

## MOTION PLANNING IN CARTESIAN PRODUCT GRAPHS

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### Abstract

Let  $G$  be an undirected graph with  $n$  vertices. Assume that a robot is placed on a vertex and  $n - 2$  obstacles are placed on the other vertices. A vertex on which neither a robot nor an obstacle is placed is said to have a hole. Consider a single player game in which a robot or obstacle can be moved to adjacent vertex if it has a hole. The objective is to take the robot to a fixed destination vertex using minimum number of moves. In general, it is not necessary that the robot will take a shortest path between the source and destination vertices in graph  $G$ . In this article we show that the path traced by the robot coincides with a shortest path in case of Cartesian product graphs. We give the minimum number of moves required for the motion planning problem in Cartesian product of two graphs having girth 6 or more. A result that we prove in the context of Cartesian product of  $P_n$  with itself has been used earlier to develop an approximation algorithm for  $(n^2 - 1)$ -puzzle.

**Keywords:** robot motion in a graph, Cartesian product of graphs.

**2010 Mathematics Subject Classification:** 05C85, 05C75, 68R10, 91A43.

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<sup>1</sup>This research was done while the author was a Ph.D. student at the Department of Mathematics, Indian Institute of Technology Guwahati, India. The author thanks Sikkim Manipal Institute of Technology and the All India Council for Technical Education (AICTE) to give paid-leave and provide financial support, respectively, for this study.

## 1. INTRODUCTION

Consider the following scenario: Given a graph,  $G$ , with a robot placed at one of its vertex (labeled as  $s$ ) and obstacles at some of the other vertices. A vertex at which the robot or an obstacle is not placed is said to have a hole or equivalently empty. Assume that we are allowed to slide an obstacle or the robot to an adjacent vertex if it is empty. The objective is to find a minimum sequence of moves that take the robot from  $s$  to a target vertex  $t$ . This problem of motion planning on a graph was introduced in [10]. It is a simple abstraction of the robot motion planning problem, where the geometry is replaced by the adjacency relation in the graph. The problem and its variations has also been studied in [1, 2, 8, 9, 11, 12, 14]. In [10], Papadimitriou *et al.* have shown that the decision version of the problem with arbitrary number of holes is NP-complete. The problem remains in the same complexity class even when restricted to planar graphs.

In this article we investigate the problem of motion planning with a single hole. In particular, we study this problem for a special class of graphs, the Cartesian product of two given graphs. In general, a minimal set of moves that take the robot to a fixed destination vertex will not take the robot through a shortest path between the source and destination vertices in the given graph. However, we show that this motion planning problem can be solved efficiently when restricted to Cartesian product graphs. We give the minimum number of moves required for the motion planning problem in Cartesian product of two graphs having girth 6 or more. A result that we prove in the context of Cartesian product of  $P_n$  with itself has been used earlier to develop an approximation algorithm for  $(n^2 - 1)$ -puzzle.

The rest of the paper is organized as follows. In next section we describe the definitions and basic results. In Sections 3 and 4, we investigate the properties with regard to local moves of the hole and trace of the robot, respectively, in Cartesian product of two graphs. Then we use these results to find minimum number of required moves in Section 5. Finally, conclusion and future work are discussed in Section 6.

## 2. BACKGROUND

Let  $G$  be a graph with vertex set  $V(G)$  and edge set  $E(G)$ . We refer to  $|V(G)|$  and  $|E(G)|$  as the order and the size of  $G$ , respectively. A graph  $G$  is called non-trivial if  $|V(G)| > 1$ . The study in this article is restricted to simple finite non-trivial graphs. For two vertices  $u, v \in V(G)$ , let  $d_G(u, v)$  denotes the distance between  $u$  and  $v$  in  $G$ . When there is no confusion about the underlying graph  $G$ , we use  $d(u, v)$  instead of  $d_G(u, v)$  to represent the distance between the vertices  $u$  and  $v$  in  $G$ . We denote the path, the cycle and the complete graph on  $n$  vertices

by  $P_n, C_n$  and  $K_n$ , respectively. The number of edges in a path is called its length. The *girth* of a graph  $G$ , denoted by  $g(G)$ , is the length of a shortest cycle contained in the graph  $G$ . For any graph  $G$  and  $u, v \in V(G)$ , we use the notation  $C_v^u$  to denote the configuration of the graph  $G$  with the robot at  $u$ , the hole at  $v$  and obstacles in the remaining vertices. Also if  $u$  and  $v$  are adjacent in  $G$ , we use  $v \xleftarrow{r} u$  and  $v \xleftarrow{o} u$  to denote respectively, the robot move and the obstacle move from the vertex  $u$  to the adjacent vertex  $v$  with a hole. We refer to the book [5] for the standard terms used in this article.

Given two graphs  $G$  and  $H$ , there are several ways to construct their product having vertex set as the Cartesian product of the vertex sets of  $G$  and  $H$ . Many of these graphs arise naturally in various contexts such as direct, Cartesian, strong and the lexicographic products. These product graphs are widely used in network design [3] and other fields. For details the reader may refer to monograph [6].

