

ON THE COMPLEXITY OF THE 3-KERNEL PROBLEM IN SOME CLASSES OF DIGRAPHS¹

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Abstract

Let D be a digraph with the vertex set $V(D)$ and the arc set $A(D)$. A subset N of $V(D)$ is k -independent if for every pair of vertices $u, v \in N$, we have $d(u, v), d(v, u) \geq k$; it is l -absorbent if for every $u \in V(D) - N$ there exists $v \in N$ such that $d(u, v) \leq l$. A k -kernel of D is a k -independent and $(k - 1)$ -absorbent subset of $V(D)$. A 2-kernel is called a *kernel*.

It is known that the problem of determining whether a digraph has a kernel (“the kernel problem”) is NP-complete, even in quite restricted families of digraphs. In this paper we analyze the computational complexity of the corresponding 3-kernel problem, restricted to three natural families of digraphs.

As a consequence of one of our main results we prove that the kernel problem remains NP-complete when restricted to 3-colorable digraphs.

Keywords: kernel, 3-kernel, NP-completeness, multipartite tournament, cyclically 3-partite digraphs, k -quasi-transitive digraph.

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1. INTRODUCTION

We will denote by D a finite digraph without loops or multiple arcs in the same direction, with vertex set $V(D)$ and arc set $A(D)$. All walks, paths and cycles will be considered to be directed. For undefined concepts and notation we refer the reader to [1] and [5].

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Let u and v be distinct vertices of a digraph D . We say that u is k -absorbed by v if $d(u, v) \leq k$. We say that v absorbs u if u is 1-absorbed by v , that is, if $(u, v) \in A(D)$. We denote this by $u \rightarrow v$, and $u \not\rightarrow v$ denotes that $(u, v) \notin A(D)$. We also use $S \rightarrow S'$ to denote that $u \rightarrow v$ for all $u \in S, v \in S'$, where S and S' are sets of vertices of D ; if S or S' consists of only a vertex v we use the notation $v \rightarrow S'$ respectively $S \rightarrow v$. For an integer $k \geq 1$, we define the k -closure $\mathcal{C}^k(D)$ of the digraph D to be the digraph with vertex set $V(\mathcal{C}^k(D)) = V(D)$ and arc set $A(\mathcal{C}^k(D))$ such that $(u, v) \in A(\mathcal{C}^k(D))$ if and only if u is k -absorbed by v in D .

If D is a digraph, the *underlying graph* of D is the unique graph $UG(D)$ with vertex set $V(D)$ and edge set $E(UG(D))$ such that $xy \in E(UG(D))$ if and only if $x \rightarrow y$ or $y \rightarrow x$.

For a subset $X \subseteq V(D)$ and an integer $k \geq 1$ we define $N^{+k}[X]$ to be the set $X \cup \{u \in V(D) : d(X, u) \leq k\}$. For a vertex $v \in V(D)$ and an integer $k \geq 1$ we define $N^{+k}(v)$ as the set $\{u \in V(D) : 0 < d(v, u) \leq k\}$; in particular, when $k = 1$ we call $N^{+1}(v)$ the *out-neighborhood* $N^+(v)$ of v . The *out-degree* $d^+(v)$ of a vertex v is defined as $d^+(v) = |N^+(v)|$. Definitions of *in-neighborhood* and *in-degree* of a vertex v are analogous. The *degree* $d(v)$ of a vertex v is defined to be $d(v) = d^+(v) + d^-(v)$. A *sink* will be a vertex $v \in V(D)$ such that $d^+(v) = 0$. An arc $(u, v) \in A(D)$ is a *symmetric arc* of D if $(v, u) \in A(D)$.

A set X of vertices of a digraph D is a *homogeneous set* if all vertices of X have the same out-neighbourhood and the same in-neighbourhood outside of the set X , i.e., for any $v \notin X$ the set $N^+(v)$ either contains X or is disjoint from X , and similarly for $N^-(v)$.

For a walk $W = (x_0, x_1, \dots, x_n)$ let $x_i W x_j$, $0 \leq i < j \leq n$, denote the sub-walk $(x_i, x_{i+1}, \dots, x_{j-1}, x_j)$. Union of walks will be denoted by concatenation or with \cup . The *circumference* of a digraph D is the length of a longest cycle in D , or infinity if there are no cycles in D .

A *biorientation* of a graph G is a digraph obtained from G by replacing each edge $xy \in E(G)$ by either the arc (x, y) or the arc (y, x) or the pair of symmetric arcs $(x, y), (y, x)$. An *orientation* of G is a biorientation without symmetric arcs. A *semicomplete* digraph is a biorientation of a complete graph; a *tournament* is an orientation of a complete graph. A *semicomplete multipartite digraph* is a biorientation of a complete m -partite graph for some $m \geq 2$; a *multipartite tournament* is an orientation of a complete m -partite graph for some $m \geq 2$. In either case, we replace *multipartite* by *m-partite* if the value of m needs to be emphasized.

A digraph D is *cyclically m-partite* if there is a homomorphism of D to the directed cycle on m vertices, or equivalently, if there exists a partition (V_0, \dots, V_{m-1}) of the vertices of D such that for every arc $(u, v) \in A(D)$, we have $u \in V_i$ if and only if $v \in V_{i+1} \pmod{m}$. We again say that D is *cyclically multipartite* if it is

cyclically m -partite for some $m \geq 2$. It is easy to see that the length of any cycle in a cyclically m -partite digraph is divisible by m .

When $m = 2$, a semicomplete 2-partite digraph is a biorientation of a complete bipartite graph, while a cyclically 2-partite graph is a biorientation of any bipartite graph. Thus the second class contains the first one, and it is known that each digraph in the second class has a kernel [16]. However, when $m = 3$, these two families of biorientations of 3-partite graphs present an interesting contrast. In the semicomplete 3-partite case, the orientations are completely arbitrary, but the underlying graph is complete 3-partite. In the cyclically 3-partite case, the underlying graph is an arbitrary 3-partite graph, but the orientations of the edges are restricted to go only from the first partite set to the second partite set, from the second partite set to the third partite set, and from the third partite set to the first partite set. Thus neither class contains the other one, and, as we will see, neither is guaranteed to contain a 3-kernel.

