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# MAXIMUM HYPERGRAPHS WITHOUT REGULAR SUBGRAPHS

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**Abstract**

We show that an $n$-vertex hypergraph with no $r$-regular subgraphs has at most $2^{n-1}+r-2$ edges. We conjecture that if $n > r$, then every $n$-vertex hypergraph with no $r$-regular subgraphs having the maximum number of edges contains a full star, that is, $2^{n-1}$ distinct edges containing a given vertex. We prove this conjecture for $n \geq 425$. The condition that $n > r$ cannot be weakened.

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## 1. Introduction

A natural question in graph theory is: What are the graphs not containing $r$-regular subgraphs? For $r \in \{1, 2\}$, the answer is easy, but for $r \geq 3$ it is not. It was a breakthrough when Tashkinov [7] proved the conjecture by Berge that every 4-regular graph contains a 3-regular subgraph. The questions on existence of $r$-regular subgraphs in regular or near-regular graphs were also considered in [1, 8]. Let $F(r, n)$ denote the maximum number of edges an $n$-vertex graph with no $r$-regular subgraphs have. For $r \geq 3$, it is not fully resolved how big $F(r, n)$ is. Pyber [4] showed that for every fixed $r$, $F(r, n) = O(n \ln n)$. On the other hand, Pyber, Rödl and Szemerédi [5] proved that $F(3, n) \geq cn \ln \ln n$.

Similar questions are also natural for hypergraphs. We view a hypergraph as a family $\mathcal{F}$ of its edges, so $|\mathcal{F}|$ is the number of edges of $\mathcal{F}$. An edge $e$ of $\mathcal{F}$ is a $k$-*edge* if $|e| = k$. Note that we do not consider empty set as an edge. If, for some $k$, every edge of $\mathcal{F}$ is a $k$-edge, then $\mathcal{F}$ is $k$-*uniform*. A hypergraph $\mathcal{F}$ is $r$-*free* if it has no $r$-regular sub(hyper)graphs. Mubayi and Verstraëte [3] proved that for every even integer $k \geq 4$, there exists $n_k$ such that for each $n \geq n_k$, each $n$-vertex $k$-uniform 2-free hypergraph $\mathcal{F}$ has at most $\binom{n-1}{k-1}$ edges, and equality holds if and only if $\mathcal{F}$ is a *full* $k$-*star*, that is, $\mathcal{F}$ consists of all $\binom{n-1}{k-1}$ edges of size $k$ containing a given vertex. They also proved the following simpler result for non-uniform hypergraphs.

**Theorem 1.1** [3]. *For $n \geq 3$, every $n$-vertex 2-free hypergraph $\mathcal{F}$ satisfies $|\mathcal{F}| \leq 2^{n-1}$, and equality holds if and only if $\mathcal{F}$ is a full star, that is, $\mathcal{F}$ consists of $2^{n-1}$ distinct edges containing a given vertex.*

Our first result is the following (simple) generalization of Theorem 1.1.

**Theorem 1.2.** *If $2 \leq r \leq 2^{n-1}$, then the maximum number of edges in an $n$-vertex $r$-free hypergraph is $2^{n-1} + r - 2$.*

Many examples of $n$-vertex $r$-free hypergraphs with $2^{n-1} + r - 2$ edges are formed by a full star with $r - 2$ other edges. If $r \geq n$, then some extremal examples do not contain full stars. For $r = 2$, Theorem 1.1 says that if $n \geq 3$, then the only $n$-vertex 2-free hypergraph with $2^{n-1}$ edges is a full star. We conjecture the following.

**Conjecture 1.3.** *Let $\mathcal{F}$ be an $n$-vertex $r$-free hypergraph with $|\mathcal{F}| = 2^{n-1} + r - 2$. If $n > r$ and $r \geq 2$, then $\mathcal{F}$ contains a full star.*

The main results of this paper are the following.

**Theorem 1.4.** *Suppose $\mathcal{F}$ is an $n$-vertex $r$-free hypergraph with $|\mathcal{F}| = 2^{n-1} + r - 2$. If $n \geq r + 2\lceil \log r \rceil + 1$, then $\mathcal{F}$ contains a full star.*

**Theorem 1.5.** *Suppose $\mathcal{F}$ is an $n$-vertex $r$-free hypergraph with $|\mathcal{F}| = 2^{n-1}+r-2$. If $n > r$ and $n \geq 425$, then $\mathcal{F}$ contains a full star.*

In the next section we prove Theorem 1.2 and derive simple properties of dense $r$-free hypergraphs. In Section 3 we show that dense $r$-free hypergraphs have no small transversals. In Section 4 we prove Theorem 1.4. In the last two sections we prove Theorem 1.5.

## 2. Preliminaries

***Proof of Theorem* 1.2.** Let $\mathcal{F}$ be an $n$-vertex $r$-free hypergraph with ground set $N$. Consider all $2^{n-1}$ pairs $\{A, N - A\}$ of subsets of $N$. In at most $r - 1$ pairs of sets both sets are edges in $\mathcal{F}$, otherwise we get an $r$-regular subgraph of $\mathcal{F}$ with vertex set $N$. If there are exactly $r - 1$ such pairs, $N$ cannot be an edge in $\mathcal{F}$, since $N$ together with those $r - 1$ pairs would form an $r$-regular subgraph of $\mathcal{F}$. Thus $|\mathcal{F}| \leq 2^{n-1} + r - 2$. If $2 \leq r \leq 2^{n-1}$, then equality can be achieved.

Let $N = [n]$ and

$$\mathcal{F} = \{e : 1 \in e\} \cup \{r - 2 \text{ smallest nonempty distinct subsets of } [n] - \{1\}\ \}.$$

Suppose that $\mathcal{F}$ has an $r$-regular subgraph $\mathcal{G}$. Let $C_1, C_2, \ldots, C_r$ be the edges of $\mathcal{G}$ that contain 1, and $D_1, D_2, \ldots, D_s$ be the remaining edges of $\mathcal{G}$. Let $C = \bigcup_{i=1}^{r} C_i$ and for $i \in [r]$ let $C'_i = C - C_i$. Since all edges are distinct, $\sum_{i=1}^{r} |C'_i| = r|C| - \sum_{i=1}^{r} |C_i| = \sum_{j=1}^{s} |D_j|$ should hold. The left-hand side is the sum of cardinalities of at least $r - 1$ nonempty distinct sets (possibly one $C'_i$ is empty) not containing 1, and the right-hand side is the sum of cardinailities of at most $r - 2$ smallest distinct sets not containing 1, and so, the right-hand side is less than the left-hand side. This contradiction shows that $H$ has no $r$-regular subgraphs. ■

Let $N$ be a finite set, and $n = |N|$. Let $3 \leq r < n$. A hypergraph $\mathcal{F}$ is $(N, r)$-*strange* if $\mathcal{F}$ is an $r$-free hypergraph with $V(\mathcal{F}) = N$ and $|\mathcal{F}| = 2^{n-1} + r - 2$ such that $\mathcal{F}$ does not contain a *full star*, i.e., $2^{n-1}$ sets containing a given element.

For a set $A \subseteq N$, $\overline{A}$ is the *complement* of $A$ to $N$, i.e., $\overline{A} = N - A$. A *full pair* in $\mathcal{F}$ is a pair $\{A, \overline{A}\}$ such that both $A$ and $\overline{A}$ are in $\mathcal{F}$. We let the set $N$ by itself form a full pair.