**Definition** Cartesian Product. The Cartesian product  $G \square H$  of two graphs  $G$  and  $H$  is a graph with vertex set  $V(G) \times V(H)$  in which  $(u_i, v_j)$  and  $(u_p, v_q)$  are adjacent if one of the following conditions holds:

- (i)  $u_i = u_p$  and  $\{v_j, v_q\} \in E(H)$ ,
- (ii)  $v_j = v_q$  and  $\{u_i, u_p\} \in E(G)$ .

The graphs  $G$  and  $H$  are known as the factors of  $G \square H$ . Cartesian product has been widely investigated and is arguably the most natural one among all products. It is associative, commutative and distributes over disjoint union. Also the graph  $G \square H$  is connected if and only if both  $G$  and  $H$  are connected [4]. Now onwards,  $G$  and  $H$  are connected finite simple graphs with  $V(G) = \{1, 2, \dots, m\}$  unless otherwise stated.

Suppose we are dealing with  $r$ -copies of a graph  $G$  and we are denoting these  $r$ -copies of  $G$  by  $G^i$ , where  $i = 1, 2, \dots, r$ . Then, for each vertex  $u \in V(G)$  we denote the corresponding vertex in the  $i^{\text{th}}$  copy  $G^i$  by  $u^i$ .

**Example 1.** Consider the Cartesian product of  $P_2$  with  $C_4$ . The graph  $P_2 \square C_4$  is shown in Figure 1. For clarity,  $P_2$  and  $C_4$  are displayed above and to the right of the product graph, respectively. The dotted lines indicate the edges corresponding to graph  $P_2$  in the product graph.

**Remark 2.** One well known fact about Cartesian product is that, we may also view  $G \square H$  as the graph obtained from  $G$  by replacing each vertex  $i \in V(G)$  by a copy  $H^i$  (say) of  $H$  and each of its edges  $\{i, j\}$  with  $|V(H)|$  edges joining corresponding vertices of  $H^i$  and  $H^j$ .

In view of Remark 2, for any vertex  $i \in V(G)$  we refer the copy of  $H$ , denoted by  $H^i$ , in  $G \square H$  corresponding to the vertex  $i$  as the  $i^{\text{th}}$  copy of  $H$  in  $G \square H$ . Also for

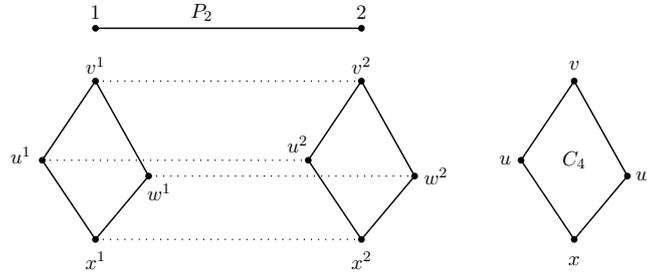


Figure 1.  $P_2 \square C_4$ .

any  $i \in V(G)$ , we use  $u^i$  to denote the vertex in  $H^i$  corresponding to the vertex  $u \in V(H)$ . Also, commutativity of the Cartesian product allows us to view  $G \square H$  as the graph obtained from  $H$  by replacing each of its vertices by a copy of  $G$  and each of its edges  $\{u, v\}$  with  $|V(G)|$  edges joining corresponding vertices of  $G$  in the two copies  $G^u$  and  $G^v$ .

Observe that the Cartesian Product  $P_n \square P_n$  is a grid graph on  $n^2$  vertices that has been extensively studied in the context of  $(n^2 - 1)$ -puzzle [7, 11–14]. Ranter and Warmuth [12] have shown that finding a solution with minimum number of moves for the  $(n^2 - 1)$ -puzzle is NP-hard. They give an approximation algorithm that produces a solution of length at most constant times the length of the optimum solution. Parberry, in [11], gives an algorithm that solves the  $(n^2 - 1)$ -puzzle in at most  $5n^3 + O(n^2)$  moves. We show that some of the ideas presented in [11,12] can be generalized to all Cartesian products (see Propositions 5, 7, 9 and 10). We apply these generalized results to compute the minimum number of moves required in motion planning problem for the Cartesian product of two given graphs.

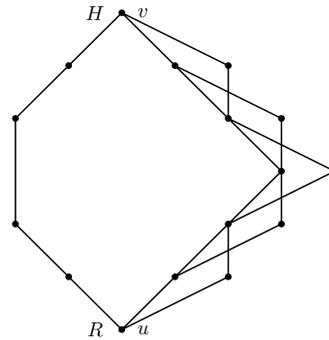


Figure 2. Path traced by robot moves is not a shortest path.

**Example 3.** Consider the configuration  $C_v^u$  as shown in Figure 2. It easy to see

that the minimum number of moves required to take the robot from  $u$  to  $v$  is 21 and the path traced by the robot moves in any minimum sequence of moves is not the shortest path from  $u$  to  $v$ . In fact, to move the robot from  $u$  to  $v$  along the shortest path requires at least 45 moves.

**Remark 4.** In case of the Cartesian product graphs, the path traced by the robot moves in a minimum sequence of moves that takes the robot from a source vertex to a target vertex is a shortest path.

### 3. LOCAL MOVES OF THE HOLE

**Definition.** An edge  $\{u^i, v^j\}$  in  $G \square H$  is said to be a  $G$ -edge (respectively,  $H$ -edge) if  $u = v$  and  $\{i, j\} \in E(G)$  (respectively, if  $i = j$  and  $\{u, v\} \in E(H)$ ).

