otherwise $\{x_1, x_2, v_3\}$ is 3-element independent set. x_1 cannot be adjacent to v_4 so x_2 is adjacent to v_4 otherwise $\{x_1, x_2, v_4\}$ is a 3-element independent set. Moreover v_1 must be adjacent to v_3, also v_2 must be adjacent to v_4, otherwise $\{v_1, v_3, x_2\}$ or $\{v_2, v_4, x_1\}$ is a 3-element independent. To this end, one can note that $v_3x_1v_1v_{n-1}\cdots v_5v_4v_2v_3v_1$ forms θ_n (see Figure 1). This is a contradiction. This observation complete the proof. ∎

The following theorem will be used in our coming result:

Theorem 3 (Chvátal and Harary[1])**.** $r(\theta_4, K_4) = 11$.

Theorem 4. *For all* $n \geq 5, r(\theta_n, K_4) = 3n-2$.

Proof. It is sufficient to prove that every graph of order $3n-2$ contains either θ_n or a 4-element independent set. We prove it by induction on n. For $n = 5$, suppose G is a graph of order 13 that contains neither θ_5 nor a 4-element independent set. Our aim is to show that G is a 5-regular graph. But this contradicts the fact that

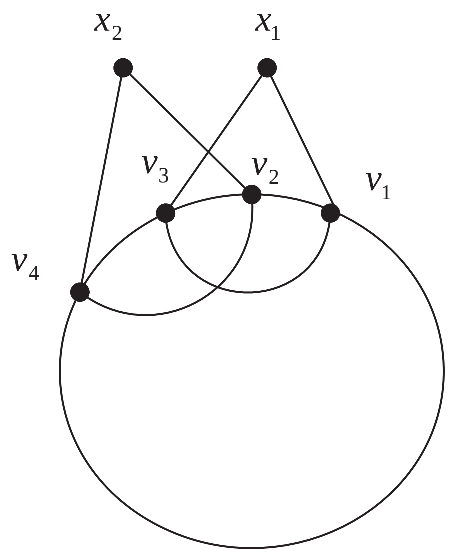

Figure 1. Depicts the situation in Theorem 2.2.

in any graph there is an even number of vertices of odd degree. Hence, G must contain either θ_5 or 4-element independent set, which implies that $r(\theta_5, K_4) = 13$. We accomplish that throughout proving four claims according to the possible degrees of G.

Claim 1. *G contains no vertex of degree at least* 7.

Proof. Suppose that G has a vertex u of degree at least 7. Let $\{v_1, v_2, \ldots, v_7\} \subseteq N(u)$ and $H = \langle \{v_1, v_2, \ldots, v_7\} \rangle$. H contains neither P_4 nor a 4-element independent set, and so $H = 2C_3 \cup K_1$ or $H = C_3 \cup 2K_2$. Let $S = \{v_8, v_9, v_{10}, v_{11}, v_{12}\}$ be theset of remaining vertices of G. We now consider the following two cases of H.

Case 1. $H = 2C_3 \cup K_1$. Let $v_1v_2v_3v_1$ and $v_4v_5v_6v_4$ be cycles of H and $K_1 = v_7$ which is shown in Figure 2. Observe that any vertex of S cannot be adjacent to two vertices of $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ as otherwise θ_5 is produced. Now, we consider two subcases:

Subcase 1.1. There is a vertex of S, say v_8, adjacent to one vertex of $\{v_1, v_2, v_3, v_4, v_5, v_6\}$, say v_1. Then v_8 is not adjacent to any vertex of $\{v_2, v_3, v_4, v_5, v_6, v_7\}$. Hence $\{v_8, v_2, v_6, v_7\}$ is a 4-element independent set. This is a contradiction.

Subcase 1.2. No vertex of S is adjacent to any vertex of $\{v_1, v_2, v_3, v_4, v_5, v_6\}$.

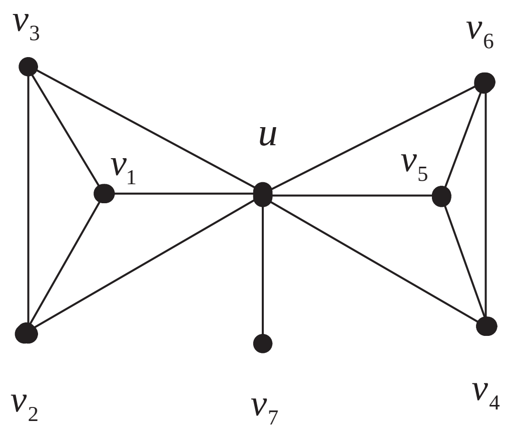

Figure 2. Represents $\langle V(H) \cup \{u\}\rangle$ in the Case 1 of Claim 1.