A digraph D is *transitive* if $(u, w) \in A(D)$ whenever (u, v, w) is a path in D , and *quasi-transitive* if $(u, w) \in A(D)$ or $(w, u) \in A(D)$ whenever (u, v, w) is a path in D . A digraph is *t-quasi-transitive* if $(u_0, u_t) \in A(D)$ or $(u_t, u_0) \in A(D)$ whenever (u_0, \dots, u_t) is a path in D . (Thus 2-quasi-transitive digraphs are just quasi-transitive digraphs.)

A digraph is *strongly connected* (or *strong*) if for every pair of vertices $u, v \in V(D)$, there exists a uv -path. A *strong component* (or *component*) of D is a maximal strong subdigraph of D . The *condensation* of D is the digraph D^* with $V(D^*)$ equal to the set of all strong components of D , and such that $(S, T) \in A(D^*)$ if and only if there is an ST -arc in D . A *terminal component* of D is a strong component T of D such that $d_{D^*}^+(T) = 0$.

In [6] Chvátal proved that recognizing digraphs that have a kernel is an NP-complete problem; later, Fraenkel proved in [7] that this so-called *kernel problem* remains NP-complete even when restricted to planar digraphs with $d^+ \leq 2$, $d^- \leq 2$, and $d \leq 3$. We propose to investigate the complexity of the analogous *k-kernel problem*, i.e., the problem of recognizing digraphs that have a k -kernel. Although there are many known sufficient conditions for the existence of k -kernels in digraphs [9, 14, 10, 11, 12], very little is known about the complexity of the k -kernel problem for $k \geq 3$.

In this paper we focus on the complexity of the 3-kernel problem, and its restriction to three natural families of digraphs. In Section 2, we will prove that the 3-kernel problem is polynomial-time solvable when restricted to semicomplete multipartite digraphs (and if a 3-kernel exists, it can be found also in polynomial time). In particular, this means that the 3-kernel problem is polynomial time solvable in the first of the classes alluded to above, that of semicomplete 3-partite digraphs. By contrast, we prove in Section 3 that the 3-kernel problem is NP-complete in the second of these classes, namely the class of cyclically 3-partite

digraphs. In fact, the 3-kernel problem remains NP-complete even when restricted to cyclically 3-partite digraphs of circumference 6. On the other hand, it follows from a result in Section 4, that a cyclically 3-partite digraph of circumference 3 always has a 3-kernel. (Recall that the circumference has to be divisible by 3, so the circumference 6 constraint is best possible.) In fact, we prove in Section 4 that any digraph in which every cycle has length 3 has a 3-kernel. We also prove in Section 4 that a cyclically 3-partite digraph has a 3-kernel provided at least one of its partite sets (in some cyclic partition) has no sinks. Thus, the 3-kernel problem in natural further restrictions of the class of cyclically 3-partite digraphs is trivial, because a 3-kernel always exists. Finally, in Section 5 we discuss k -kernels in the class of k -quasi-transitive digraphs. It is known that transitive digraphs always have a kernel [3]; we show that for quasi-transitive digraphs the existence of a kernel can be recognized in polynomial time. Analogously, we prove that the 3-kernel problem is polynomial time solvable when restricted to 3-quasi-transitive digraphs, and ask whether a corresponding general statement holds for k -kernels and k -quasi-transitive digraphs.

We note that as a byproduct of one of our proofs we deduce the fact that the kernel problem remains NP-complete when restricted to 3-colorable digraphs.

2. MULTIPARTITE TOURNAMENTS

In [9] the following theorem is proved.

Theorem 1. *Let T be a multipartite tournament. Then the following assertions are equivalent:*

- (1) T has a 3-kernel.
- (2) There is a vertex $v \in V(T)$ such that $\{v\}$ is a 2-absorbing set of $T - (X \setminus \{v\})$, where X is the partite set of T that contains v .
- (3) There is a vertex $v \in V(T)$ such that every vertex $x \in T - (X \setminus \{v\})$ contained in a directed 4-cycle of T is 2-absorbed by $\{v\}$ in T , where X is the partite set of T that contains v .

It is easy to derive a polynomial time algorithm to recognize multipartite tournaments with a 3-kernel, based on condition (2). We first extend this result to semicomplete multipartite digraphs.

Making a simple observation elaborating on Theorem 1 we obtain the following lemma.

Lemma 2. *Let D be a semicomplete multipartite digraph. Let $v \in V(D)$ and X be the partite set of D that contains v . If $\{v\}$ is a 2-absorbing set of $D - (X \setminus \{v\})$, then either $\{v\}$ is a 3-kernel of D , or there is a vertex $w \in X$ not 2-absorbed by*

v such that $\{w\}$ is a 3-kernel of D , or there is a homogeneous set of vertices $K \subseteq X$ such that K is a 3-kernel of D .

Proof. Let $v \in V(D)$ and $X \subseteq V(D)$ be like in the hypothesis. If $\{v\}$ is not a 3-kernel of D , then the set $N_1 \subseteq X$ of vertices not 2-absorbed by $\{v\}$ is non-empty. Let w be a vertex of minimum out-degree among the vertices of maximum in-degree of N_1 . If $\{w\}$ is not a 3-kernel of D , then the set $N_2 \subseteq V(T)$ of vertices not 2-absorbed by $\{w\}$ is non-empty. We will show that $K = N_2 \cup \{w\}$ is a homogeneous set of T .

First, it is easy to observe that $N^-(v) \subseteq N^-(w)$. Since w is not 2-absorbed by v , if $x \rightarrow v$ for some $x \in V(D) \setminus X$, then $x \rightarrow w$, otherwise w would be 2-absorbed by v . Since $N^-(v) \subseteq N^-(w)$, any vertex 2-absorbed by v is also 2-absorbed by w , and hence $N_2 \subseteq N_1$. With an argument similar to the one previously used for v and w , we see that for every $x \in N_2$ we have $N^-(w) \subseteq N^-(x)$. Hence, for every $x \in N_2$ it is clear that $d^-(w) \leq d^-(x)$. Since w was chosen to have the maximum in-degree in N_1 , and $N_2 \subseteq N_1$, we have $d^-(w) = d^-(x)$, and thus $N^-(w) = N^-(x)$, for every $x \in N_2$. From here it is clear that if $x \in N_2$ is arbitrarily chosen, then $N^+(x) \cap N^-(x) = \emptyset$, otherwise, x would be 2-absorbed by w . Finally, since w was chosen with minimum out-degree among the vertices of maximum in-degree, we also have $N^+(w) \cap N^-(w) = \emptyset$. Recalling that D is semicomplete multipartite we conclude that $N^+(x) = N^+(y)$ for every $x, y \in K$. Therefore, K is a homogeneous set.