**Definition.** For any path  $P$  in  $G \square H$ , by  $G$ -length and  $H$ -length of  $P$  we mean the number of  $G$ -edges and  $H$ -edges in  $P$ , respectively. We use  $l_G(P)$  and  $l_H(P)$  to denote the  $G$ -length and  $H$ -length of  $P$ , respectively.

**Definition.** Given two graphs  $G$  and  $H$ . For any  $u^i, v^j \in V(G \square H)$ , we call the distance between  $u$  and  $v$  in  $H$  to be the  $H$ -distance between  $u^i$  and  $v^j$  in  $G \square H$ , and the distance between  $i$  and  $j$  in  $G$  to be the  $G$ -distance between  $u^i$  and  $v^j$  in  $G \square H$ . We use  $d_G(u^i, v^j)$  and  $d_H(u^i, v^j)$  to denote the  $G$ -distance and  $H$ -distance between  $u^i$  and  $v^j$  in  $G \square H$ , respectively.

As mentioned earlier, when there is no confusion about the underlying graph  $G$ , we use  $d(u, v)$  instead of  $d_G(u, v)$  to represent the distance between the vertices  $u$  and  $v$  in  $G$ .

**Proposition 5.** *Given two graphs  $G$  and  $H$ . Let  $\{u, v\}, \{v, w\} \in E(H)$  and  $i \in V(G)$ . Then  $d_{G \square H - v^i}(u^i, w^i) = \min\{d_{H-v}(u, w), 4\}$ .*

**Proof.** Let  $P$  be a shortest path connecting  $u^i$  and  $w^i$  in  $G \square H - v^i$ . We need to show that  $|P| = \min\{d_{H-v}(u, w), 4\}$ . We consider the following cases:

*Case I.*  $V(P) \cap V(H^i) = V(P)$ . In this case  $V(P) \subseteq V(H^i - v^i)$  and so  $|P| = d_{H-v}(u, w)$ .

*Case II.*  $V(P) \cap V(H^i) \neq V(P)$ . We claim that  $|P| = 4$ . Notice that for any  $\{i, j\} \in E(G)$ , the vertices  $x^i, y^j$  are adjacent in  $G \square H$  if and only if  $x = y$ . Therefore if we are moving away from the copy  $H^i$  using the path  $P$  we must also come back to the copy  $H^i$ . Hence  $G$ -distance covered along the path  $P$  must be at least two. Also  $d(u, w) = 2$ , otherwise  $\{u, w\} \in E(H)$  and this implies  $|P| = 1$ , which is not possible. So  $H$ -distance traveled along the path  $P$  must be at least two. Hence  $|P| \geq 4$ . Now for any  $\{i, j\} \in E(G)$  the path  $[u^i, u^j, v^j, w^j, w^i]$  connects  $u^i$  and  $w^i$  in  $G \square H$ . This proves our claim. ■

**Corollary 6.** *Given two graphs  $G$  and  $H$ . Let  $\{u, v\}, \{v, w\} \in E(H)$  and  $i \in V(G)$ , where  $u, v, w$  are distinct. Then starting from the configuration  $C_{u^i}^{v^i}$  of  $G \square H$ , we require at least  $\min\{1 + d_{H-v}(u, w), 5\}$  moves to move the robot to  $w^i$ . In particular, if  $g(H) \geq 6$  then we need at least 5 moves to move the robot to  $w^i$ .*

**Proof.** Notice that,  $\{u^i, v^i\}, \{v^i, w^i\} \in E(G \square H)$ . In order to move the robot from  $v^i$  to  $w^i$ , before it, the hole must be moved from  $u^i$  to  $w^i$ . This takes  $\min\{d_{H-v}(u, w), 4\}$  moves, since  $d_{G \square H - v^i}(u^i, w^i) = \min\{d_{H-v}(u, w), 4\}$ . Then the move  $v^i \xleftarrow{x} w^i$  takes the robot from  $v^i$  to  $w^i$ . Hence the result follows.

If  $g(H) \geq 6$  then  $d_{H-v}(u, w) \geq 4$  and so  $\min\{1 + d_{H-v}(u, w), 5\} = 5$ . Thus, at least five moves are required to take the robot from  $v^i$  to  $w^i$ . ■

As Cartesian product of graphs is commutative, so the proof of the following proposition can be drawn in the same line as that of Proposition 5.

**Proposition 7.** *Given two non-trivial graphs  $G$  and  $H$ . Let  $\{i, j\}, \{j, k\} \in E(G)$  and  $u \in V(H)$ . Then  $d_{G \square H - u^j}(u^i, u^k) = \min\{d_{G-j}(i, k), 4\}$ .*

**Corollary 8.** *Given two graphs  $G$  and  $H$ . Let  $\{i, j\}, \{j, k\} \in E(G)$  and  $u \in V(H)$ . Then starting from the configuration  $C_{u^j}^{u^i}$  of  $G \square H$ , we require at least  $\min\{1 + d_{G-j}(i, k), 5\}$  moves to move the robot to  $u^k$ . In particular, if  $g(G) \geq 6$ , then we need at least 5 moves to move the robot to  $u^k$ .*

**Proposition 9.** *Given two graphs  $G$  and  $H$ . Let  $\{i, j\} \in E(G)$  and  $\{u, v\} \in E(H)$ . Then, starting from the configuration  $C_{u^i}^{u^j}$  of  $G \square H$  we need at least three moves to move the robot to  $v^j$ .*

**Proof.** To move the robot from  $u^j$  to  $v^j$ , before it, the hole must be moved from  $u^i$  to  $v^j$ . This takes two steps, since  $d_{G \square H - u^j}(u^i, v^j) = 2$ . Then the move  $v^j \xleftarrow{x} u^j$  takes the robot to  $v^j$ . Hence the result follows. ■

As Cartesian product of graphs is commutative, so the proof of the following proposition can be drawn in the same line as that of Proposition 9.