Thus, each vertex of S is adjacent to v_7 (as otherwise, if one vertex of S is not adjacent to v_7, say v_8, then $\{v_1, v_4, v_7, v_8\}$ is a 4-element independent set). Moreover, $\langle S\rangle = K_5$ (as otherwise, two non-adjacent vertices of $\langle S\rangle$ with a vertex of each of $v_1v_2v_3v_1$ and $v_4v_5v_6v_4$ form a 4-element independent set). But $\theta_5 \subset K_5$ and so θ_5 is a subgraph of G. This is a contradiction.

Case 2. $H = C_3 \cup 2K_2$. Let $v_1v_2v_3v_1, v_4v_5$ and v_6v_7 be the cycle and the two edges of H, respectively. Note that any vertex of S cannot be adjacent to two vertices of the cycle or adjacent to a vertex of the cycle and a vertex of $\{v_4, v_5, v_6, v_7\}$ or a vertex of $\{v_4, v_5\}$ and a vertex of $\{v_6, v_7\}$ as otherwise θ_5 is produced. Now, if a vertex of S, say v_8, is adjacent to a vertex of $\{v_1, v_2, v_3\}$, say v_1, then $\{v_2, v_4, v_6, v_8\}$ is a 4-element independent set, a contradiction. Similarly, if no vertex of S is adjacent to a vertex of $\{v_1, v_2, v_3\}$ but there is a vertex adjacent to at most one vertex of $\{v_4, v_5, v_6, v_7\}$, then a 4-element independent set is produced. Thus, every vertex of S is adjacent either to both v_4 and v_5 or to both v_6 and v_7. Since $|S| = 5$, without loss of generality we may assume that v_8 and v_9 are adjacent to both v_4 and v_5. To this end, if $v_8v_9 \in E(G)$, then $v_4v_8v_9v_5uv_4v_5$ is a θ_5, a contradiction. Thus, $v_8v_9 \notin E(G)$, which implies that $\{v_1, v_6, v_8, v_9\}$ is a 4-element independent set. This is a contradiction. □

Claim 2. *G contains no vertex of degree* 6.

Proof. Suppose that G has a vertex u of degree 6. Let $N(u) = \{v_1, v_2, \ldots, v_6\}$ and $H = \langle N(u)\rangle$. Since H contains neither P_4 nor a 4-element independent set, $H = 2C_3$ or $H = C_3 \cup P_3$ or $H = C_3 \cup K_2 \cup K_1$ or $H = 3K_2$. As above we consider four cases of H.

Case 1. $H = 2C_3$. Let $v_1v_2v_3v_1$ and $v_4v_5v_6v_4$ be the two cycles of H and $S = \{v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$ be the set of remaining vertices in G. Observe that any vertex of S is not adjacent to two vertices of H as otherwise θ_5 is produced. Note that there are at least two non adjacent vertices of S, say v_7 and v_8 (as otherwise $\langle S\rangle = K_6$ and so θ_5 is produced). Since vertex v_7 (also v_8) is adjacent to one vertex of $N(u)$, say v (also v'), then by choosing vertices $w \in \{v_1, v_2, v_3\} - \{v\ , v'\}$ and $w' \in \{v_4, v_5, v_6\} - \{v\ , v'\}$ we obtain the 4-element independent set $\{w, w'v,\ _7, v_8\}$, which is impossible. Figure 3 depicts the situations.

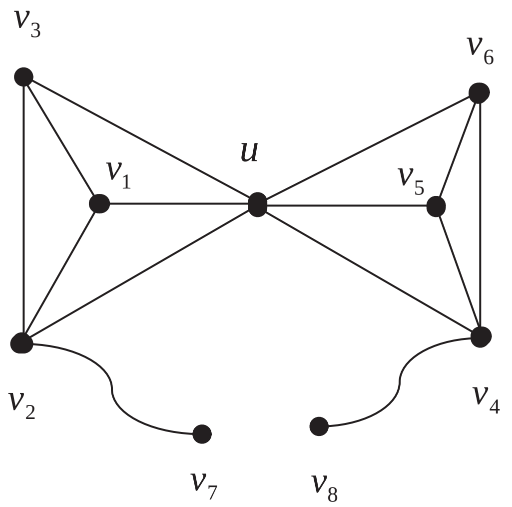

Figure 3. Represents the situation in Case 1 of Claim 2
(the case $v = v_2$ and $v^{'} = v_4$.)

Case 2. $H = C_3 \cup P_3$. The proof of this case follows by the same lines as of the proof of Case 1.