Since $\{w\}$ is 2-absorbent in $D - (X \setminus \{w\})$, we have that K is 2-absorbent in D . Besides, since K is homogeneous, with $N^+(x) \cap N^-(x) = \emptyset$ for every $x \in K$, it is also 3-independent. So, K is a 3-kernel of D . ■

From here, it is easy to obtain a characterization theorem for semicomplete multipartite digraphs having a 3-kernel.

Theorem 3. *Let D be a semicomplete multipartite digraph. Then D has a 3-kernel if and only if there is a vertex $v \in V(D)$ such that $\{v\}$ is a 2-absorbing set of $D - (X \setminus \{v\})$, where X is the partite set of D containing v .*

Proof. For the implication not covered by Lemma 2, let K be a 3-kernel of D and $v \in K$ an arbitrarily chosen vertex. Since K is an independent set, it is contained in some partite set X of D ; let u be any vertex of $V(D) \setminus X$. If $u \not\rightarrow v$, then $v \rightarrow u$. Also, there must exist $x \in K$ such that $d(u, x) \leq 2$. If $x = v$ we are done. If $x \neq v$ it follows from the 3-independence of K that $u \not\rightarrow x$, otherwise $d(v, x) = 2$. Hence, there must exist $y \in V(D) \setminus X$ such that $u \rightarrow y \rightarrow x$. Since D is a semicomplete multipartite digraph, $v \rightarrow y$ or $y \rightarrow v$. The 3-independence of K prevents the former case from happening, hence $y \rightarrow v$ and $d(u, v) = 2$. Therefore, $\{v\}$ is a 2-absorbing set of $D - (X \setminus \{v\})$. ■

This result yields a polynomial time algorithm for the 3-kernel problem in semi-complete multipartite digraphs.

Corollary 4. *The 3-kernel problem can be decided in polynomial time when restricted to the class of semicomplete multipartite digraphs.*

It will be seen below that our algorithm actually finds a 3-kernel (if one exists) in the claimed time.

Proof of Corollary 4. By virtue of Theorem 3, to determine whether a semi-complete m -partite digraph D has a 3-kernel it suffices to conduct a backwards Breadth First Search (BFS) from every vertex v to determine if a 2-absorbing vertex exists in $D - (X \setminus \{v\})$, where X is the partite set of D containing v . If $|V(D)| = n$ and $|A(T)| = m$, this can be performed in time $O(nm)$.

If such a vertex v exists, and the backwards BFS shows that $\{v\}$ is a 3-kernel, then we are done. Otherwise, we can choose a vertex w of minimum out-degree among the vertices with maximum in-degree not 2-absorbed by v and check with a backwards BFS whether or not w is a 3-kernel of D . If not, according to Lemma 2, the set of vertices of maximum in-degree not absorbed by w together with w , is a 3-kernel of D . Let us observe that this work can be accomplished by performing at most n backwards BFS searches, hence, a 3-kernel of D , if it exists, can be found in time $O(nm)$. ■

In the special case of semicomplete bipartite digraphs the structure of a 3-kernel is simple enough to obtain a linear time recognition algorithm.

Theorem 5. *Let $D = (X, Y)$ be a semicomplete bipartite digraph. A 3-kernel of D consists either of all the sinks of D , or of a vertex $v \in X$ such that $Y \rightarrow v$, or of a vertex $v \in Y$ such that $X \rightarrow v$.*

Proof. Due to the 3-independence of a 3-kernel K of D , it must be the case that $K \subseteq X$ or $K \subseteq Y$.

Let $Z = \{v \in V(D) : d^+(v) = 0\}$. If $Z \neq \emptyset$, then $Z \subseteq X$ or $Z \subseteq Y$, because D is semicomplete bipartite. Let us assume without loss of generality that $Z \subseteq X$. Clearly Z is 3-independent. Also, $Y \rightarrow Z$ and, if $x \in X \setminus Z$, then $x \rightarrow y$ for some $y \in Y$. Since $x \rightarrow y \rightarrow Z$, the vertex x is 2-absorbed by Z , and Z is a 3-kernel of D .

If $Z = \emptyset$ and K is a 3-kernel of D , let us assume without loss of generality $K \subseteq X$, and let $v \in K$. If there is some $y \in Y$ such that $y \not\rightarrow v$, then there must be $v' \in X \cap K$ such that $y \rightarrow v'$. Since D is semicomplete bipartite, and $y \not\rightarrow v$ implies $v \rightarrow y$, we must have $v \rightarrow y \rightarrow v'$, contradicting the 3-independence of K . As a consequence, $Y \rightarrow v$. Also, since $Z = \emptyset$, for every $v \neq x \in X$ there is $y \in Y$ such that $x \rightarrow y$. Recalling that $Y \rightarrow v$, it is now clear that $K = \{v\}$. ■

Let us observe that in the previous proof we have shown that, in general, if $Z = \emptyset$ and $v \in X$ is a vertex such that $Y \rightarrow v$, then $\{v\}$ is a 3-kernel of D . So, we have found a way to determine if a semicomplete bipartite digraph has a 3-kernel and, at the same time, to find a 3-kernel.

Corollary 6. *It can be determined in linear time whether a semicomplete bipartite digraph has a 3-kernel.*

In this case also our algorithm actually finds a 3-kernel (if one exists) in the claimed time.

Proof of Corollary 6. Assume that $|X| = n_1$ and $|Y| = n_2$. By Theorem 5, to decide if D has a 3-kernel, we only have to determine if a sink exists in D , or if a vertex with in-degree equal to n_2 exists in X , or if a vertex with in-degree equal to n_1 exists in Y . This can be done in time $O(|V(D)| + |A(D)|)$.