In order to prove Theorems 1.4 and 1.5, we derive some properties of $(N, r)$-strange hypergraphs. If $\mathcal{F}$ is $(N, r)$-strange, then it contains at most $r - 1$ full pairs, and so, since $|\mathcal{F}| = 2^{n-1} + r - 2$,

(1) it contains exactly $r - 1$ full pairs.

Moreover,

(2) for each $A \subset N$ with $N \neq A \neq \emptyset$, either $A \in \mathcal{F}$ or $\overline{A} \in \mathcal{F}$.

Furthermore, the following statements hold for each $(N, r)$-strange hypergraph $\mathcal{F}$.

**Lemma 2.1.** *If $A, B \in \mathcal{F}$, $A \cap B = \emptyset$ and both $A$ and $B$ are not in full pairs, then $A \cup B \in \mathcal{F}$.*

***Proof.*** If $A \cup B \notin \mathcal{F}$, then $\overline{A \cup B} \in \mathcal{F}$ by (2). Thus $A, B, \overline{A \cup B}$ with $r-1$ full pairs form an $r$-regular subfamily of $\mathcal{F}$, a contradiction. ■

**Lemma 2.2.** *If $A \in \mathcal{F}$ and $B$ and $C$ are disjoint nonempty subsets of $A$ such that $A = B \cup C$, then at least one of $B$ and $C$ is in $\mathcal{F}$.*

***Proof.*** Suppose that $A = B \cup C$ is a partition of $A$ into nonempty sets and $B, C \notin \mathcal{F}$. Then by (2), $\overline{B}$ and $\overline{C}$ are in $\mathcal{F}$ but not in full pairs. Thus the sets $A$, $\overline{B}$ and $\overline{C}$ together with $r-2$ full pairs different from $\{A, \overline{A}\}$ form an $r$-regular subgraph of $\mathcal{F}$, a contradiction. ■

**Corollary 2.3.** *Every edge $A$ of $\mathcal{F}$ contains an element $x_A$ such that $\{x_A\} \in \mathcal{F}$. In particular, the union $S$ of 1-edges of $\mathcal{F}$ intersects each edge of $\mathcal{F}$.*

**Lemma 2.4.** *Let $A$ and $B$ be edges of $\mathcal{F}$ such that $A \cap B \neq \emptyset$. If at least one of $A$ and $B$ is not in a full pair, then either $A \cap B$ or $A \cup B$ is in $\mathcal{F}$.*

***Proof.*** Suppose that $A \cap B, A \cup B \notin \mathcal{F}$. Then $\overline{A \cap B}$ is in $\mathcal{F}$, and $\overline{A \cup B}$ is either empty or in $\mathcal{F}$. In both cases, the sets $A, B, \overline{A \cap B}$, and $\overline{A \cup B}$ cover every element of $N$ exactly twice. Adding $r-2$ full pairs containing neither $A$ nor $B$ will give an $r$-regular subgraph of $\mathcal{F}$. ■

## 3. Sizes of Transversals of $(N, r)$-strange Hypergraphs

A set $A \subset V(H)$ is a *transversal* of a hypergraph $H$ if every edge of $H$ intersects $A$.

Let $S$ be a minimum transversal of a hypergraph $\mathcal{F}$. Then $S$ contains all 1-edges of $\mathcal{F}$. If $\mathcal{F}$ is $(N, r)$-strange, then by Corollary 2.3, $S$ contains no other vertices. Thus $S$ is exactly the union of 1-edges of $\mathcal{F}$. It has several useful properties.

The goal of this section is to prove the following fact. Throughout the paper, $k$ denote $\lceil \log_2 r \rceil$.

**Theorem 3.1.** *Let $3 \leq r < n$ and $N$ be a finite set with $|N| = n$. If $S$ is the smallest transversal of an $(N, r)$-strange hypergraph $\mathcal{F}$, then $|S| \geq n - 3k - 2$.*

Let $S$ be the smallest transversal of an $(N, r)$-strange hypergraph $\mathcal{F}$.

**Lemma 3.2.** *If a nonempty $S' \subset S$ is not in $\mathcal{F}$, then every $S' \subseteq B \subseteq N-(S-S')$ is not in $\mathcal{F}$, and hence every $S - S' \subseteq A \subseteq N - S'$ is in $\mathcal{F}$.*

***Proof.*** Suppose that such $B$ is in $\mathcal{F}$. By Lemma 2.2, either $S'$ or $B - S'$ is in $\mathcal{F}$. But $(B - S') \cap S = \emptyset$, and we know that $S'$ is not in $\mathcal{F}$, a contradiction. ■

From now on, in this section, we will assume that

(3) $|S| \leq n - 2k - 2.$

Note that to prove Theorem 3.1, we could make the stronger assumption that $|S| \leq n - 3k - 3$, but we plan to use these lemmas also in the next section.

For $S' \subseteq S$ and $M \subseteq N - S$, we say that $M$ *belongs to* $S'$ if $S' \cup M \in \mathcal{F}$. A nonempty proper subset $S'$ of $S$ is *firm* if some $M \subset N - S$ with $|M| \geq 1 + k$ belongs to $S'$. In particular, $S$ is firm by the following reason. For a set $A \subset N-S$ with $|A| = k+1$, one of $A \cup S$ and $N-S-A$ is in $\mathcal{F}$ by (2). Since $S$ is a transversal, $N - S - A$ is not in $\mathcal{F}$. Thus $S \cup A \in \mathcal{F}$, so $A$ belongs to $S$ and $S$ is firm.

**Lemma 3.3.** *Let $S' \subseteq S$ and $M \subseteq N - S$. If $M$ belongs to $S'$, then every $M' \subset M$ belongs to $S'$*

***Proof.*** Since $M \cup S' \in \mathcal{F}$, by Lemma 2.2, either $S' \cup M' \in \mathcal{F}$ or $M - M' \in \mathcal{F}$. But the latter does not hold, since $M \cap S = \emptyset$. This proves the lemma. ■

**Lemma 3.4.** *For every partition $S = S' \cup S''$ of $S$ into nonempty subsets, exactly one of $S'$ and $S''$ is firm.*

***Proof.*** Assume first that neither of $S'$ and $S''$ is firm. Let $M$ be a subset of $N-S$ with $|M| = 1+k$. Since $S'$ is not firm, $S' \cup M \notin \mathcal{F}$. Then $N - (S' \cup M) \in \mathcal{F}$, and $N - (S' \cup M) = S'' \cup (N - S - M)$. So by (3), $|N - S - M| \geq 2k+2-(1+k)$, and thus $S''$ is firm. Assume now that both $S'$ and $S''$ are firm. If a set $M \subset N - S$ with $|M| \geq k + 1$ belongs to both $S'$ and $S''$, then we will find an $r$-regular subgraph $\mathcal{H}$ of $\mathcal{F}$.

Since $2^{|M|} \geq r$, there are at least $r$ subsets of $M$. Call them $A_1, A_2, \ldots, A_r$. Let $\mathcal{H} = \{A_i \cup S' : 1 \leq i \leq r\} \cup \{(M - A_i) \cup S'' : 1 \leq i \leq r\}$, it is a subgraph of $\mathcal{F}$ by Lemma 3.3. By construction, $\mathcal{H}$ is $r$-regular, a contradiction.