Case 3. $H = C_3 \cup K_2 \cup K_1$. Let $C_3 = v_1v_2v_3v_1$, $K_2 = v_5v_6$ and $K_1 = v_4$. Let $S = \{v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$ be the set of remaining vertices. Note that at least one vertex of S is not adjacent to v_4 as otherwise $d(v_4) = 7$. Without loss of generality, we may assume that v_7 is not adjacent to v_4. As in Case 1, every vertex of S is adjacent to at most one vertex of C_3. Thus, v_7 is adjacent to both vertices of v_5 and v_6 (as otherwise a non-adjacent vertex to v_7 on C_3 and a non-adjacent vertex to v_7 from $\{v_5, v_6\}$ with $\{v_4, v_7\}$ form a 4-element independent set, a contradiction). Now, if there is another vertex of $S - \{v_7\}$, say v_8, that is not adjacent to v_4, then v_8 is adjacent to both of v_5 and v_6 and so $v_7v_8 \notin E(G)$ (as otherwise $v_5uv_6v_7v_8v_5v_6$ is θ_5, a contradiction). Therefore, by choosing a non adjacent vertex to any of v_7 and v_8 from the cycle C_3 with $\{v_4, v_7, v_8\}$ we form a

4-element independent set, this is a contradiction. Now, if every vertex of $S-\{v_7\}$ is adjacent to v_4, then $S-\{v_7\}$ contains at least two non-adjacent vertices, say v_8 and v_9 (otherwise $\langle S-\{v_7\}\rangle = K_5$ which contains θ_5, a contradiction). Note that neither v_8 nor v_9 is adjacent to any vertex of v_5 and v_6 (to see that, without loss of generality, we may assume that v_8 is adjacent to v_5, then $uv_6v_5v_8v_4uv_5$ is θ_5, this is a contradiction). Thus, by choosing a non-adjacent vertex to any of v_8 and v_9 from the cycle C_3 with $\{v_5, v_8, v_9\}$ we form a 4-element independent set. This is a contradiction.

Case 4. $H = 3K_2$. Let $N(u) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and v_1v_2, v_3v_4 and v_5v_6 be the edges of H. Let $S = \{v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$ be the set of remaining vertices. One can notice that no vertex of S is adjacent to two vertices of two different edges of H. Moreover, if there is a vertex of S, say v_7, which is adjacent to only one vertex of $N(u)$, say v_1, then $\{v_2, v_3, v_5, v_7\}$ is a 4-element independent set. Thus, each vertex of S must be adjacent to the vertices of exactly one edge of H. Since $|S| = 6$, there are at least two vertices of S, say v_7 and v_8, that are adjacent to the vertices of the same edge of H, say v_1 and v_2. If $v_7v_8 \in E(G)$, then as in Case 3 we see that θ_5 is produced. If $v_7v_8 \notin E(G)$, then as in Case 3 a 4-element independent set is obtained. This is a contradiction. □

Claim 3. *G contains no vertex of degree* 4.

Proof. Suppose that G has a vertex u of degree 4. Let $N(u) = \{v_9, v_{10}, v_{11}, v_{12}\}$ and $S = \{v_1, v_2, \ldots, v_8\}$ be the set of remaining vertices. Since $r(C_4, K_3) = 7$, $\langle S\rangle$ must contain a cycle of length 4, otherwise $\alpha(\langle S\rangle) \geq 3$, and so three independent vertices of $\langle S\rangle$ with u form a 4-element independent set. Let the cycle be $v_1v_2v_3v_4v_1$. Note that any vertex of $\{v_5, v_6, v_7, v_8\}$ cannot be adjacent to two consecutive vertices of the cycle $v_1v_2v_3v_4v_1$ since otherwise θ_5 is produced. Now, fix a vertex of $\{v_5, v_6, v_7, v_8\}$, say v_5. According to the adjacency of v_5 we consider three cases:

Case 1. v_5 is adjacent to two nonconsecutive vertices of the cycle, say v_1 and v_3. If $v_2v_4 \in E(G)$, then $v_1v_5v_3v_4v_2v_1v_4$ is θ_5. A contradiction. If $v_2v_4 \notin E(G)$, then $\{v_2, v_4, v_5, u\}$ is a 4-element independent set, a contradiction.