**Proposition 10.** *Given two graphs  $G$  and  $H$ . Let  $\{i, j\} \in E(G)$  and  $\{u, v\} \in E(H)$ . Then starting from the configuration  $C_{u^i}^{v^i}$ , we need at least three moves to move the robot to  $v^j$ .*

**Definition.** A robot move in  $G \square H$  is said to be a  $G$ -move (respectively,  $H$ -move) if the edge along which the move took place is a  $G$ -edge (respectively,  $H$ -edge).

**Definition.** Let  $S$  be a sequence of moves that take the robot from  $u^i$  to  $v^j$  in  $G \square H$ . An  $H$ -move (respectively  $G$ -move) in  $S$  of the robot preceded by another  $H$ -move (respectively  $G$ -move) of the robot is said to be a secondary  $H$ -move (respectively secondary  $G$ -move). An  $H$ -move (respectively  $G$ -move) of the robot

preceded by a  $G$ -move (respectively  $H$ -move) of the robot is said to be a primary  $H$ -move (respectively  $G$ -move). Also the edge corresponding to a primary  $G$ -move (respectively primary  $H$ -move) in  $S$  is said to be a primary  $G$ -edge (respectively primary  $H$ -edge).

In view of the above definitions we summarize the results of this section in terms of the following remark.

**Remark 11.** Given two graphs  $G$  and  $H$ , each having girth six or more.

- (i) In view of Corollary 6 and Corollary 8, to perform each secondary  $G$ -move (or  $H$ -move) of the robot we require at least five moves.
- (ii) In view of the Propositions 9 and 10, to perform each primary  $G$ -move (or  $H$ -move) of the robot we require at least three moves.
- (iii) In a minimum sequence of moves, the robot should take as many primary moves as possible.

#### 4. TRACE OF THE ROBOT

**Proposition 12.** Let  $G$  and  $H$  be two graphs such that  $i, j \in V(G)$  and  $u, v \in V(H)$ . Further, let  $S$  be a sequence of moves that take the robot from  $u^i$  to  $v^j$  in  $G \square H$ . Then

- (i) the number of  $H$ -moves of the robot in  $S$  is at least  $d_H(u, v)$ ,
- (ii) the number of  $G$ -moves of the robot in  $S$  is at least  $d_G(i, j)$ .

**Proof.** Let  $x \in V(H)$  and  $\{r, s\} \in E(G)$ . Then we observe that

$$d_H(x^r, v^j) = d_H(x^s, v^j) = d_H(x, v) \quad \text{and} \quad \{x^r, x^s\} \in E(G \square H).$$

Thus a  $G$ -move of the robot from  $x^r$  to  $x^s$  does not alter the  $H$ -distance of the robot from  $v^j$ .

Also if  $r \in V(G)$  and  $\{x, y\} \in E(H)$  then,

$$d_H(x^r, v^j) = \begin{cases} d_H(y^r, v^j) - 1, & \text{if } y^r \text{ is on the shortest path connecting } x^r \text{ to } v^j, \\ d_H(y^r, v^j) + 1, & \text{if } x^r \text{ is on the shortest path connecting } y^r \text{ to } v^j, \\ d_H(y^r, v^j), & \text{otherwise.} \end{cases}$$

Therefore any  $H$ -move of the robot can reduce its  $H$ -distance from  $v^j$  at most by one. Hence to take the robot from  $u^i$  to  $v^j$  we need at least  $d_H(u, v)$  number of  $H$ -moves. We can argue the second statement in similar manner because the Cartesian product is commutative. ■

**Lemma 13.** *Consider the graphs  $G$  and  $H$  each having girth six or more. Let  $i, j \in V(G)$  and  $\{u, v\}, \{u, w\} \in E(H)$ . Then, each robot move in a minimum sequence of moves that takes  $C_{v^i}^{u^i}$  to  $C_{w^j}^{u^j}$  in  $G \square H$  is a  $G$ -move. Also, such a minimum sequence involves exactly  $k$  number of  $G$ -moves and  $5k$  moves in total, where  $k = d(i, j) \geq 1$ .*

**Proof.** Let  $S$  be a sequence of moves that takes  $C_{v^i}^{u^i}$  to  $C_{w^j}^{u^j}$  in  $G \square H$ . First assume that the number of robot moves in  $S$  is  $t$  and each of these robot moves in  $S$  is a  $G$ -move. By Proposition 10, we need at least three moves to accomplish the first  $G$ -move of the robot. Notice that each of the remaining  $t - 1$  robot moves in  $S$  is a secondary  $G$ -moves. So by Remark 11, we need minimum  $5(t - 1)$  moves to accomplish these  $t - 1$  secondary  $G$ -moves. Now, if  $u^j \xleftarrow{r} u^k$  is the  $t^{\text{th}}$  robot move in  $S$ , it will leave the graph  $G \square H$  with the configuration  $C_{u^k}^{u^j}$ . Since  $d_{G \square H - w^j}(u^k, w^j) = 2$ , so we need minimum two more moves to take the hole from  $u^k$  to  $w^j$ . Hence  $S$  involves minimum  $5t$  moves. Notice that, the expression  $5t$  takes the minimum value when  $t$  is minimum.