If a set $M \subset N - S$ with $k \leq |M| \leq k + 2$ belongs to neither $S'$ nor $S''$, then $N - S - M$ belongs to both, and again $\mathcal{F}$ has an $r$-regular subgraph. Thus each $M \subset N - S$ with $|M| = k + 1$ belongs to exactly one of $S'$ and $S''$. Let $\mathcal{R}_{S'}$ (respectively, $\mathcal{R}_{S''}$) denote the family of $M \subset N - S$ with $|M| = k + 1$ that belongs to $S'$ (respectively, to $S''$). By our assumption, both $\mathcal{R}_{S'}$ and $\mathcal{R}_{S''}$ are nonempty. Then there exist $M' \in \mathcal{R}_{S'}$ and $M'' \in \mathcal{R}_{S''}$ with $|M' \cap M''| = k$. Thus $M' \cap M''$ belongs to both $S'$ and $S''$, and so $\mathcal{F}$ has an $r$-regular subgraph, a contradiction. ■

**Corollary 3.5.** *If $S'$ is a firm subset of $S$, then every $M \subset N - S$ with $1 \leq |M| \leq n - s - (k+1)$ belongs to $S'$.*

**Corollary 3.6.** *Every two firm subsets of $S$ intersect each other.*

***Proof.*** Suppose that $S_1$ and $S_2$ are two disjoint firm subsets of $S$. Let $M \subset N-S$ with $|M| = k+1$. By Corollary 3.5, $M$ belongs to both $S_1$ and $S_2$. Then as in the proof of Lemma 3.4, $\mathcal{F}$ has an $r$-regular subgraph with vertex set $S_1 \cup S_2 \cup M$, a contradiction. ■

**Lemma 3.7.** $s \leq r - 1$

***Proof.*** Suppose $s \geq r$.

*Case* 1. $s \geq r + k$. Since the number of 1-edges in full pairs is at most $r-1$, we can choose $k+1 (\leq s - (r-1))$ 1-edges of $\mathcal{F}$ that are not in full pairs. Let $S'$ be the union of these edges. If some $A \subseteq S'$ is not in $\mathcal{F}$, then $\overline{A}$ and the 1-edges contained in $A$ cover $N$ once, and together with the $r-1$ full pairs (that exist by (1)) we obtain an $r$-regular subgraph of $\mathcal{F}$ covering $N$, a contradiction. Thus all nonempty subsets of $S'$ are in $\mathcal{F}$, and the number of nonempty proper subsets of $S'$ is at least $2^{k+1} - 2 \geq 2r - 2$. We can pair them up so that they are partitions of $S'$. At least $r-1$ of such pairs exist, so together with $S'$ they form an $r$-regular subgraph of $\mathcal{F}$, a contradiction.

*Case* 2. $r \leq s \leq r + k - 1$. By (3), $n - s \geq k + 1$. If there are $v_1, v_2 \in S$ such that $S - v_1, S - v_2 \notin \mathcal{F}$, then by Lemma 3.2, every $B \subseteq N - S$ satisfies $B + v_1 \in \mathcal{F}$ and $B + v_2 \in \mathcal{F}$. Since there are at least $2^{n-s} \geq r$ possible sets for $B$, we can find $r$ pairs of sets $v_1 + B, v_2 + (N - S - B)$, and they will form an $r$-regular subgraph of $\mathcal{F}$ on $(N - S) + v_1 + v_2$.

Thus for some $r-1$ vertices $v_1, \ldots, v_{r-1} \in S$, the sets $S - v_i$ are in $\mathcal{F}$. Then the family $\{v_1, \ldots, v_{r-1}, S - v_1, \ldots, S - v_{r-1}, S\}$ covers every $v \in S$ exactly $r$ times, a contradiction. ■

**Lemma 3.8.** *No* 1*-edge of* $\mathcal{F}$ *is firm.*

***Proof.*** Let $S = \{v_1, v_2, \ldots, v_s\}$. Suppose that $S_1 := \{v_1\}$ is firm. Then by Corollary 3.6, no subset of $S - v_1$ is firm. Hence by Lemma 3.4, the firm subsets of $S$ are exactly the sets containing $v_1$.

Since not every subset of $N$ containing $v_1$ is in $\mathcal{F}$ and $s \leq r-1$, there are at least $r - 1 - (s-1) = r - s$ edges $W_1, \ldots, W_{r-s}$ that are in $\mathcal{F}$, not 1-edges and do not contain $v_1$. For $j = 1, \ldots, r-s$, let $M_j = W_j - S$. Let $M = \bigcup_{j=1}^{r-s} M_j$. Choose $W_1, \ldots, W_{r-s}$ so that to minimize $|M|$.

*Case* 1. $|M| \leq n - s - k - 1$. Denote by $\mathcal{F}'$ the family $\{S \cup M, \{v_2\}, \ldots, \{v_s\}, S \cup M - v_2, \ldots, S \cup M - v_s, W_1, \ldots, W_{r-s}, S \cup M - W_1, \ldots, S \cup M - W_{r-s}\}$.

Since $\mathcal{F}'$ forms an $r$-regular hypergraph, $\mathcal{F}'$ is not a subgraph of $\mathcal{F}$. But since $\{v_1\}$ is firm, by the choice of $W_j$ and Corollary 3.5, every member of $\mathcal{F}'$ is in $\mathcal{F}$, a contradiction. This proves Case 1.

Let $t = \max\{|A - S| : A \in \mathcal{F} \text{ and } v_1 \notin A\}$ and let $A_0 \in \mathcal{F}$ be such that $v_1 \notin A_0$ and $|A_0 - S| = t$.

*Case* 2. $t \geq k$. Let $M_0$ be any $k$-element subset of $A_0 - S$ and $W_0 = M_0 \cup (A_0 \cap S)$. Since $A_0 \in \mathcal{F}$ and $S \cap (A_0 - W_0) = \emptyset$, $W_0 \in \mathcal{F}$. Since $2^k \geq r$, $M_0$ contains some $r$ distinct subsets $M'_1, \ldots, M'_r$. Let $W'_i = M'_i \cup (A_0 \cap S)$ for $i = 1, \ldots, r$. Since $(M_0 - M'_i) \cap S = \emptyset$, each of $W'_i$ is in $\mathcal{F}$. Moreover, since $|S| \leq n - 2k - 2$, $|M_0| = k$, and $\{v_1\}$ is firm, for every $1 \leq i \leq r$, the set $(S \cup M_0) - W'_i$ contains $v_1$ and has at most $n - s - (k+1)$ vertices in $N - S$. This means that $(S \cup M_0) - W'_i$ is also in $\mathcal{F}$. So, the family $\{W'_1, \ldots, W'_r, (S \cup M_0) - W'_1, \ldots, (S \cup M_0) - W'_r\}$ forms an $r$-regular hypergraph, a contradiction.

*Case* 3. $\log_2(r - s) \leq t \leq k - 1$. Let $M_0 = A_0 - S$. In our case, $2^{|M_0|} = 2^t \geq r - s$. Let $M'_1, \ldots, M'_{r-s}$ be any distinct subsets of $M_0$, and for $i = 1, \ldots, r - s$, let $W'_i = M'_i \cup (A_0 \cap S)$. Similarly to Case 2, since $(M_0 - M'_i) \cap S = \emptyset$, each of $W'_i$ is in $\mathcal{F}$. Moreover, since $|S| \leq n - 2k - 2$, $|M_0| \leq k - 1$, and $\{v_1\}$ is firm, for every $1 \leq i \leq r$, the set $(S \cup M_0) - W'_i$ is also in $\mathcal{F}$. By the same reason, for every $2 \leq j \leq s$, the set $(S \cup M_0) - v_j$ is in $\mathcal{F}$. So, the family $\{W'_1, \ldots, W'_{r-s}, (S \cup M_0) - W'_1, \ldots, (S \cup M_0) - W'_{r-s}, \{v_2\}, \ldots, \{v_s\}, S \cup M_0 - v_2, \ldots, S \cup M_0 - v_s, S \cup M_0\}$ forms an $r$-regular hypergraph, a contradiction.