Case 2. v_5 is adjacent to exactly one vertex of the cycle, say v_1. If $v_2v_4 \notin E(G)$, then $\{v_2, v_4, v_5, u\}$ is a 4-element independent set, a contradiction. If $v_2v_4 \in E(G)$, then $\{v_6, v_7, v_8\}$ must contain a vertex that is not adjacent to v_4, say v_6, otherwise $d_G(v_4) \geq 6$. Note that v_6 must be adjacent to v_5, otherwise $\{v_4, v_5, v_6, u\}$ is a 4-element independent set, a contradiction. Now, we consider the following subcases:

Subcase 2.1. At least one of v_7 and v_8 is adjacent to v_4, say v_7. Then v_7 cannot be adjacent to v_5 since otherwise $\theta_5 = v_1v_2v_4v_7v_5v_1v_4$ is produced. Also,

v_7 cannot be adjacent to v_3, otherwise $\theta_5 = v_3v_2v_1v_4v_7v_3v_4$ is produced. Hence, $\{v_3, v_5, v_7, u\}$ is a 4-element independent set. This is a contradiction.

Subcase 2.2. Non of v_7 and v_8 is adjacent to v_4. Then v_7 is adjacent to v_5, otherwise $\{v_4, v_5, v_7, u\}$ is a 4-element independent set. By the symmetry, v_8 is adjacent to v_5. Similarly, each of v_7 and v_8 is adjacent to v_6 (otherwise, $\{v_4, v_6, v_7, u\}$ is a 4-element independent set if v_7 is not adjacent to v_6 and $\{v_4, v_6, v_8, u\}$ is a 4-element independent set if v_8 is not adjacent to v_6). Finally, $v_7v_8 \in E(G)$ (otherwise, $\{v_4, v_7, v_8, u\}$ is a 4-element independent set). One can easily check that non of v_6, v_7, v_8 is adjacent to v_2, otherwise θ_5 is produced. To this end, if $v_1v_3 \notin E(G)$, then each vertex $v \in \{v_6, v_7, v_8\}$ is adjacent to v_3, otherwise $\{v, v_1, v_3, u\}$ is a 4-element independent set, a contradiction. But in this case $\theta_5 = v_8v_5v_6v_7v_3v_8v_7$ is produced (see Figure 4). This a contradiction.

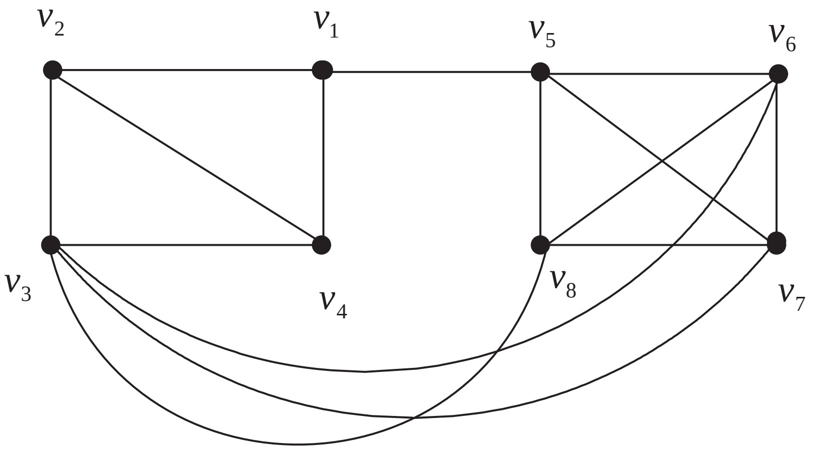

Figure 4. Depicts the situation in Subcase 2.2 of Claim 3 in case $v_1v_3 \notin E(G)$.

Now we need to consider the case $v_1v_3 \in E(G)$. Let $v_9, v_{10} \in N(u)$ such that $v_9v_{10} \notin E(G)$. Then each of v_9, v_{10} is adjacent to at most one vertex of each of $\{v_1, v_2, v_3, v_4\}$ and $\{v_5, v_6, v_7, v_8\}$, since otherwise θ_5 is produced. Hence, there are vertices $v \in \{v_1, v_2, v_3, v_4\}$ and $v' \in \{v_5, v_6, v_7, v_8\}$ which are adjacent to neither v_9 nor v_{10} with $\{v_9, v_{10}\}$. Thus,$\{v_9, v_{10}, v_9, v_{10}\}$ is a 4-element independent set (see Figure 5). This is a contradiction.

Case 3. v_5 is adjacent to no vertex of the cycle. Then by using the last argument from Subcase 2.2 we get the same contradiction. □

Claim 4. *G contains no vertex of degree less than or equal to* 3.