If there are no sinks in D , then finding a vertex in the remaining cases will give us the 3-kernel we want. If there is a sink in D , then we have to find all such vertices, which also can be performed in time $O(|V(D)| + |A(D)|)$. ■

If we restrict our analysis to bipartite tournaments, we can be more precise. We have the following simple corollary.

Corollary 7. *Let T be a bipartite tournament. Then T has a 3-kernel if and only if there is a sink in T .*

Proof. In a bipartite tournament $T = (X, Y)$, a vertex $v \in X$ can only absorb vertices in Y at odd distances, hence, if it 2-absorbs Y , it absorbs Y . The result follows directly from Theorem 5 and the fact that T is a bipartite tournament. ■

We wonder whether one can obtain linear time recognition algorithms for semicomplete m -partite digraphs with a 3-kernel also for $m \geq 3$.

3. AN NP-COMPLETENESS PROOF

A classical result in kernel theory states that every bipartite digraph has a kernel [16]. Since bipartite digraphs are cyclically 2-partite digraphs, it is natural to ask if every cyclically k -partite digraph has a k -kernel. The answer is no. The digraph depicted in Figure 1 is given in [17] as an example of a cyclically 3-partite digraph without a 3-kernel. So, the next natural question is to ask for the complexity of the k -kernel problem restricted to the class of cyclically k -partite digraphs. As we have already observed, for $k = 2$ a cyclically 2-partite digraph is simply a bipartite digraph, which always has a kernel [16]. Hence, the 2-kernel problem

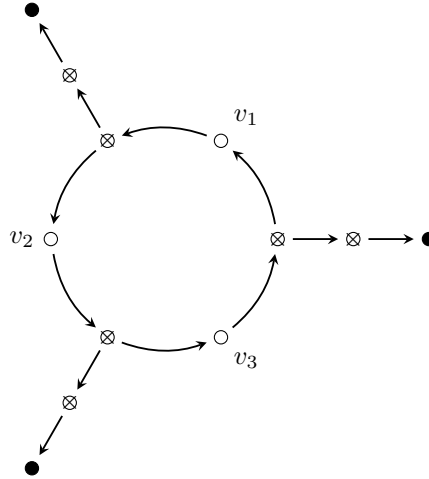


Figure 1. The digraph H , a cyclically 3-partite digraph without a 3-kernel.

can be decided in constant time for the family of cyclically 2-partite digraphs. In this section and the next one we will explore the case when $k = 3$.

The following lemma is proved in [17].

Lemma 8. *The digraph H of Figure 1 does not have a 3-kernel.*

This section is devoted to prove the following result.

Theorem 9. *The 3-kernel problem for the class of cyclically 3-partite digraphs is NP-complete, even when restricted to cyclically 3-partite digraphs of circumference 6.*

We will provide a reduction from the 3-coloring problem to the 3-kernel problem. Given a graph G , we will construct a cyclically 3-partite digraph D_G with circumference 6 and such that G has a 3-coloring if and only if D_G has a 3-kernel. So, for every vertex $v \in V(G)$ we will consider the digraph D_v depicted in Figure 2. The digraph D_v consists of the digraph $H_v \cong H$ joined towards a directed triangle $T_v = (v_0, v_1, v_2, v_0)$. The three depicted types of vertices (circles, triangles and squares) represent the cyclical partition of D_v . (The arcs go from the vertices that are circles to the triangles to the squares and then back to the circles.)

Observation 10. *The digraph in Figure 2 has exactly three distinct 3-kernels, each of them consisting of one vertex v of the directed triangle, one vertex, in the same partite set as v (triangle, square or circle), of the directed 6-cycle and all the sinks. One of these 3-kernels is represented by the black vertices in Figure 2.*

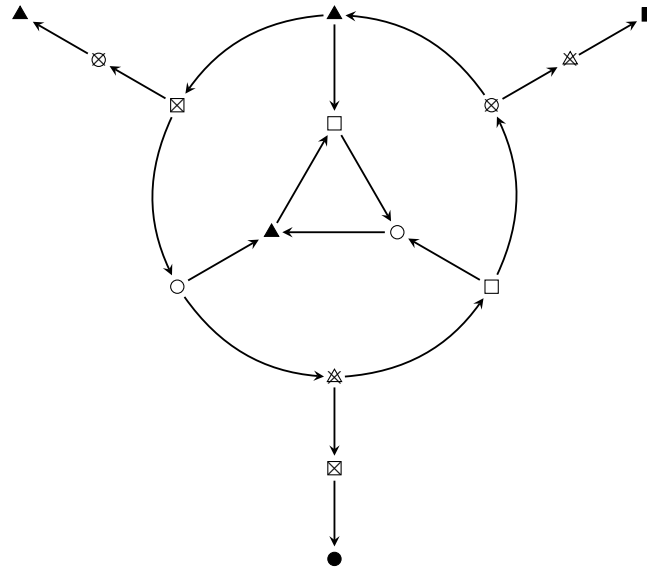


Figure 2. D_v : gadget for every $v \in V(G)$.

Proposition 11. *If D is a digraph with a 3-kernel and having D_v (Figure 2) as an induced subdigraph such that the vertices in H_v are incident to no other vertices in D , at least one vertex in the directed triangle T_v of D_v must be chosen in such a 3-kernel.*

The previous observation is clear, since, otherwise, a 3-kernel should be chosen for H_v , which, in virtue of Lemma 8, cannot be done. Also, the 3-independence of a 3-kernel implies that at most one vertex of T_v can be chosen. So, for $v \in V(G)$, exactly one vertex of T_v will be chosen for a 3-kernel. The obvious interpretation is that the vertex v is colored with color i if and only if the vertex chosen for the 3-kernel in T_v belongs to the i -th partite set of the cyclical partition of D_v .