*Case* 4. $|M| \geq n - k - s$ and $t < \min\{k - 1, \log_2(r - s)\}$. Let $M_0 = A_0 - S$. We claim that $|M| \leq r - s - 2^t + t + 1$. To prove the claim, we show a way to choose $W_1, \ldots, W_{r-s}$ so that

$$(4) \qquad \left|\bigcup_{j=1}^{i} W_j - S\right| \leq i - 2^t + t + 1,$$

for every $2^t - 1 \leq i \leq r - s$. The sets $W_1, \ldots, W_{2^t - 1}$ are all the sets of the form $A_0 - X$ where $X \subseteq M_0, X \neq M_0$. So, for $i = 2^t - 1$, (4) holds. Suppose that for some $2^t - 1 \leq i_0 \leq r - s - 1$, we have found $W_1, \ldots, W_{i_0}$ satisfying (4) for $i = i_0$. Let $\mathcal{C}$ be the family of members of $\mathcal{F}$ not containing $v_1$ that are distinct from $W_1, \ldots, W_{i_0}$. Since $i_j \leq r - s - 1$, there is $C \in \mathcal{C} \neq \emptyset$. Let $C' = C - S - \bigcup_{j=1}^{i_0} W_j$. If $C' = \emptyset$, then we let $C := W_{i_0+1}$ and (4) holds for $i = i_0 + 1$. Suppose $x \in C'$. Since $(C' - x) \cap S = \emptyset$, the set $C - C' + x$ is in $\mathcal{F}$, does not contain $v_1$, and is distinct from $W_1, \ldots, W_{i_0}$. So, letting $W_{i_0+1} = C - C' + x$ we again have that (4) holds for $i = i_0 + 1$. This proves the claim.

Let $\mathcal{F}'$ be the family defined in Case 1. Since it is $r$-regular, some $W' \in \mathcal{F}'$ is not in $\mathcal{F}$. Then $\overline{S \cup M - W'} \in \mathcal{F}$ by (2). By the definition of $t$, $|N - (S \cup M)| \leq |\overline{S \cup M - W'} - S| \leq t$. Thus by (4), $n = |N| = |M| + |S| + |N - (M \cup S)| \leq (r - s - 2^t + t + 1) + s + t = r - 2^t + 2t + 1$.

If $t \geq 3$, we get $n \leq r-1$, a contradiction.

If $t = 2$, we get $n \leq r+1$, $|M| = r-s-1$ and $|A_0 - S| = 2$. Then $(A_0 - S) \subset M$ with $|M| = r-s-1$ and $|\overline{M \cup S - W'} - S| \geq 2$. Thus there are distinct $A_0, A_1$ with $|A_i - S| = 2$, $(A_0 - S) \cap (A_1 - S) = \emptyset$ and $v_1 \notin A_i$. Let $\{v_1, v_2, \ldots, v_{r-s-6}\} \subset M - A_0 - A_1$. For $x = 1, 2, \ldots, r-s-6$, let $v_x \in M_{j_x}$. By Lemma 3.3, $W_{j_x} - M_{j_x} + v_x \in \mathcal{F}$ for every $x_1, \ldots, r-s-6$. These edges with $A \cup (A_i \cap S)$ for nonempty $A \subseteq A_i - S$ yield that $|M| \leq r-s-2$, a contradiction.

Thus $t = 1$. Then $n = r+1$, $|M| = r-s$ and every edge not containing $v_1$ is either a 2-edge or a 1-edge. And $\overline{S \cup M - W'}$ is also a 2-edge, so $W'$ is a 1-edge. Let $W' = \{v_2\}$.
Since $|M| = r-s$ and $\overline{S \cup M - v_2} = \{v_2\} \cup (N - M - S)$, for each $w \in N - S$ there is exactly one 2-edge not containing $v_1$ containing $w$. Since $|N - S| = r-s+1$ and the number of 1-edges in $S - v_1$ is $s-1$, $\mathcal{F}$ has exactly $r-s+1+s-1 = r$ edges not containing $v_1$.

So we have exactly two sets containing $v_1$ that are not in $\mathcal{F}$. Call them $D_1, D_2$. We have $D_1 = S \cup M - v_2$. Since $|N - S| = n - s \geq 2k+2 \geq 4$ and every vertex in $N - S$ is contained in exactly one 2-edge not containing $v_1$, there are at least 4 different ways to choose $W_i$s to get minimum $|M|$. Each way gives different $M$, let two of them be $M$ and $M'$. Then, by the above logic, $D_2 = S \cup M' - v_3$ for some $v_3$, and $|M \cap M'| \geq 2$. Thus $D_1 \cap D_2 - S$ is not empty. Let $w \in D_1 \cap D_2 - S$. Then there are $2^{n-1} - 2$ edges in $\mathcal{F}$ containing both $w$ and $v_1$, and exactly one edge containing $w$ but not $v_1$. Thus $\mathcal{F} - w = \{E \in \mathcal{F} : w \notin E\}$ is an $(n-1)$-vertex hypergraph with $2^{n-1} + r - 1$ edges. By Theorem 1.2, $\mathcal{F} - w$ contains an $r$-regular subgraph, a contradiction. ■

**Lemma 3.9.** $s \leq r-2$.

***Proof.*** Suppose that $s \geq r-1$. Then by Lemma 3.7, $s = r-1$. By Lemma 3.8, $S - v_i$ is firm (and so is in $\mathcal{F}$) for every $i = 1, \ldots, s$. Then the $2r-1$ sets

$$\{v_1\}, \ldots, \{v_{r-1}\}, S - v_1, \ldots, S - v_{r-1}, S$$

form an $r$-regular subgraph of $\mathcal{F}$, a contradiction. ■

**Lemma 3.10.** *Let $k \geq 2$ and let $B$ be a set with $|B| \geq 2k+2$. Then there are at least $2^{k-2}+1$ partitions $(B_{i,1}, B_{i,2}, B_{i,3})$ of $B$ such that $|B_{i,1}| = \lceil (k+1)/2 \rceil$, $|B_{i,2}| = \lceil (k+1)/2 \rceil$, $|B_{i,3}| \geq k+1$ and all $3\lceil 2^{k-2} \rceil + 3$ parts of these partitions are distinct.*

***Proof.*** We will choose $B_1$ of size $\lceil (k+1)/2 \rceil$ and $B_2$ of size $\lceil (k+1)/2 \rceil$, so that $|B_3| = |B| - 2\lceil k/2 \rceil \geq k+1$.