Next, let  $d(i, j) = k$  and  $[i = i_0, i_1, \dots, i_k = j]$  be a path of length  $k$  connecting  $i$  and  $j$  in  $G$ . Then  $[u^i = u^{i_0}, u^{i_1}, \dots, u^{i_k} = u^j]$  is a path of length  $k$  in  $G \square H$  joining  $u^i$  to  $u^j$ . So, the sequence of moves

$$v^i \xleftarrow{o^*} u^{i_1} \xleftarrow{r} u^{i_0} \xleftarrow{o^*} u^{i_2} \xleftarrow{r} u^{i_1} \xleftarrow{o^*} u^{i_3} \dots u^{i_{k-2}} \xleftarrow{o^*} u^{i_k} \xleftarrow{r} u^{i_{k-1}} \xleftarrow{o^*} w^j$$

takes the robot from  $u^i$  to  $u^j$  along this path and each move in this sequence is a  $G$ -move. Also it involves exactly  $k$  number of  $G$ -moves of the robot. Therefore by Proposition 12, a minimum sequence of moves  $S$  (not involving  $H$ -moves of the robot) that takes the configuration  $C_{v^i}^{u^i}$  to  $C_{w^j}^{u^j}$  involves exactly  $5k$  moves.

Finally, assume that the sequence  $S$  involves  $H$ -moves also and let  $p$  be the number of primary  $H$ -moves in  $S$ . It is enough to show that the sequence  $S$  involves more than  $5k$  moves. We consider the following cases:

*Case I.* The first move of the robot is an  $H$ -move Note that to make the first move of the robot requires at least one move. Since the first move of the robot in  $S$  takes it away from  $G^u$ , and  $G$ -moves always keep the robot in the same copy of  $G$ , so to bring the robot back to  $G^u$  we need one more  $H$ -move. Therefore, minimality of  $S$  implies that  $S$  involves at least one primary  $H$ -move. That is  $p \geq 1$ . Also the maximum number of primary  $G$ -move possible in  $S$  is  $p + 1$  or  $p$  according as the last move of the robot is a  $G$ -move or  $H$ -move. If the last move of the robot is a  $G$ -move then we will be required two more moves to take the hole to  $w^j$ . So the number of moves involved in  $S$  is at least  $1 + 3p + 3(p + 1) + 5(k - p - 1) + 2$  or  $1 + 3p + 3p + 5(k - p)$  according as the last move of the robot is a  $G$ -move or  $H$ -move. That is,  $S$  involves at least  $5k + p + 1$  moves if the first move of the robot is an  $H$ -move.

*Case II.* The first move of the robot is a  $G$ -move. In this case the first move of the robot requires at least three moves and the first  $H$ -move of the robot is primary and so  $p \geq 2$ . Also the maximum number of primary  $G$ -move possible in  $S$  is  $p$  or  $p - 1$  according as the last move is a  $G$ -move or a  $H$ -move. If the last move is a  $G$ -move then we will be required two more moves to take the hole to  $w^j$ . So the number of moves required is at least  $3 + 3p + 3p + 5(k - p - 1) + 2$  or  $3 + 3p + 3(p - 1) + 5(k - p)$  according as the last move is a  $G$ -move or a  $H$ -move. That is,  $S$  involves at least  $5k + p$  moves if the first move of the robot is a  $G$ -move. Thus the number of moves in  $S$  is at least  $5k + 2$ , if it involves  $H$ -moves of the robot.

This completes the proof. ■

Since the Cartesian product of graphs is commutative, so proof of the following lemma can be drawn in the same line as that of Lemma 13.

**Lemma 14.** *Consider the graphs  $G$  and  $H$  each having girth six or more. Let  $\{i, j\}, \{i, k\} \in E(G)$  and  $u, v \in V(H)$ . Then, each robot move in a minimum sequence of moves that takes  $C_{u^j}^{u^i}$  to  $C_{v^k}^{v^i}$  in  $G \square H$  is an  $H$ -move. Also, such a minimum sequence involves exactly  $p$  number of  $H$ -moves and  $5p$  moves in total, where  $p = d(u, v) \geq 1$ .*

The Lemma 15 gives the minimum number of moves required to take the robot from a given factor to another factor of  $G \square H$ . The proof of this lemma is immediate from Lemma 13 and Lemma 14.