Proof. Suppose that G has a vertex u of degree less than or equal to 3. Then

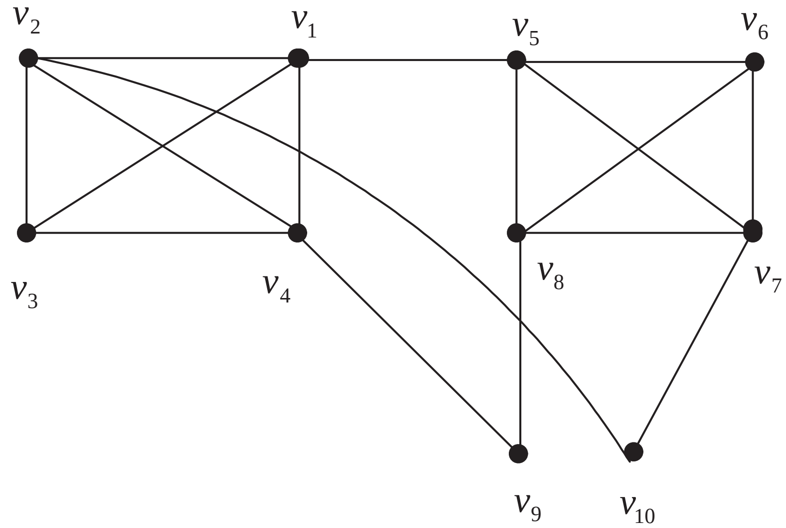

Figure 5. Depicts the situation in Subcase 2.2 of Claim 3 in case $v_1v_3 \in E(G)$.

there is a set S of nine vertices in G which are distinct from u and not adjacent to u. The subgraph $\langle S\rangle$ of G contains no θ_5. By Theorem 2.2, $\langle S\rangle$ contains a 3-element independent set, say v_1, v_2, v_3. Hence $\{v_1, v_2, v_3, u\ \}$ is a 4-element independent set. This is a contradiction. This observation completes the proof of the claim. □

Now, by inductive hypothesis, $r(\theta_{n-1}, K_4) = 3n - 5$ for $n > 5$. Suppose that G is a graph of order $3n - 2$ that contains neither θ_n nor a 4-element independent set. Since, $r(\theta_{n-1}, K_4) = 3n - 5$, G contains θ_{n-1} as a subgraph, say $\theta_{n-1} : v_1v_2 \cdots v_{n-1}v_1v_m$ for some $3 \leq m \leq n - 2$ of length $n - 1$. Also, using $r(\theta_n, K_3) = 2n - 1$ and $|G - \theta_{n-1}| = 2n - 1$, we get that $G - \theta_{n-1}$ contains 3-element independent set $X = \{x_1, x_2, x_3\}$. Since G has no 4-element independent set, each vertex on θ_{n-1} is adjacent to at least one vertex of X. No vertex in X is adjacent to two consecutive vertices of θ_{n-1}, since otherwise θ_n is produced. Moreover, if $x \in X$ is adjacent to v_i and v_j, then $v_{i+1}v_{j+1} \notin E(G)$, as otherwise $v_ixv_jv_{j-1} \cdots v_{i+1}v_{j+1} \cdots v_{i-1}v_iv_{i+1}$ forms a theta graph of order n.

Claim 5. *No vertex of X is adjacent to more than two vertices of θ_{n-1}.*

Proof. Suppose there is a vertex $x \in X$ such that x is adjacent to v_i, v_j and v_k. Then $v_{i+1}v_{j+1} \notin E(G), v_{i+1}v_{k+1} \notin E(G)$ and $v_{j+1}v_{k+1} \notin E(G)$. Moreover, x cannot be adjacent to any vertex of $\{v_{i+1}, v_{j+1}, v_{k+1}\}$ which implies that

$\{x, v_{i+1}, v_{j+1}, v_{k+1}\}$ is a 4-element independent set. The proof of the claim is complete. □

Now, since $n - 1 > 4$, at least one vertex of X is adjacent to two vertices of θ_{n-1}, we may assume that x_1 is adjacent to v_i and v_j only, thus $x_1v_{i+1} \notin E(G)$ and $x_1v_{j+1} \notin E(G)$. Since v_{j+2} is adjacent to some vertex of X, we may assume that $x_2v_{j+2} \in E(G)$, it is clear that x_2 cannot be adjacent to v_{i+1}, since otherwise $v_ix_1v_jv_{j-1} \cdots v_{i+1}x_2v_{j+2}v_{j+3} \cdots v_{i-1}v_iv_{i+1}$ forms a theta graph of order n. Moreover, x_2 cannot be adjacent to v_{j+1}, thus $\{x_1, x_2, v_{i+1}, v_{j+1}\}$ is a 4-element independent set, a contradiction. The contradiction completes the proof of the theorem. ■

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