For $k \geq 4$, we have

$$\binom{2k+2-\lceil\frac{k+1}{2}\rceil}{\lceil\frac{k+1}{2}\rceil} = \frac{(2k+2-\lceil\frac{k+1}{2}\rceil)(2k+1-\lceil\frac{k+1}{2}\rceil)(2k-\lceil\frac{k+1}{2}\rceil)}{\lceil\frac{k+1}{2}\rceil(2k+2-2\lceil\frac{k+1}{2}\rceil)(2k+1-2\lceil\frac{k+1}{2}\rceil)} \tag{5}$$
$$\binom{2k-2-\lceil\frac{k-2+1}{2}\rceil}{\lceil\frac{k-2+1}{2}\rceil} \geq 4\binom{2(k-2)+2-\lceil\frac{k-1}{2}\rceil}{\lceil\frac{k-1}{2}\rceil}.$$

So, for even $k$,

$$\begin{aligned}\binom{2k+2-\lceil(k+1)/2\rceil}{\lceil(k+1)/2\rceil} &\geq 4\binom{2(k-2)+2-\lceil(k-1)/2\rceil}{\lceil(k-1)/2\rceil}\\ &\geq \cdots \geq 4^{\frac{k-2}{2}}\binom{4}{2} = 6\cdot 2^{k-2}.\end{aligned}$$

For odd $k$,

$$\begin{aligned}\binom{2k+2-\lceil(k+1)/2\rceil}{\lceil(k+1)/2\rceil} &\geq 4\binom{2(k-2)+2-\lceil(k-1)/2\rceil}{\lceil(k-1)/2\rceil} \geq \cdots \geq 4^{\frac{k-3}{2}}\binom{6}{2}\\ &= 15\cdot 2^{k-3} \geq 6\cdot 2^{k-2}.\end{aligned}$$

First for each $i = 1, \ldots, 2^{k-2}+1$, choose a set $B_{i,1}$ of size $\lceil k/2\rceil$ so that all chosen sets are distinct. Then one by one for each $i = 1, \ldots, 2^{k-2}+1$, choose a set $B_{i,2}$ of size $\lceil k/2\rceil$ so that

(a) $B_{i,2}$ is distinct from all $2^{k-2}+1$ sets $B_{i',1}$ and previously chosen $B_{i',2}$, and

(b) $B_{i,1} \cup B_{i,2}$ is distinct from all already chosen $B_{i',1} \cup B_{i',2}$.

Even at the last step (step $2^{k-2}+1$), the number of forbidden sets is at most $3\cdot 2^{k-2}+1 < 6\cdot 2^{k-2}$. So, by (5), we finish the construction. ■

**Corollary 3.11.** *Let $B$ be a set with $|B| \geq 3k+3$. Then there are at least $2^{k-2}+1$ partitions $(B_{i,1}, B_{i,2}, B_{i,3})$ of $B$ such that all $3\lceil 2^{k-2}\rceil + 3$ parts of these partitions are distinct, and each $B_{i,j}$ has size at least $k+1$.*

***Proof.*** Let $A \subseteq B$ and $|A| = k+1$. Let $B' = B - A$. Partition $A$ into $A_1 \cup A_2 \cup A_3$ with $|A_1| = |A_2| = k+1-\lceil\frac{k+1}{2}\rceil$. By Lemma 3.10, there are at least $2^{k-2}+1$ partitions $(B'_{i,1}, B'_{i,2}, B'_{i,3})$ of $B'$ such that all parts of these paritions are distinct, and $|B'_{i,1}| = |B'_{i,2}| = \lceil\frac{k+1}{2}\rceil, |B'_{i,3}| \geq k+1$. Take $B_{i,j} = B'_{i,j} \cup A_i$. Then partitions $(B_{i,1}, B_{i,2}, B_{i,3})$ satisfy all the conditions. ■

***Proof of Theorem* 3.1.** Suppose $s \leq n-3k-3$. Then all lemmas in this section hold, since $s \leq n-3k-3 \leq n-2k-2$.

Let $S'$ be a smallest firm subset of $S$. Note that $S'$ is not a 1-edge. Partition $S'$ into nonempty subsets $S_1$ and $S_2$. By the minimality of $S'$, sets $S_1$ and $S_2$ are not firm, and so $S-S_1$ and $S-S_2$ are firm.

Let $B := N - S$. Then $|B| = n - s \geq 3k + 3$. So, by Corollary 3.11, there are $K := \lceil 2^k/6 \rceil + 1$ partitions

$$(B_{1,1}, B_{1,2}, B_{1,3}), (B_{2,1}, B_{2,2}, B_{2,3}), \ldots, (B_{K,1}, B_{K,2}, B_{K,3})$$

of $B$ such that all $B_{i,j}$ are distinct and $|B_{i,j}| \geq k + 1$. For every $i \in \{1, \ldots, K\}$ and every $j \in \{1, 2, 3\}$, the three sets $S' \cup (B - B_{i,j})$, $S_1 \cup (B - B_{i,j+1})$, and $S_2 \cup (B - B_{i,j+2})$ (where $j$ counts modulo 3) are in $\mathcal{F}$ (by Corollary 3.5 and the fact that $|B - B_{i,j}| \leq n - s - (k + 1)$) and cover every vertex in $N$ exactly twice. Using such triples for $i = 1, \ldots, K$ and $j = 1, 2, 3$, we cover every vertex exactly $6K \geq 2^k \geq r$ times and every set appears at most once. If $r < 6K$ and is even, then we use not all triples.

If $r$ is odd, then we pick a full pair $(A, N - A)$. There are at most two triples $(S' \cup (B - B_{i,j}), S_1 \cup (B - B_{i,j+1}), S_2 \cup (B - B_{i,j+2}))$ containing $A$ or $N - A$. Then we cover the set $N$ once by the set $A$ and $N - A$ and $r - 1$ times with $\frac{r-1}{2} \leq 3K - 2$ triples $(S' \cup (B - B_{i,j}), S_1 \cup (B - B_{i,j+1}), S_2 \cup (B - B_{i,j+2}))$ containing neither $A$ nor $N - A$. This contradicts the choice of $\mathcal{F}$. Therefore $|S| \geq n - 3k - 2$. ∎

## 4. Proof of Theorem 1.4

If the theorem does not hold, then for some $3 \leq r < n$, $k = \lceil \log_2 r \rceil$ with $n \geq r + 2k + 1$, and for some $n$-vertex set $N$, there exists an $(N, r)$-strange hypergraph $\mathcal{F}$. Let $S$ be the union of 1-edges in $\mathcal{F}$. By Lemma 3.9, $|S| \leq r - 2 \leq n - 2k - 3$.

Let $\mathcal{S}_{nf}$ denote the family of non-firm subsets of $S$. For every $S' \in \mathcal{S}_{nf}$, let

$$\mathcal{F}_{S'} := \{W \in \mathcal{F} : W \cap S = S'\}.$$

Furthermore, let

$$\mathcal{F}_{nf} := \bigcup_{S' \in \mathcal{S}_{nf}} \mathcal{F}_{S'}.$$

**Lemma 4.1.** *Let $M := \bigcup_{W \in \mathcal{F}_{nf}} W - S$. Then*
(a) $|M| \leq r - s - 2$;
(b) $|\mathcal{F}_{nf}| \leq r - 2$.

***Proof.*** Assume that (a) does not hold and that $w_1, \ldots, w_{r-s-1}$ are in $M$. Let $M' := \{w_1, \ldots, w_{r-s-1}\}$. For $j = 1, \ldots, r - s - 1$, let $W_j$ be a member of $\mathcal{F}_{nf}$ such that $w_j \in W_j$, and let $S_j = W_j \cap S$. By Lemma 3.3, $W'_j := S_j + w_j$ is in $\mathcal{F}$ for every $j = 1, \ldots, r - s - 1$. Since each $S_j$ and 1-edges are non-firm, $S - S_j$ and $S - v_j$ are firm. Also $|N - S - M'| = n - s - (r - s - 1) = n - r + 1 \geq 2k + 2$, thus by Corollary 3.5, every set of the form $S \cup M' - S_j - w_j$ or of the form $S \cup M' - v_i$ is in $\mathcal{F}$. So, every member of the family $\{S \cup M', \{v_1\}, \ldots, \{v_s\}, S \cup M' - \{v_1\}, \ldots, S \cup M - \{v_s\}, W'_1, \ldots, W'_{r-s-1}, S \cup M' - W'_1, \ldots, S \cup M' - W'_{r-s-1}\}$

is in $\mathcal{F}$. Moreover, together they cover every vertex in $S \cup M'$ exactly $r$ times. This proves (a).