**Lemma 15.** *Consider the graph  $G \square H$  with the initial configuration  $C_{v^i}^{u^i}$ , where  $G$  and  $H$  are connected and have girth six or more. Then*

- (i) *to move the robot from  $H^i$  to  $H^j$  we require  $l + 2 + 5(k - 1)$  moves.*
- (ii) *to move the robot from the  $G^u$  to  $G^v$  we require at least  $l + 5(l - 1)$  moves, where  $k = d(i, j)$  and  $l = d(u, v)$ .*

**Definition.** Let  $\{u, v\} \in E(H)$  and  $i, j \in V(G)$ . Then a pair of moves of the robot of the form  $v^i \xleftarrow{r} u^i$  and  $u^j \xleftarrow{r} v^j$  is said to be a pair of opposing  $H$ -moves of the robot in  $G \square H$ . Similarly, we can define a pair of opposing  $G$ -moves of the robot in  $G \square H$ .

**Lemma 16.** *Given the graphs  $G$  and  $H$  each having girth six or more. Let  $S$  be a sequence of moves that takes the robot from  $u^i$  to  $v^j$  in  $G \square H$ . If  $S$  is minimum then there is no opposing moves of the robot in  $S$ .*

**Proof.** If possible suppose that  $S$  involves a pair of opposing  $H$ -moves of the robot. Let this pair of moves be  $v^i \xleftarrow{r} u^i$  and  $u^j \xleftarrow{r} v^j$ . Let  $S_1$  be the subsequence of  $S$  consisting of all moves starting from the move  $v^i \xleftarrow{r} u^i$  up to the move  $u^j \xleftarrow{r} v^j$ . Clearly  $S_1$  takes the configuration  $C_{v^i}^{u^i}$  to the configuration  $C_{v^j}^{u^j}$  and

it involves the  $H$ -moves  $v^i \stackrel{r}{\leftarrow} u^i$  and  $u^j \stackrel{r}{\leftarrow} v^j$  of the robot, a contradiction (see Lemma 13). Therefore  $S$  cannot involve a pair of opposing  $H$ -moves of the robot. Similarly using the Lemma 14, we can conclude that  $S$  cannot involve a pair of opposing  $G$ -moves of the robot. ■

**Lemma 17.** *Consider the graphs  $G$  and  $H$  each having girth six or more. Let  $S$  be a sequence of moves that takes the robot from  $u^i$  to  $v^j$  in  $G \square H$ . Then the  $H$ -moves (respectively,  $G$ -moves) of the robot in  $S$  trace a walk in  $H$  (respectively in  $G$ ). If  $S$  is minimum then these walks are paths and the number of  $H$ -moves (respectively  $G$ -moves) of the robot in  $S$  is  $l_H(P)$  (respectively,  $l_G(P)$ ).*

**Proof.** Let  $P$  be the sub-graph of  $H$  induced by the edges in  $H$  corresponding to the  $H$ -moves of the robot in  $S$ . We claim that  $P$  is a walk in  $H$  connecting  $u$  and  $v$ . Let  $x^i \stackrel{r}{\leftarrow} w^i$  and  $z^j \stackrel{r}{\leftarrow} y^j$  be two consecutive  $H$ -moves in  $S$ , i.e., all other robot moves in  $S$  taken place between these two moves are  $G$ -moves. Clearly  $\{w^i, x^i\}, \{y^j, z^j\} \in E(G \square H)$  and hence  $\{w, x\}, \{y, z\} \in E(H)$ . Notice that  $C_{z^j}^{y^j}$  is reachable from  $C_{w^i}^{x^i}$  by means of  $H$ -moves and obstacle moves only, so  $x = y$ . So the edges in  $P$  corresponding to two consecutive  $H$ -moves  $x^i \stackrel{r}{\leftarrow} w^i$  and  $z^j \stackrel{r}{\leftarrow} y^j$  are incident with each other at  $x$  in  $H$ . It follows by similar argument that if  $y^p \stackrel{r}{\leftarrow} x^p$  and  $w^q \stackrel{r}{\leftarrow} z^q$  are the first and the last  $H$ -moves in  $S$  respectively, then  $x = u$  and  $w = v$ . Therefore,  $P$  is a walk in  $H$  connecting  $u$  and  $v$ .

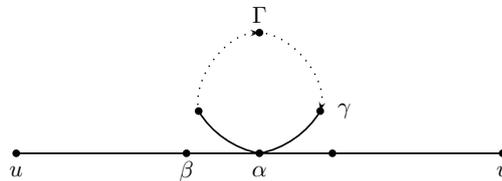


Figure 3. The walk  $P(u, v)$  in  $H$ .

Now assume that  $S$  is minimum. We claim that  $P$  is a path. On the contrary, assume that  $P$  contains a cycle, say  $\Gamma$ . While moving along  $P$  from  $u$  to  $v$  in  $H$ , let  $\alpha$  be the vertex on  $P$  at which it enters  $\Gamma$ ,  $\beta$  be the vertex on  $P$  just before entering  $\Gamma$  and  $\gamma$  be the vertex on  $\Gamma$  adjacent to  $\alpha$  at which it reaches just before re-entering  $\alpha$  after moving along the cycle  $\Gamma$  (see Figure 3). Then there exist a sub-sequence  $S_1$  of  $S$  such that for some for some  $r, s \in V(G)$ , the sequence  $S_1$  takes the configuration  $C_{\beta^r}^{\alpha^r}$  to  $C_{\gamma^s}^{\alpha^s}$ . Since  $\Gamma$  has at least three vertices, so the sub-sequence  $S_1$  must involve at least one  $H$ -move, a contradiction (see Lemma 13). Therefore we can conclude that  $P$  is a path.