Hence, for an edge $uv \in E(G)$, we need to link D_u and D_v in such a way that u_i and v_i cannot be chosen at the same time for a 3-kernel. We will begin by choosing an arbitrary acyclic orientation for G , \vec{G} . Then, for every $(u, v) \in E(G)$, we will consider the construction $D_{(u,v)}$ depicted in Figure 3. For the sake of clarity, some vertices have been omitted from Figure 3: every crossed vertex should have a “tail” of length 2, like in Figure 2.

Thus for every graph G we can construct in polynomial time the digraph D_G as follows.

1. Obtain \vec{G} orienting G acyclicly.
2. For every $v \in V(G)$, construct the digraph D_v .

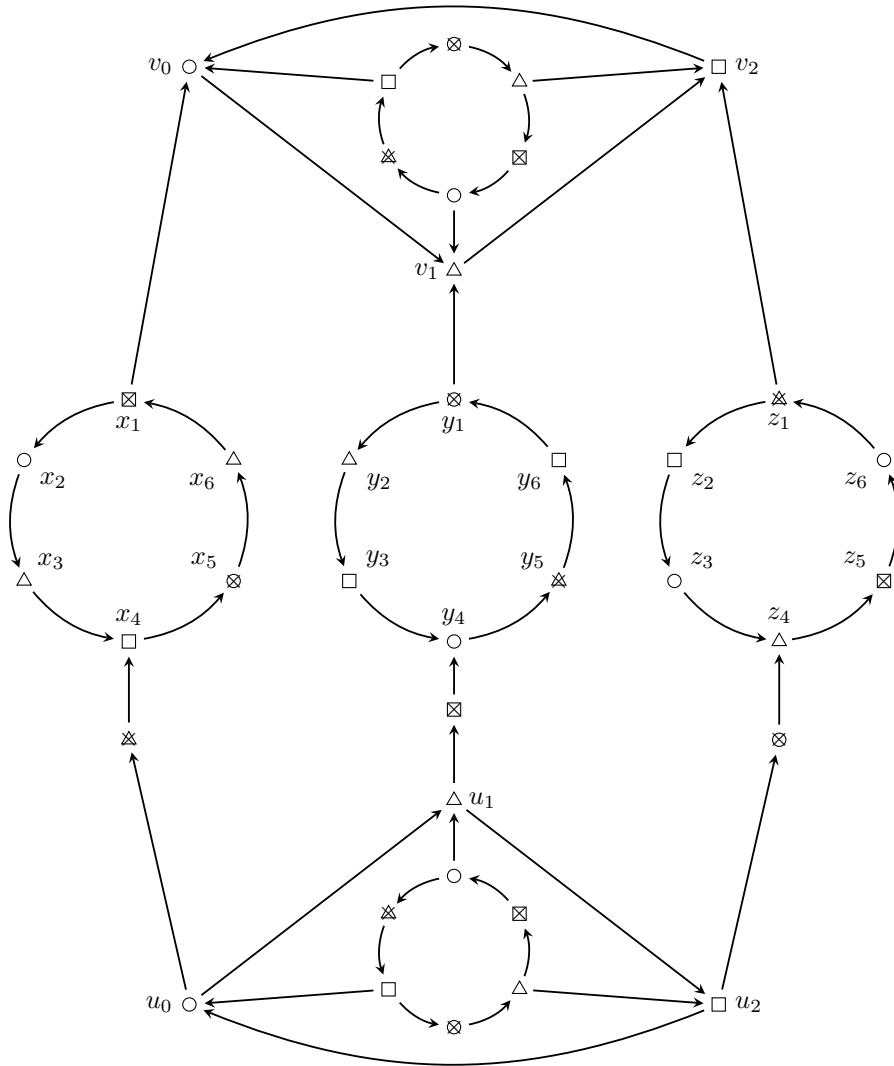


Figure 3. $D_{(u,v)}$: gadget for every (u,v) in the acyclic orientation of G .

3. For every $(u, v) \in A(\vec{G})$, join D_u and D_v using $D_{(u,v)}$.

Hence the digraph D_G depends on the acyclic orientation \vec{G} . Nonetheless, all the results we will prove about D_G remain valid regardless of which acyclic orientation is chosen, so we will consider that an arbitrary acyclic orientation has been already chosen whenever we talk about D_G .

Lemma 12. *Let G be a graph. Then D_G is a cyclically 3-partite digraph with circumference 6.*

Proof. The digraph $D_{(u,v)}$ has a cyclical 3-partition depicted in Figure 3: each partite set consists of all the vertices with the same shape (circles, triangles or squares). Clearly, these cyclical 3-partitions are compatible for every pair of adjacent arcs of \vec{G} and thus, the union of the corresponding partite sets can be taken to obtain a cyclical 3-partition for the digraph D_G .

Also, since \vec{G} is acyclic, every directed cycle of D is contained in some D_v . It is easy to confirm that the longest directed cycle of D_v has length 6. Hence, D_G has circumference 6. ■

Lemma 13. *Let D be a digraph with a 3-kernel K . If $D_{(u,v)}$ is an induced subdigraph of D such that only the vertices u_i, v_i may be incident to other vertices in D , then for every $i \in \{0, 1, 2\}$ the following statements hold”.*

- (i) *If $v_i \in K$, then $u_i \notin K$.*
- (ii) *If $u_i \in K$, then $v_i \notin K$.*

Proof. We will prove the statements for v_0 and u_0 , the remaining cases have similar proofs. If $v_0 \in K$, then the vertex x_6 is 2-absorbed by v_0 and hence cannot be included in K . Note that $N^+(x_4) = \{x_5\}$ and $N^+(x_5) = \{x_6\}$. Since x_5 is also already absorbed by K (recall that we are omitting the “tails” pending from the crossed vertices in Figure 3), the only possibility is that $x_4 \in K$ (otherwise x_4 would not be 2-absorbed by K). Now, x_2, x_3 and u_0 are 2-absorbed by x_4 . Thus, $u_0 \notin K$.