Suppose now that $W_1, \ldots, W_{r-1}$ are in $\mathcal{F}_{nf}$. Since $|M| \leq r - s - 2$, every member of the family $\{S \cup M, W_1, \ldots, W_{r-1}, S \cup M - W_1, \ldots, S \cup M - W_{r-1}\}$ is in $\mathcal{F}$. Moreover, together they cover every vertex in $S \cup M$ exactly $r$ times. This proves (b). ■

**Remark 4.2.** Since no member of $\mathcal{F}_{nf}$ contains any element in $N - S - M$, for every $w \in N - M - S$, every subset of $N - S - w$ belongs to every firm $S' \subset S$.

Let $S'$ be a smallest firm subset of $S$. By Lemma 3.8 $S'$ is not an 1-edge. Choose a partition $S' = S_1 \cup S_2$ of $S'$ into nonempty subsets. By the minimality of $S'$, sets $S_1$ and $S_2$ are not firm, and so $S - S_1$ and $S - S_2$ are firm.
Fix any element $z \in N - S - M$ and let $B := N - S - z$. Since $s \leq r - 2$, $|B| \geq n - (r-2) - 1 \geq 2k + 2$. So, by Lemma 3.10, there are $K := \lceil 2^k/6 \rceil + 1$ partitions $(B_{1,1}, B_{1,2}, B_{1,3}), (B_{2,1}, B_{2,2}, B_{2,3}), \ldots, (B_{K,1}, B_{K,2}, B_{K,3})$ of $B$ such that all $B_{i,j}$ are distinct. For every $i \in \{1, \ldots, K\}$ and every $j \in \{1, 2, 3\}$, the three sets $S' \cup (B - B_{i,j})$, $S_1 \cup (B - B_{i,j+1})$, and $S_2 \cup (B - B_{i,j+2})$ (where $j$ counts modulo 3) are in $\mathcal{F}$ (by Remark 4.2) and cover every vertex in $N - z$ exactly twice. Using such triples for $i = 1, \ldots, K$ and $j = 1, 2, 3$, we cover every vertex exactly $6K \geq 2^k \geq r$ times and every set appears at most once. If $r < 6K$ and is even, then we use not all triples. If $r$ is odd, then we pick a full pair $(A, N - A)$. Then we cover the set $N$ once by the set $A$ and $N - A$ and $r - 1$ times with the triples $(S' \cup (B - B_{i,j}), S_1 \cup (B - B_{i,j+1}), S_2 \cup (B - B_{i,j+2}))$ for $\frac{r-1}{2} (\leq 3K - 2)$ triples containing neither $A$ nor $N - A$.

## 5. SIZE OF ALMOST $\mathcal{F}$-FREE SUBSETS

A set $A$ is *almost $\mathcal{F}$-free* if every $B \in \mathcal{F}$ such that $B \subseteq A$ has size 1.

The aim of this section is to prove the following theorem.

**Theorem 5.1.** *If $n \geq 425$, then $|T| \leq n - 15k - 6$ for each almost $\mathcal{F}$-free $T \subseteq N$.*

Observe that for $n \geq 425$,

$$(6) \quad n - 15k - 6 \geq \frac{n}{2} > 0 \text{ and } n > (4k + 4)(\lceil \log(k) \rceil + 6) + 2k + 6.$$

We need some notation and lemmas. Let $T$ be a maximum almost $\mathcal{F}$-free set, and $Q = N - T$. Assume that $|Q| < 15k + 6$, i.e., $|T| > n - 15k - 6$. For $Q' \subseteq Q$ and $T' \subseteq T$, we say that $T'$ *belongs to* $Q'$ if $Q' \cup T' \in \mathcal{F}$. A nonempty subset $Q'$ of $Q$ is *solid* if some $T' \subset T'$ with $|T'| \geq 3 + k$ belongs to $Q'$.

To show that $Q$ is solid, let $B \subset T$ with $|B| = 2$. Since $T$ is almosts $\mathcal{F}$-free, $B \notin \mathcal{F}$. Then $N - B = (T - B) \cup Q \in \mathcal{F}$. By (6), $|T - B| \geq n/2 - |B| = n/2 - 2 \geq k + 3$, and so $Q$ is solid.

**Lemma 5.2.** *Let $Q' \subseteq Q$ and $T' \subseteq T$. If $T'$ belongs to $Q'$, then every $T'' \subset T'$ with $|T''| \le |T'| - 2$ belongs to $Q'$.*

***Proof.*** Since $T' \cup Q' \in \mathcal{F}$, by Lemma 2.2, either $Q' \cup T'' \in \mathcal{F}$ or $T - T'' \in \mathcal{F}$. But the latter does not hold, since $T$ is almost $\mathcal{F}$-free. This proves the lemma. ∎

**Lemma 5.3.** *For every partition $Q = Q' \cup Q''$ of $Q$ into nonempty subsets, exactly one of $Q'$ and $Q''$ is solid.*

***Proof.*** Assume first that $Q'$ is not solid. By (6), there exists a set $M \subset T$ with $|M| = 3 + k$. Since $Q'$ is not solid, $Q' \cup M \notin \mathcal{F}$. Then $N - (Q' \cup M) \in \mathcal{F}$, and $N - (Q' \cup M) = Q'' \cup (T - M)$. So, since $|T - M| \ge n - 15k - 6 - (3 + k) \ge k + 3$, $Q''$ is solid.

Assume now that both $Q'$ and $Q''$ are solid. We will show that if a set $M \subset T$ with $|M| \ge k + 3$ belongs to both $Q'$ and $Q''$, then $\mathcal{F}$ has an $r$-regular subgraph with vertex set $Q \cup M$.

If $a \in M$, then the number of distinct subsets $A_1, A_2, \ldots, A_r$ of $M$ containing $a$ with $2 \le |A_i| \le |M| - 2$ is at least

$$2^{|M|-1} - (|M| + 1) = 2^{k+2} - k - 4 = 4r - k - 4 \ge r.$$

Note that $r \ge 2$, and $M - A_i \neq A_j$, since $a \in A_j$ and $a \notin M - A_j$. Let $\mathcal{H} = \{A_i \cup Q' : 1 \le i \le r\} \cup \{(M - A_i) \cup Q'' : 1 \le i \le r\}$. By construction, $\mathcal{H}$ is $r$-regular, a contradiction.

If a set $M \subset T$ with $|M| = k+4$ belongs to neither of $Q'$ and $Q''$, then $T - M$ belongs to both, and again $\mathcal{F}$ has an $r$-regular subgraph. Thus each $M \subset T$ with $|M| = k + 4$ belongs to exactly one of $Q'$ and $Q''$. Let $\mathcal{R}_{Q'}$ (respectively, $\mathcal{R}_{Q''}$) denote the family of $M \subset T$ with $|M| = k + 4$ that belong to $Q'$ (respectively, to $Q''$). By our assumption, both $\mathcal{R}_{Q'}$ and $\mathcal{R}_{Q''}$ are nonempty. Then there exist $M' \in \mathcal{R}_{Q'}$ and $M'' \in \mathcal{R}_{Q''}$ with $|M' \cap M''| = k + 3$. By Lemma 5.2, $M' \cap M''$ belongs to both $Q'$ and $Q''$, and so $\mathcal{F}$ has an $r$-regular subgraph, a contradiction. ∎

**Corollary 5.4.** *If $Q'$ is a solid subset of $Q$, then every $M \subset T$ with $k + 3 \le |M| \le |T| - (k + 3)$ belongs to $S'$.*

**Lemma 5.5.** *The number of 1-edges not in full pairs of $\mathcal{F}$ is at most $k$.*

***Proof.*** Assume that there are $k + 1$ distinct 1-edges $\{a_1\}, \{a_2\}, \ldots, \{a_{k+1}\}$ not in full pairs. If some nonempty $B \subset A = \{a_1, a_2, \ldots, a_{k+1}\}$ is not in $\mathcal{F}$, then $\overline{B} \in \mathcal{F}$ by (2). Then $\overline{B}$ together with 1-edges contained in $B$ cover $N$ once and none of these is in a full pair. These sets together with $r - 1$ full pairs cover $N$ exactly $r$ times, a contradiction. Thus every nonempty subset of $A$ is in $\mathcal{F}$.