Since the graph Cartesian product is commutative so the proof of the remaining part of the lemma can be argued as above. ■

In view of the results obtained in this section we have the following concluding remark.

**Remark 18.** Given two graphs  $G$  and  $H$  each having girth six or more. Let  $S$  be a minimum sequence of moves that takes the robot from  $u^i$  to  $v^j$  in  $G \square H$ . Then, by Lemmas 16 and 17, the robot moves in  $S$  traces a path  $P(u^i, v^j)$  in  $G \square H$  such that

- (i)  $l_H(P)$  = the length of the  $uv$ -path in  $H$  traced by the  $H$ -moves in  $S$ ,
- (ii)  $l_G(P)$  = the length of the  $ij$ -path in  $G$  traced by the  $G$ -moves in  $S$ .

## 5. MINIMUM NUMBER OF MOVES

**Definition.** Given a path  $P$  connecting  $u^i$  and  $v^j$  in  $G \square H$ . By a *minimal* sequence of moves with *trace*  $P$  we mean a sequence with minimum number of moves that takes the robot from  $u^i$  to  $v^j$  along the path  $P$  in  $G \square H$ .

**Definition.** By a *minimal*  $u^i v^j$ -path in  $G \square H$  we mean a  $u^i v^j$ -path  $P$  such that the  $G$ -edges in  $P$  induces a  $ij$ -path in  $G$  and the  $H$ -edges in  $P$  induces a  $uv$ -path in  $H$ .

In view of the above definitions, we have the following remark.

**Remark 19.** Given two graphs  $G$  and  $H$  each having girth six or more.

- (i) A shortest path in  $G \square H$  is a minimal path but a minimal path in  $G \square H$  need not be a shortest path.
- (ii) By Remark 18, the path traced by the robot moves in a minimum sequence of moves in  $G \square H$  is a minimal path.

**Definition.** Give two graphs  $G, H$  and a path  $P$  in  $G \square H$ . By a *primary edge* in  $P$  we mean an  $H$ -edge that is preceded by a  $G$ -edge or a  $G$ -edge that is preceded by an  $H$ -edge. By a *secondary edge* in  $P$  we mean an  $H$ -edge that is preceded by an  $H$ -edge or a  $G$ -edge that is preceded by a  $G$ -edge.

We now state the following lemma without proof. This lemma gives the maximum number of primary edges that a path can have in  $G \square H$  with given  $H$ -length and  $G$ -length.

**Lemma 20.** *Given two graphs  $G$  and  $H$ . Let  $P$  be a path connecting  $u^i$  and  $v^j$  in  $G \square H$  such that  $l_G(P) = a$  and  $l_H(P) = b$ . Then, the maximum number of primary edges  $P$  can have is*

- (i)  $2a - 1$ , if  $a = b$ ,
- (ii)  $2b$ , if  $a > b$  and first edge in  $P$  is a  $G$ -edge,
- (iii)  $2b - 1$ , if  $a > b$  and first edge in  $P$  is an  $H$ -edge,
- (iv)  $2a$ , if  $b > a$  and first edge in  $P$  is an  $H$ -edge,

(v)  $2a - 1$  if  $b > a$  and first edge in  $P$  is a  $G$ -edge.

**Theorem 21.** *Given two graphs  $G$  and  $H$  each having girth six or more. Consider the configuration  $C_{v^i}^{u^i}$  of  $G \square H$ . For some  $j \in G \square H$ , let  $P$  be a minimal path connecting  $u^i$  and  $v^j$  in  $G \square H$ . Let  $S$  be a minimal sequence with trace  $P$ . If  $l_G(P) = a$  and  $l_H(P) = b$ , then  $S$  involves at least*

- (i)  $k + 5a + 5b - 2m - 5$  moves if the first move of the robot is an  $H$ -move,
  - (ii)  $k + 5a + 5b - 2m - 3$  moves if the first move of the robot is a  $G$ -move,
- where  $m$  is the number of primary moves of the robot in  $S$  and  $k = d(u, v)$ .

**Proof.** Since  $S$  is minimal so it involves exactly  $a + b$  robot moves.

*Case I.* The first edge in  $P$  is an  $H$ -edge. In this case the first robot move is an  $H$ -move, say  $w^i \xleftarrow{r} u^i$ . In order to realize this move, before it, the hole must move from  $v^i$  to  $w^i$ . Therefore, we require  $k$  moves to realize the first robot move, since  $d_{G \square H - u^i}(v^i, w^i) = k - 1$  ( $k - 1$  moves to bring the hole at  $w^i$  plus the robot move  $w^i \xleftarrow{r} u^i$ ). Since  $m$  is the number of primary moves in  $S$ , so the number of secondary robot moves in  $S$  is  $a + b - m - 1$ . Hence, by Remark 11, the number of moves in  $S$  is  $k + 3m + 5(a + b - m - 1)$ , i.e.,  $k + 5a + 5b - 2m - 5$ .