If $u_0 \in K$, then $x_4 \notin K$, because $d(u_0, x_4) = 2$. If $x_2 \in K$, then x_3 cannot be included in K and also, no vertex that 2-absorbs x_3 can be included in K . So, $x_2 \notin K$ and the only possibility is that $x_3, x_6 \in K$. Finally $d(x_6, v_0) = 2$ and thus $v_0 \notin K$. ■

Lemma 14. *Let G be a graph and K a 3-kernel of D_G . Then, for every vertex $v \in G$, exactly one vertex v_i of $T_v \subset D_v$, $i \in \{0, 1, 2\}$, belongs to K . Moreover, if for every such $v_i \in K$ the vertex $v \in V(G)$ is colored with color i , we obtain a 3-coloring of G .*

Proof. The first statement is just Observation 11. The second statement follows directly from Lemma 13. ■

Lemma 15. *Let G be a graph with a 3-coloring $f : V(G) \rightarrow \{0, 1, 2\}$. The set $\{v_{f(v)}\}_{v \in V(G)}$ consisting in exactly one v_i of every T_v in D_G can be extended in a unique way to a 3-kernel of D_G .*

Proof. Let $v \in V(G)$ be arbitrarily chosen. It follows from Observation 10 that a 3-kernel containing $v_{f(v)}$ can be chosen for D_v .

If there is $u \in V(G)$ such that $(u, v) \in A(\vec{G})$, we can assume without loss of generality that $f(v) = 0$. It can be directly verified that the set $\{v_0, x_4, y_3, y_6, z_3, z_6, u_{f(u)}\}$ together with all the sinks of $D_{(u,v)}$ and the necessary vertices to complete a 3-kernel for D_v and D_u , form a 3-kernel for $D_{(u,v)}$.

Also, using an argument similar to the one for Lemma 13, it can be shown that this is the only way to extend $\{v_{f(v)}, u_{f(u)}\}$ to a 3-kernel of $D_{(u,v)}$. Since \vec{G} is acyclic, it is easy to observe that a unique 3-kernel containing $\{v_{f(v)}\}_{v \in V(G)}$ can be built for D_G . ■

Lemma 16. *Let G be a graph. There is a one to one correspondence between the 3-colorings of G and the 3-kernels of D_G .*

Proof. This now follows from Lemmas 14 and 15. ■

We are now ready to prove Theorem 9.

Proof of Theorem 9. If D is a digraph and $K \subseteq V(D)$, it can be verified in polynomial time whether or not K is a 3-kernel of D . Also, Lemma 16 shows a polynomial reduction from the 3-coloring problem to the 3-kernel problem restricted to the class of cyclically 3-partite digraphs with circumference 6. ■

To obtain the last result of this section we will use the following theorem, which can be found in [15].

Theorem 17. *Let D be a digraph and $k \geq 2$ an integer. Then $K \subseteq V(D)$ is a k -kernel of D if and only if K is a kernel of $\mathcal{C}^{k-1}(D)$.*

Now, we are ready to derive the following fact about the kernel problem as a byproduct.

Corollary 18. *The kernel problem restricted to the class of 3-colorable digraphs is NP-complete.*

Proof. Let G be a graph. Lemma 16 shows that G has a 3-coloring if and only if D_G has a 3-kernel. Since D_G is a cyclically 3-partite digraph, the 2-closure of D , $\mathcal{C}^2(D)$, is 3-colorable. Also, Theorem 17 guarantees that $\mathcal{C}^2(D)$ has a kernel if and only if D has a 3-kernel if and only if G has a 3-coloring. Hence, we have found a polynomial reduction of the 3-coloring problem to the kernel problem restricted to the class of 3-colorable digraphs. ■

4. TWO SUFFICIENT CONDITIONS FOR THE EXISTENCE OF 3-KERNELS IN CYCLICALLY 3-PARTITE DIGRAPHS

In the previous section we have shown that the 3-kernel problem is NP-complete, even when restricted to the class of cyclically 3-partite digraphs of circumference 6. In this section we will show that, with very simple additional conditions, the 3-kernel problem restricted to the family of cyclically 3-partite digraphs becomes trivial, because a 3-kernel always exists. Our first condition involves the distribution of the sinks in the digraph.

We will denote by \mathcal{Z} the family of cyclically 3-partite digraphs that admit a cyclic 3-partition $D = (V_0, V_1, V_2)$ in which at least one V_i , for $i \in \{0, 1, 2\}$, has no sink.

Theorem 19. *If D belongs to \mathcal{Z} , then D has a 3-kernel K .*

Proof. Let $X_0 = \{v \in V(D) : d^+(v) = 0\} \subseteq V(D)$. If $X_0 = \emptyset$, then V_0, V_1 and V_2 are 3-kernels of D . If $X_0 \subseteq V_i$ for some $0 \leq i \leq 2$, then V_i is a 3-kernel of D . So, let us suppose that $X_0 \subseteq V_0 \cup V_1$ and $X_0 \cap V_0 \neq \emptyset \neq X_0 \cap V_1$.

We will construct a subset of $V(D)$ that must be contained in every 3-kernel of D . We begin by recursively defining a family of subdigraphs of D and a family of subsets of $V(D)$.

- $D_0 = D$.
- $X_0 = \{v \in V(D) : d^+(v) = 0\} \subseteq V(D)$.
- $D_{n+1} = D[V(D_n) \setminus N_{D_n}^{-2}[X_n]]$.
- $X_{n+1} = \left\{ v \in V(D_{n+1}) : d_{D_{n+1}}^+(v) = 0 \right\}$.

Clearly, $X = \bigcup_{n \in \mathbb{N}} X_n$ must be contained in every 3-kernel of D (otherwise, no other vertex could absorb a vertex in X). Also, we have by construction that X is 3-independent. A very important observation is that $X \subseteq V_0 \cup V_1$. Proceeding by induction on n we can observe that $X_0 \subseteq V_0 \cup V_1$ by hypothesis. Now, let us suppose that $X_{n-1} \subseteq V_0 \cup V_1$. If $v \in V_2 \cap V(D_n)$, and $d_{D_n}^+(v) = 0$, then every

out-neighbor of v in D_n is 2-absorbed by X_{n-1} . Note that $X_{n-1} \subseteq V_0 \cup V_1$, and $N_{D_n}^+(v) \subseteq V_0$. Recalling that D is cyclically 3-partite, we can conclude that every out-neighbor of v is absorbed (at distance 1) by X_{n-1} . Thus, v is 2-absorbed by X_{n-1} and, by construction, $v \notin V(D_n)$, a contradiction. Since the contradiction arose from the assumption $d_{D_n}^+(v) = 0$, then $v \notin X_n$. As v was arbitrarily chosen in $V_2 \cap V(D_n)$, we can conclude that $X_n \cap V_2 = \emptyset$, i.e., $X_n \subseteq V_0 \cup V_1$, as we desired.