There are $2^k$ distinct nonempty subsets of $A$ containing $a_1$, call them $B_1, B_2, \ldots, B_{2^k}$. Then all nonempty sets among $B_1, B_2, \ldots, B_r, A - B_1, A - B_2, \ldots, A - B_r$ are in $\mathcal{F}$, and they form an $r$-regular subgraph of $\mathcal{F}$, a contradiction. Therefore the number of 1-edges not in full pairs of $\mathcal{F}$ is at most $k$. ■

**Lemma 5.6.** *The number of* 1*-edges in full pairs in* $\mathcal{F}$ *is at least* $n - 4k - 2$. *Thus at most* $8k - 2$ *elements in full pairs are neither* 1*-edges nor* $(n-1)$*-edges.*

***Proof.*** By Theorem 3.1, $|S| \geq n - 3k - 2$, so the number of 1-edges is at least $n - 3k - 2$. If fewer than $n - 4k - 2$ of them are in full pairs, then we get $k + 1$ distinct 1-edges $a_1, a_2, \ldots, a_{k+1}$ not in full pairs, a contradiction to Lemma 5.5. ■

**Lemma 5.7.** *For each* $a \in Q$, *there is* $A \in \mathcal{F}$ *with* $2 \leq |A| \leq 3$ *such that* $\{a\} = A \cap Q$.

***Proof.*** Since $T$ is a maximum almost $\mathcal{F}$-free set, $T \cup \{a\}$ is not almost $\mathcal{F}$-free. So, there is $B \subset T \cup \{a\}$ such that $B \in \mathcal{F}$ and $|B| \geq 2$. Take a smallest such $B$.

If $|B| = b \geq 4$, then there is $B' \subset B$ with $|B'| = b - 2 > 1$ and $B' \subset T$. Then $B' \notin \mathcal{F}$, and by Lemma 2.2, $B - B' \in \mathcal{F}$, so $A = B - B'$ is what we need. ■

**Lemma 5.8.** *The set* $Q$ *contains at least one solid* 1*-edge.*

***Proof.*** Let $B$ be a smallest solid set in $Q$. Suppose $|B| \geq 2$. Then there are disjoint nonempty $B'_1, B'_2 \subset B$ with $B'_1 \cup B'_2 = B$. By Lemma 5.3, $B_1 = Q - B'_1$ and $B_2 = Q - B'_2$ are solid.

By (6), $T \geq n - 15k - 6 \geq 3k + 9$. Let $K := \lceil 2^k/6 \rceil + 1$. Similarly to the proofs of Lemma 3.10 and Corollary 3.11, for each $i = 1, 2, \ldots, K$ there are partitions $(T_{i,1}, T_{i,2}, T_{i,3})$ of $T$ such that all $T_{i,j}$ are distinct and $|T_{i,j}| \geq k + 3$ for all $i = 1, 2, \ldots, K$ and $j = 1, 2, 3$.

For every $i \in \{1, \ldots, K\}$ and every $j \in \{1, 2, 3\}$, the three sets $B \cup (T - T_{i,j})$, $B_1 \cup (T - T_{i,j+1})$, and $B_2 \cup (T - T_{i,j+2})$ (where $j$ counts modulo 3) are in $\mathcal{F}$ (by Corollary 5.4, and the fact that $|T - T_{i,j}| \leq |T| - (k+3)$) and cover every vertex in $N$ exactly twice. Using such triples for $i = 1, \ldots, K$ and $j = 1, 2, 3$, we cover every vertex exactly $6K \geq 2^k \geq r$ times and every set appears at most once. If $r < 6K$ and is even, then we use not all triples.

If $r$ is odd, then we pick a full pair $(A, N - A)$. There are at most two triples $(B \cup (T - T_{i,j}), B_1 \cup (T - T_{i,j+1}), B_2 \cup (T - T_{i,j+2}))$ containing $A$ or $N - A$. Then we cover the set $N$ once by the sets $A$ and $N - A$ and $r - 1$ times by $\frac{r-1}{2} \leq 3K - 2$ triples $(B \cup (T - T_{i,j}), B_1 \cup (T - T_{i,j+1}), B_2 \cup (T - T_{i,j+2}))$ containing neither $A$ nor $N - A$. This contradicts the choice of $\mathcal{F}$. ■

**Lemma 5.9.** $|Q| < 4k + 4$.

***Proof.*** Suppose $|Q| \geq 4k+4$. By Lemma 5.8, $Q$ contains a solid 1-edge $\{a\}$. Let $Q-a = \{b_1, b_2, \dots, b_{4k+3}, \dots, b_{|Q|-1}\}$. By Lemma 5.7, for each $i = 1, 2, \dots, 4k+3$, we can find $B_i$ with $2 \leq |B_i| \leq 3$ such that $B_i \cap Q = \{b_i\}$. Let $L := N - a - \bigcup_{i=1}^{4k+3} B_i$. By definition, $|\bigcup_{i=1}^{4k+3} B_i| \leq 12k+9$. Since $n \geq 13k+13$, $|L| \geq 13k+3-1-(12k+9) = k+3$. Let $L' \subseteq L$ with $|L'| = k+3$. Let $M = N - L'$. Then $\mathcal{F}$ contains at least $n-4k-5$ edges $\{a_1\}, \{a_2\}, \dots, \{a_{n-4k-5}\}$ such that all $M - a_i$ are also in $\mathcal{F}$, since $a \in M - a_i$ and $k+3 \leq |M - a_i| \leq |T| - k - 3$. Recall that for each $i = 1, \dots, 4k+3$, $B_i \in \mathcal{F}$ and $M - B_i \in \mathcal{F}$. Since $r \leq n-1$, the edges $\{a_1\}, \dots, \{a_{r+1-4k-5}\}, M - a_1, \dots, M - a_{r+1-4k-5}, B_1, \dots, B_{4k+3}, M - B_1, \dots, M - B_{4k+3}, M$ form an $r$-regular subgraph of $\mathcal{F}$, a contradiction. ∎

**Lemma 5.10.** *If $\{a\}$ is a solid 1-edge and $B \in \mathcal{F}$ with $a \notin B$, then $|B \cap T| < \lceil \log k \rceil + 5$.*

***Proof.*** If there is a set $B$ with $a \notin B$ and $|B \cap T| \geq \lceil \log k \rceil + 5$, then by Lemma 5.2, we can find $B_1, B_2, \dots, B_{8k} \in \mathcal{F}$ such that $B_i \subset B$, $B \cap Q = B_i \cap Q$, since $2^{\lceil \log k \rceil + 4} - (\lceil \log k \rceil + 5) \geq 8k$. Let $X \subset N - (B \cup Q)$ with $|X| = k+3$ and let $M = N - X$. Since at least $n-3k-2-(k+3) = n-4k-5$ of 1-edges $\{a_i\}$ are in $M$, the sets $B_1, B_2, \dots, B_{4k+4}, M - B_1, \dots, B - B_{4k+4}, \{a_1\}, \{a_2\}, \dots, \{a_{r-4k-5}\}, M - a_1, M - a_2, \dots, M - a_{r-4k-5}, M$ form an $r$-regular subgraph of $\mathcal{F}$, a contradiction. ∎