*Case II.* The first edge in  $P$  is a  $G$ -edge. In this case the first robot move is a  $G$ -move. Let this move be  $u^k \xleftarrow{r} u^i$ . So, to perform this move we must first take the hole from  $v^i$  to  $u^k$ . Clearly  $d_{G \square H - u^i}(v^i, u^k) = k + 1$  and so we require  $k + 2$  moves to perform the first robot move ( $k + 1$  moves to bring the hole at  $u^k$  plus the robot move  $u^k \xleftarrow{r} u^i$ ). Since  $m$  is the number of primary moves in  $S$ , so the number of secondary robot moves in  $S$  is  $a + b - m - 1$ . Hence, by Remark 11, the number of moves in  $S$  is  $k + 2 + 3m + 5(a + b - m - 1)$ , i.e.,  $k + 5a + 5b - 2m - 3$ . ■

Notice that, among all minimal paths  $P$  with  $l_G(P) = a$  and  $l_H(P) = b$ , the two expressions  $k + 5a + 5b - 2m - 5$  and  $k + 5a + 5b - 2m - 3$  in the above theorem attains the minimum when  $P$  has the maximum number of primary edges. Thus, we have the following corollary.

**Corollary 22.** *Given two graphs  $G$  and  $H$  each having girth six or more. Consider the configuration  $C_{v^i}^{u^i}$  of  $G \square H$ . For some  $j \in G \square H$ , let  $S$  be a minimum sequence of moves that takes the robot from  $u^i$  to  $v^j$ . Let  $P$  be the trace of  $S$ , with  $l_G(P) = a$  and  $l_H(P) = b$ . Then among all minimal paths  $P'$  with  $l_G(P') = a$  and  $l_H(P') = b$ , the path  $P$  has maximum number of primary edges. Also, the number of moves involved in  $S$  is*

- (i)  $k + 5a + b - 3$ , if  $a \geq b$ ,
  - (ii)  $k + a + 5b - 5$ , if  $a < b$ ,
- where  $k = d(u, v)$ . Further, if  $a \geq b$  then the first edge in  $P$  must be an  $H$ -edge.

**Proof.** We consider the following cases:

*Case I.*  $a = b$ . In this case maximum number of primary moves possible is  $2a - 1$  and it is independent of the type of the first edge in  $P$  (see Lemma 20). So, by Theorem 21, the first move of the robot in  $S$  must be an  $H$ -move and the number of moves involved in  $S$  is  $k + 5a + 5b - 2(2a - 1) - 5$ , i.e.,  $k + 6a - 3$  moves.

*Case II.*  $a > b$ . In this case maximum number of primary moves  $S$  can have is  $2b$  or  $2b - 1$  according as the first edge in  $P$  is  $G$ -edge or a  $H$ -edge (see Lemma 20). And, by Theorem 21, in either case the number of moves involved in  $S$  is  $k + 5a + b - 3$ .

*Case III.*  $a < b$ . In this case maximum number of primary moves  $S$  can have is  $2a$  or  $2a - 1$  according as the first edge in  $P$  is an  $H$ -edge or a  $G$ -edge. So, by Theorem 21, the first move in  $S$  must be an  $H$ -move and the number of moves involved in  $S$  is  $k + a + 5b - 5$ .

Hence the proof is complete. ■

Finally, the two expressions  $k + 5a + b - 3$  and  $k + a + 5b - 5$  attains the minimum when  $a$  and  $b$  are minimum. That is, when  $P$  is a shortest path with maximum number of primary edges. Thus, we have the following corollary.

**Corollary 23.** *Given two graphs  $G$  and  $H$  each having girth six or more. Consider the configuration  $C_{v^i}^{u^i}$  of  $G \square H$ . For some  $j \in G \square H$ , let  $S$  be a minimum sequence of moves that takes the robot from  $u^i$  to  $v^j$ . Let  $P$  be the trace of  $S$ , with  $l_G(P) = a$  and  $l_H(P) = b$ . Then  $P$  is a shortest path with maximum number of primary edges. Also, the number of moves involved in  $S$  is*

- (i)  $5a + 2b - 3$ , if  $a \geq b$ ,
- (ii)  $a + 6b - 5$ , otherwise.

*Further, if  $a \geq b$  then the first edge in  $P$  must be an  $H$ -edge.*

## 6. CONCLUSION AND FUTURE WORK

We have investigated the minimum number of moves required for the motion planning problem in Cartesian product graphs. We summarize the results in this article as the following theorem.

**Theorem 24.** *Consider the graph  $G \square H$  with the configuration  $C_{v^i}^{u^i}$ , where  $G$  and  $H$  are connected and have girth six or more. Let  $S$  be a minimum sequence of moves that takes the robot from  $u^i$  to  $v^j$ . If  $P$  is the path in  $G \square H$  traced by the robot moves in the sequence  $S$ , then*

- (i)  $P$  is a shortest path in  $G \square H$  connecting  $u^i$  and  $v^j$ ,

(ii) among all shortest paths connecting  $u^i$  and  $v^j$ ,  $P$  has the maximum number of primary edges.

Also minimum number of moves required is  $2d(u, v) + 5d(i, j) - 3$  or  $6d(u, v) + d(i, j) - 5$  according as  $d(i, j) \geq d(u, v)$  or  $d(i, j) < d(u, v)$ .

As future work, we plan to investigate other graph products, in particular strong, lexicographic and direct products.

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Received 10 April 2012  
Revised 5 February 2013  
Accepted 5 February 2013