Set $D' = \bigcap_{n \in \mathbb{N}} D_n$. Now $V(D')$ is conformed by precisely those vertices in D that are not 2-absorbed by X . So, if $V(D') = \emptyset$, then X is a 3-kernel of D . Otherwise, we can consider $V'_0 = V_0 \cap V(D')$. Since $V'_0 \subseteq V_0$, V'_0 is 3-independent in D . Also by construction, $X \cup V'_0$ is 3-independent. So, let $v \in V(D') \setminus V'_0$ be arbitrarily chosen.

If $v \in V_2$, and there is no $u \in V'_0$ such that $v \rightarrow u$, then there must exist $u \in V_0 \setminus V'_0$ and $w \in V'_1 = V_1 \cap V(D')$ such that $v \rightarrow u \rightarrow w$. Otherwise, $d_{D_{n+1}}^+(v) = 0$ and v should be in X_i for some $i \in \mathbb{N}$. Hence, u is 2-absorbed by X . Since $u \in V_0$ and $X \subseteq V_0 \cup V_1$, so u is absorbed (at distance 1) by X . Therefore, there must exist $i \in \mathbb{N}$ such that u is absorbed by X_i , and then, v is 2-absorbed by X_i . This implies that $v \notin D_{i+1}$, implying that $v \notin D'$, a contradiction. Thus for every $v \in V_2 \cap V(D')$ there is $u \in V'_0$ such that $v \rightarrow u$. Finally, if $v \in V'_1$, then $d_{D'}^+(v) \neq 0$, so v is 2-absorbed by V'_0 , and so, V'_0 is 2-absorbent in D' .

We have proved that $X \cup V'_0$ is a 3-kernel of D . ■

Our second restriction is on the length of the directed cycles of D . The next fact is easy to see.

Lemma 20. *Let D be a digraph with circumference k . If $(u, v) \in A(D)$ and there is a vu -directed path in D , then $d(v, u) \leq k - 1$.*

Next we note the following fact.

Lemma 21. *Let D be a digraph with circumference 3. Then, every 3-cycle in $H = \mathcal{C}^2(D)$ has at least one symmetric arc.*

Proof. Let $C = (v_0, v_1, v_2, v_0)$ be a directed cycle in H . By Lemma 20, if any arc of C is an arc of D , then it is symmetric in H .

So, we can assume that none of the arcs of C is an arc of D , then there are vertices v_{ij} such that (v_i, v_{ij}, v_j) are directed paths in D for $(i, j) \in \{(0, 1), (1, 2), (2, 0)\}$.

If $v_{01} = v_{12} = v_{20}$, then (v_1, v_{12}, v_0) is a directed path in D and $(v_1, v_0) \in A(H)$. If $v_{01} \neq v_{12}$, then either $(v_{01}, v_1, v_{12}, v_2, v_{20}, v_0)$, $(v_0, v_{01}, v_1, v_{12}, v_2, v_0)$ or $(v_0, v_{01}, v_1, v_{12}, v_2, v_{20}, v_0)$ (depending whether $v_{01} = v_{20}$, $v_{12} = v_{20}$ or $v_{01} \neq v_{20} \neq v_{12}$, respectively) is a directed cycle of length greater than three, which results in a contradiction. Analogous arguments apply in the remaining cases. ■

To prove our next result we will use the following theorem, which can be found in [4].

Theorem 22. *Let D be a digraph. If every directed cycle of D has at least one symmetric arc, then D has a kernel.*

Now we are ready to prove our second sufficient condition for the existence of 3-kernels in digraphs.

Theorem 23. *If D is a digraph such that every directed cycle of D has length exactly 3, then D has a 3-kernel.*

Proof. According to Theorems 17 and 22, it suffices to prove that every directed cycle in the 2-closure of D has at least one symmetric arc. Let $H = \mathcal{C}^2(D)$ and $C = (v_0, v_1, \dots, v_{n-1}, v_0)$ be a directed cycle of H . The case $n = 3$ is covered by Lemma 21.

According to Lemma 20 we may assume that $A(C) \cap A(D) = \emptyset$ when $n \geq 4$. So, there exist vertices $v_{i(i+1)}$ such that $(v_i, v_{i(i+1)}, v_{i+1})$ is a directed path in D for $0 \leq i \leq n - 1, (\text{mod } k)$. We affirm that there are $0 \leq i \neq j \leq n - 1$ such that either $v_{i(i+1)} = v_{j(j+1)}$ or $v_{i(i+1)} = v_j$. Otherwise, $\bigcup_{i=0}^{n-1} (v_i, v_{i(i+1)}, v_{i+1})$ would be a directed cycle of length greater than 3 in D .

Now, let $0 \leq i \neq j \leq n - 1$ be such that $v_{i(i+1)} = v_{j(j+1)}$ or $v_{i(i+1)} = v_j$ and $|j - i|$ is minimal with this property. We will assume without loss of generality that $i = 0$. There are two cases.

If $v_{01} = v_{j(j+1)}$ then we have again two cases. If $j = 1$, then $v_{01} = v_{12}$, but this would imply that (v_{01}, v_1, v_{01}) is a directed 2-cycle in D , contradicting our hypothesis.

Hence, $j \geq 2$ and, for every $0 < k \neq l < j$, we have $v_{k(k+1)} \neq v_{l(l+1)}$, $v_{k(k+1)} \neq v_{01} \neq v_{l(l+1)}$ and $v_j \neq v_{k(k+1)} \neq v_l$, if not, the minimality of $|j - i|$ would be contradicted. Therefore, $C' = (v_{01}, v_1, v_{12}, v_2, \dots, v_{(j-1)j}, v_j, v_{j(j+1)})$ is a directed cycle in D . Recall that every directed cycle in D has length 3, and C' has even length, a contradiction.