**Lemma 5.11.** *There are at most $4k+3$ sets $A_i \in \mathcal{F}$ such that no $A_i$ is a 1-edge and no solid 1-edge $a$ is contained in $A_i$.*

***Proof.*** Suppose that there are $4k+4$ such sets $A_1, A_2, \dots, A_{4k+4}$. Then by Lemma 5.10,

$$\left| T \cap \bigcup_{i=1}^{4k+4} A_i \right| \leq (4k+4)(\lceil \log k \rceil + 5) \leq |T| - k - 3.$$

Thus, as in the proof of Lemma 5.10, we can find an $r$-regular subgraph of $\mathcal{F}$ by using $A_i$ instead of $B_i$. ∎

**Lemma 5.12.** *If $\{a\}$ is a solid 1-edge, then there is at most one $D \notin \mathcal{F}$ with $a \in D$.*

***Proof.*** Suppose $D_1, D_2 \notin \mathcal{F}$ with $a \in D_1 \cap D_2$. By Lemma 5.10, $|D_i \cap T| \geq |T| - k + 3$ for $i = 1, 2$. So, $|D_1 \cap D_2 \cap T| \geq |T| - 2k - 6$.

By Lemmas 5.10 and 5.11, at least $|T| - (4k+4)(\log k + 6)$ elements in $T$ are covered only by 1-edges and sets containing $a$.

By (6), $|T| - (4k+4)(\log k + 6) - 2k - 6 > 0$. So there is $c \in D_1 \cap D_2$ such that $c$ is not covered by any edge of size at least 2 not containing $a$. Since $\mathcal{F}$ is $(n, r)$-strange, Then at most $2^{n-1} + 1 - 2 = 2^{n-1} - 1$ edges of $\mathcal{F}$ contain $c$. Thus the family $\mathcal{F}_c = \{A \in \mathcal{F} : c \in A\}$ has at least $2^{n-2} + r - 1$ edges on $n-1$ vertices, and by Theorem 1.2 we get an $r$-regular subgraph of $\mathcal{F}'$ which is also a subgraph of $\mathcal{F}$, a contradiction. ∎

***Proof of Theorem* 5.1.** By Lemma 5.8, $\mathcal{F}$ has a solid 1-edge $\{a\}$. By Lemma 5.12, there is at most one set $D \notin \mathcal{F}$ with $a \in D$. Since $\mathcal{F}$ is $(n,r)$-strange, such $D$ exists and exactly $r-1$ edges of $\mathcal{F}$ do not contain $a$, call them $B_1, B_2, \ldots, B_{r-1}$.

*Case* 1. $\bigcup_{i=1}^{r-1} B_i = N - a$. Let $l$ be the minimum integer such that we can renumber $B_1, \ldots, B_{r-1}$ so that $\bigcup_{i=1}^{l} B_i = N - a$. Let $\mathcal{B} = \{B_{l+1}, B_{l+2}, \ldots, B_{r-1}\}$. Let $C_1 = B_1, C_2 = B_2 - B_1, C_3 = B_3 - B_2 - B_1, \ldots, C_l = B_l - B_1 - B_2 - \cdots - B_{l-1}$. By the minimality of $l$, $C_i \neq \emptyset$ for every $i = 1, \ldots, l$. By construction, $\{C_1, \ldots, C_l\}$ is a partition of $N - a$.

For every $i = 1, \ldots, l$, there are $2^{|C_i|} - 2$ ways to choose a nonempty proper subset $A$ of $C_i$. By Lemma 2.2, for each proper subset $A$ of $C_i$, one of $A$ and $B_i - A$ is in $\mathcal{F}$, and hence it is in $\mathcal{B}$. It follows that $\mathcal{B}$ contains at least $\frac{1}{2}(2^{|C_i|} - 2) = 2^{|C_i|-1} - 1 \geq |C_i| - 1$ sets $B$ such that (i) $0 < |B \cap C_i| < |C_i|$ and (ii) $B \cap C_j = \emptyset$ for all $i+1 \leq j \leq l$. Since all $C_i$s are disjoint, we conclude that $|\mathcal{B}| \geq \sum_{i=1}^{l}(|C_i| - 1) = n - 1 - l$. Together with $B_1, B_2, \ldots, B_l$, we have at least $n-1$ members of $\mathcal{F}$ not containing $a$. This contradicts the fact that $\mathcal{F}$ has only $r - 1 \leq n - 2$ sets not containing $a$.

*Case* 2. There is $y \in N - a - \bigcup_{i=1}^{r-1} B_i$. Since $N - D \in \mathcal{F}$ and $a \notin N - D$, $y \notin N - D$. So, $y \in D$. Thus $y$ belongs to at most $2^{n-2} - 1$ members of $\mathcal{F}$ containing $a$ and to none not containing $a$. So, the family $\mathcal{F}' = \mathcal{F} - y$ has at least $2^{n-1} + r - 2 - (2^{n-2} - 1) = 2^{n-2} + r - 1$ members. By Theorem 1.2, $\mathcal{F}'$ has an $r$-regular subgraph, which is also a subgraph of $\mathcal{F}$, a contradiction. ∎

## 6. Proof of Theorem 1.5

Suppose $\mathcal{F}$ is $(n,r)$-strange hypergraph on $N$. By Theorem 5.1,

(7) every $S \subseteq N$ with $|S| \geq n - 15k - 5$ contains some $A \in \mathcal{F}$ with $|A| \geq 2$.

Let $B_1, B_2, \ldots, B_l$ be the 1-edges not in full pairs. Let $N_1 = N - B_1 - B_2 - \cdots - B_l$. By Lemma 5.5, $|N_1| \geq n - k$. So, by (7), $N_1$ contains some $B_{l+1} \in \mathcal{F}$ with $|B_{l+1}| \geq 2$. Then by Lemma 2.2, we can choose such $B_{l+1}$ with $2 \leq |B_{l+1}| \leq 3$. Let $N_2 = N_1 - B_{l+1}$. Since $|N_2| \geq (n-k) - 3$, again by (7) and Lemma 2.2, $N_2$ contains some $B_{l+2} \in \mathcal{F}$ with $2 \leq |B_{l+2}| \leq 3$. Similarly, we find $B_{l+3}, \ldots, B_{5k+2}$. Since at least $n - 4k - 2$ of 1-edges are in full pairs, by Lemma 5.6, at most $4k+1$ full pairs have no 1-edges. Among the at most $8k+2$ sets in these full pairs, at most $4k+1$ of the sets are in $\{B_1, B_2, \ldots, B_{5k+2}\}$, since $|B_i| \leq 3$ and $n \geq 425$. Thus some $k+1$ sets among $B_1, B_2, \ldots, B_{5k+2}$ are not in full pairs. Call them

$A_1, A_2, \dots, A_{k+1}$. Then for any $I \subset [k+1]$, $A_I = \bigcup_{i \in I} A_i$ is in $\mathcal{F}$, otherwise $\overline{A_I}$ and $\{A_j : j \in I\}$ together with $r-1$ full pairs yield an $r$-regular subgraph of $\mathcal{F}$. Therefore $\mathcal{F}$ contains $2^{k+1-1} \geq r$ different pairs of edges of the kind $A_I, A_{[k+1]-I}$. They form an $r$-regular subgraph of $\mathcal{F}$ covering $A_{[k+1]}$, a contradiction.

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