Discussiones Mathematicae
Graph Theory 34 (2014) 409–413
doi:10.7151/dmgt.1706

Note

A NOTE ON *PM*-COMPACT BIPARTITE GRAPHS[1]

Jinfeng Liu

Center for Combinatorics and LPMC-TJKLC
Nankai University, Tianjin 300071, China

and

Xiumei Wang

Department of Mathematics, Zhengzhou University
Zhengzhou 450001, China

e-mail: wangxiumei@zzu.edu.cn

Abstract

A graph is called perfect matching compact (briefly, PM-compact), if its perfect matching graph is complete. Matching-covered PM-compact bipartite graphs have been characterized. In this paper, we show that any PM-compact bipartite graph G with $\delta(G) \geq 2$ has an ear decomposition such that each graph in the decomposition sequence is also PM-compact, which implies that G is matching-covered.

Keywords: perfect matching, PM-compact graph, matching-covered graph.

2010 Mathematics Subject Classification: 05C70.

1. Introduction

In this paper, graphs under consideration are loopless, undirected, finite and connected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A subset M of $E(G)$ is called a *perfect matching* of G if no two edges in M are adjacent and M covers all vertices of G. The *perfect matching graph* of G, denoted by $PM(G)$, is the graph in which each perfect matching of G is a vertex and two vertices M_1 and M_2 are adjacent in $PM(G)$ if and only if the symmetric difference of M_1 and

[1]This work is supported by NSFC (grant no. 11101383, 11201121, and 11201432).

M_2 is an alternating cycle. The *perfect matching polytope* of G is the convex hull of the incidence vectors of all perfect matchings of G. Chvátal [4] shows that two vertices of the perfect matching polytope are adjacent if and only if the symmetric difference of the two perfect matchings is a cycle. This implies that $PM(G)$ is the 1-skeleton graph of the perfect matching polytope of G. Naddef and Pulleyblank [5] show that if $PM(G)$ is bipartite then $PM(G)$ is a hypercube and otherwise $PM(G)$ is Hamilton-connected. Bian and Zhang [1] give a sharp upper bound of the number of edges for the graphs whose perfect matching graphs are bipartite. Padberg and Rao [6] show that, for $n \geq 4$, the diameter of $PM(K_{2n})$ is 2 and, for $n \in \{2,3\}$, the diameter of $PM(K_{2n})$ is 1.

Let G be a graph which has perfect matchings. If $PM(G)$ is a complete graph, i.e., the diameter of the 1-skeleton graph of the perfect matching polytope of G is 1, we call G *perfect matching compact*, or *PM-compact* for short. Clearly, K_4 and K_6 are PM-compact. Let v be a vertex of degree 2 of G which has two distinct neighbors. The *bicontraction* of v is the graph obtained from G by contracting both edges incident with v. The *retract* of G is the graph obtained from G by successively bicontracting vertices of degree 2 until either there are no vertices of degree 2 or at most two vertices remain. A graph with two vertices and at least two parallel edges is denoted by K_2^*. A graph is *matching-covered* if every edge of it appears in a perfect matching. Let $\delta(G)$ denote the minimum degree of G. For bipartite graphs, the following result is obtained in [7].

Theorem 1. (i) *Let G be a matching-covered bipartite graph. Then G is PM-compact if and only if the retract of G is $K_{3,3}$ or K_2^*.*

(ii) *The graph $K_{3,3}$ is the only simple matching-covered PM-compact bipartite graph G with $\delta(G) \geq 3$.*

Let H be a subgraph of a graph G. An *ear* of G with respect to H is a path of odd length in G which has both ends, but no edges or interior vertices, in H. We call an ear *trivial* if it is an edge. An *ear decomposition* of a bipartite graph G is a sequence of subgraphs $(G_0, G_1, \ldots, G_r)$, where $G_0 = K_2$, $G_r = G$, and for $1 \leq i \leq r$, G_i is the union of G_{i-1} and an ear P_i of G_i with respect to G_{i-1}. Clearly, G_1 is an even cycle and $G = K_2 + P_1 + \cdots + P_r$. In [3] Theorem 4.1.1 and Theorem 4.1.6 imply the following.