If $v_{01} = v_j$, then $j \geq 3$, because $A(C) \cap A(D) = \emptyset$. Also, for every $0 < k \neq l < j$, we have $v_{k(k+1)} \neq v_{l(l+1)}$, $v_{k(k+1)} \neq v_{01} \neq v_{l(l+1)}$ and $v_j \neq v_{k(k+1)} \neq v_l$, if not, the minimality of $|j - i|$ would be contradicted. Hence, $C' = (v_1, v_{12}, v_2, \dots, v_{(j-1)j}, v_j, v_1)$ is a directed cycle in D of length at least 5, which results in a contradiction.

Since the contradiction arose from the assumption $A(C) \cap A(D) = \emptyset$, at least one arc of C is an arc of D and Lemma 20 implies that such an arc is symmetric in H . ■

We can observe that Theorem 23 is not restricted to cyclically 3-partite digraphs. Nonetheless, in the case of cyclically 3-partite digraphs, it follows from Theorem 23 that the circumference constraint in Theorem 9 cannot be improved.

5. k -QUASI-TRANSITIVE DIGRAPHS

It is well known that a transitive digraph must always have a kernel [3]. We first observe that the kernel problem can be solved in polynomial time for quasi-transitive digraphs. The following result can be found in [2].

Lemma 24. *Let D be a strong quasi-transitive digraph on at least two vertices. Then the following holds:*

- (i) $\overline{UG}(D)$ is disconnected;
- (ii) *If S and S' are two connected components of $\overline{UG}(D)$, then either $V(S') \rightarrow V(S)$ or $V(S) \rightarrow V(S')$, or both $V(S') \rightarrow V(S)$ and $V(S) \rightarrow V(S')$ in which case $|V(S)| = |V(S')| = 1$.*

From here, it is easy to derive the following lemma.

Lemma 25. *Let D be a strong quasi-transitive digraph. Then D has a kernel if and only if there is an absorbing vertex in D .*

Proof. We only prove the non-trivial implication. Let K be a kernel of D . Since K is independent, it follows from Lemma 24 that it must be contained in $V(S)$ for some connected component S of $\overline{UG}(D)$. Recalling that D is strongly connected, there must be at least one connected component $S' \neq S$ of $\overline{UG}(D)$ such that $V(S) \rightarrow V(S')$. Since $K \subseteq S$, it must be the case that $V(S') \rightarrow V(S)$. Hence, Lemma 24 implies that $|V(S)| = 1$, and thus $|K| = 1$. If $K = \{v\}$, then v is an absorbing vertex of D . ■

In [14] it is observed that, in order for a k -quasi-transitive digraph D to have a k -kernel, it suffices to construct a k -kernel for every terminal strong component of D . In particular, this applies to kernels and quasi-transitive digraphs, and it allows us to conclude the following theorem.

Theorem 26. *The kernel problem restricted to the class of quasi-transitive digraphs can be solved in polynomial time.*

In this case also our algorithm actually finds a 3-kernel (if one exists) in the claimed time.

Proof of Theorem 26. Let $D = (V, A)$ be a digraph such that $|V| = n$ and $|A| = m$. The condensation of D can be obtained in time $O(n + m)$ and it can have at most $O(n)$ terminal strong components. For every terminal component C , it can be verified in time $O(n + m)$ if an absorbing vertex exists: it suffices to construct the out-degree sequence of C . Hence, the kernel problem can be decided in time $O(n^2 + nm)$. If D has a kernel, it can be found in the same time. ■

This suggests a natural generalization.

Problem 27. Is the k -kernel problem polynomial time solvable for k -quasi-transitive digraphs?

We prove this is true for $k = 3$ (as well as $k = 2$, as per the above theorem).

Let F_n be the digraph with vertex set $\{x_0, \dots, x_n\}$ and arc set $\{(x_0, x_1), (x_1, x_2), (x_2, x_0)\} \cup \{(x_0, x_{i+3}), (x_{i+3}, x_1) : 0 \leq i \leq n-3\}$, where $n \geq 3$. In [8], it is proved that every strong 3-quasi-transitive digraph is either semicomplete, or semicomplete bipartite, or isomorphic to F_n for some $n \geq 3$. Also, we know that F_n has a 3-kernel for every $n \geq 3$ (it is enough to choose $\{x_2\}$); and every semicomplete digraph has also a 3-kernel, consisting of a vertex of maximum in-degree. So, the 3-kernel problem for 3-quasi-transitive digraphs is reduced to semicomplete bipartite digraphs (we have already mentioned that, in order for D to have a 3-kernel it suffices to construct a 3-kernel for every terminal component of D).

Corollary 28. *The 3-kernel problem restricted to the class of 3-quasi-transitive digraphs can be decided in polynomial time.*

In this case also our algorithm actually finds a 3-kernel (if one exists) in the claimed time.

Proof of Corollary 28. Let $D = (V, A)$ be a digraph such that $|V| = n$ and $|A| = m$. The condensation of D can be obtained in time $O(n + m)$ and it can have at most $O(n)$ terminal strong components. For every semicomplete bipartite terminal component, according to Corollary 6, it can be verified if it has a 3-kernel and, if so, a 3-kernel can be found, both in time $O(n + m)$. For each semicomplete terminal component, a 3-kernel can be found in time $O(n + m)$. For every terminal component isomorphic to F_n , a 3-kernel can be constructed in constant time. Hence, the 3-kernel problem can be decided in time $O(n^2 + nm)$. If D has a 3-kernel, it can be found in the same time. ■

It is interesting to note that the k -kernel problem is trivial for $(k - 2)$ -quasi-transitive digraphs: every $(k - 2)$ -quasi-transitive digraph has a k -kernel [14, 13] (for $k \geq 4$). For $(k - 1)$ -transitive digraphs, it has been conjectured that a k -kernel also always exists.

Conjecture 29 [14]. *If D is a $(k - 1)$ -quasi-transitive digraph then D has a k -kernel.*

This has been verified for $k = 3$ and $k = 4$, [14, 12]. More generally, we may ask the following.

Problem 30. Determine the complexity of the k -kernel problem in $(k-1)$ -quasi-transitive digraphs.

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