Theorem 2. *A bipartite graph G is matching-covered if and only if G has an ear decomposition.*

This theorem implies that for an ear decomposition of a matching-covered bipartite graph, each member of the sequence is matching-covered. If G is a matching-covered graph, then G is 2-connected, and so has minimum degree at least 2. In this paper, we show that a PM-compact bipartite graph G with $\delta(G) \geq 2$ has an ear decomposition such that each member of the decomposition sequence is PM-compact, which implies that G is matching-covered. Thus the characterization of PM-compact bipartite graphs is complete. (Note that each pendant edge (of which one end has degree 1) of a graph is contained in all perfect matchings. Using the obtained results, it is easy to characterize PM-compact bipartite graphs with minimum degree one.)

2. Main Result

A vertex v of a graph G is said to be *pendant* if its degree is 1 in G. A bipartite graph G with bipartition (X, Y) is denoted by $G[X, Y]$. The following lemma is an immediate consequence of Exercise 16.1.13 in [2].

Lemma 3. *Let $G[X, Y]$ be a bipartite graph. Then G has a unique perfect matching if and only if*

(i) *each of X and Y contains a pendant vertex, and*

(ii) *when the pendant vertices and their neighbors are deleted, the resulting graph (if nonempty) has a unique perfect matching.*

Lemma 4. *Let G be a PM-compact graph and H a subgraph of G which has a perfect matching. If either (i) H is a spanning subgraph of G or (ii) $G - V(H)$ has a perfect matching, then H is PM-compact.*

Proof. If (i) holds, the assertion follows directly from the definition of PM-compact graphs.

If (ii) holds, let M be a perfect matching of $G - V(H)$. Suppose that M_1' and M_2' are two distinct perfect matchings of H. Then $M_1 = M_1' \cup M$ and $M_2 = M_2' \cup M$ are two perfect matchings of G. Since G is PM-compact, $M_1 \triangle M_2$ is an alternating cycle of G. So $M_1' \triangle M_2' = M_1 \triangle M_2$ is an alternating cycle of H, and hence H is PM-compact. ■

Theorem 5. *Let G be a PM-compact bipartite graph with $\delta(G) \geq 2$. Then G has an ear decomposition $(G_0, G_1, \ldots, G_r)$ such that each G_i, $1 \leq i \leq r$, is PM-compact.*

Proof. Suppose that H is a subgraph of G such that $G - V(H)$ has a unique perfect matching M^*. If a nontrivial ear P of G with respect to H is an M^*-alternating path, then we call P a *normal ear.*

Claim. *The graph G has a normal ear with respect to H.*

Proof. To show this, write $G^* = G - V(H)$. Let P^* be a longest M^*-alternating path in G^*. Let x and y be the two ends of P^*. We assert that both x and y

are covered by $M^* \cap E(P^*)$ and each have a unique neighbor in G^*, that is, their other neighbors are all in H. We show this by way of contradiction. If x is not covered by $M^* \cap E(P^*)$, let y' be the vertex matched to x under M^* (clearly, $y' \in V(G^*)$); otherwise, let y' be an arbitrary neighbor of x in $G^* - E(P^*)$. When $y' \notin V(P^*)$, $P^* + xy'$ is an M^*-alternating path which is longer than P^*. But this contradicts the choice of P^*. When $y' \in V(P^*)$, let C^* be the union of the edge xy' and the segment of P^* from x to y'. Since G is bipartite, C^* is an even cycle which is an M^*-alternating cycle. Hence $M^* \triangle E(C^*)$ is another perfect matching of G^*, which contradicts the uniqueness of M^*. Therefore x is covered by $M^* \cap E(P^*)$ and has only one neighbor in G^* (namely, a member of $V(P^*)$). By symmetry, y also has these properties. The assertion follows.

Since $\delta(G) \geq 2$, by the above assertion, x and y have neighbors in H. Let $x_1, y_1 \in V(H)$ be two neighbors of y and x, respectively. The above assertion also implies that the length of P^* is odd. Since G is bipartite, we have $x_1 \neq y_1$. Write $P = P^* + xy_1 + yx_1$. By the above assertion again, P is an M^*-alternating path with odd length. So P is a normal ear of G with respect to H. The claim follows. □

We now proceed inductively to get an ear decomposition of G. For an even cycle C of G, if $G - V(C)$ has a perfect matching, we call C a *PM-alternating cycle*.

Recall $\delta(G) \geq 2$. By Lemma 3, G has at least two perfect matchings. Since each cycle in the symmetric difference of any two perfect matchings of G is a PM-alternating cycle of G, G has PM-alternating cycles. Let C be a PM-alternating cycle of G, and set $H_1 = C$. If $G - V(H_1)$ has two perfect matchings M_1' and M_2', let E_1 and E_2 be the two disjoint perfect matchings in H_1. Then $M_1 = M_1' \cup E_1$ and $M_2 = M_2' \cup E_2$ are two perfect matchings of G. Since $M_1 \triangle M_2$ contains at least two alternating cycles, namely, C and an alternating cycle in $M_1' \triangle M_2'$, M_1 and M_2 are not adjacent in $PM(G)$. This contradicts the assumption that G is PM-compact. So either $G - V(H_1)$ has a unique perfect matching, say M', or $G - V(H_1)$ is null.

For the former case, by the above claim, G has a normal ear P_2 with respect to H_1. Set $H_2 = H_1 + P_2$. If H_2 is not spanning, then $M' \setminus E(P_2)$ is the unique perfect matching of $G - V(H_2)$. So we can proceed to find a normal ear P_3 of G with respect to H_2. Continue in this way until $H_k = H_{k-1} + P_k$, $k \geq 1$, is a spanning subgraph of G. Write $E' = E(G) \setminus E(H_k)$. Then each edge in E' is a trivial ear of G with respect to H_k. Write $r = k + |E'|$. Then we get an ear decomposition $(H_1, H_2, \ldots, H_k, \ldots, H_r)$ of G, where $H_i = H_{i-1} + P_i$ such that P_i is a normal ear of H_i with respect to H_{i-1} for each $2 \leq i \leq k$ and a trivial ear (an edge in E') of H_i with respect to H_{i-1} for each $k+1 \leq i \leq r$.

For the latter case, H_1 is a spanning subgraph of G. Then each edge in $E' = E(G) \setminus E(H_1)$ is a trivial ear of G with respect to C. Since $G = H_1 + E'$, we are done.

Let $(G_0, G_1, \ldots, G_r)$ be an arbitrary ear decomposition of G. Recall that G_0 is K_2 and G_1 is an even cycle. To complete the proof, we show that for each $1 \leq i \leq r-1$, G_i is PM-compact. Note that $G - V(G_i)$ either is null or has a perfect matching (which is unique). Thus either G_i is a spanning subgraph of G or $G - V(G_i)$ has a unique perfect matching. Since G_i also has a perfect matching, by Lemma 4, G_i is PM-compact. ■

Note that in the proof of Theorem 5, we show a stronger assertion that for each ear decomposition of a PM-compact bipartite graph G with $\delta(G) \geq 2$, each member in the decomposition sequence is PM-compact.

By Theorem 2 and Theorem 5, we get the following.

Corollary 6. *Any PM-compact bipartite graph G with $\delta(G) \geq 2$ is matching-covered.*

Acknowledgement

The authors are grateful to referee for his/her helpful comments which have improved the presentation of this paper.

References

[1] H. Bian and F. Zhang, *The graph of perfect matching polytope and an extreme problem*, Discrete. Math. **309** (2009) 5017–5023.
doi:10.1016/j.disc.2009.03.009

[2] J.A. Bondy and U.S.R. Murty, Graph Theory (Springer-Verlag, Berlin, 2008).

[3] L. Lovász and M.D. Plummer, Matching Theory (Elsevier Science Publishers, B.V. North Holland, 1986).

[4] V. Chvátal, *On certain polytopes associated with graphs*, J. Combin. Theory (B) **18** (1975) 138–154.
doi:10.1016/0095-8956(75)90041-6

[5] D.J. Naddef and W.R. Pulleyblank, *Hamiltonicity in* $(0-1)$*-polytope*, J. Combin. Theory (B) **37** (1984) 41–52.
doi:10.1016/0095-8956(84)90043-1

[6] M.W. Padberg and M.R. Rao, *The travelling salesman problem and a class of polyhedra of diameter two*, Math. Program. **7** (1974) 32–45.
doi:10.1007/BF01585502

[7] X.M. Wang, Y.X. Lin, M.H. Carvalho, C.L. Lucchesi, G. Sanjith and C.H.C. Little, *A characterization of PM-compact biparite graphs and near-bipartite graphs*, (2012) in submission.

Received 19 July 2012
Revised 22 October 2012
Accepted 22 October 2012