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TREE-LIKE PARTIAL HAMMING GRAPHS

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Abstract

Tree-like partial cubes were introduced in [B. Brešar, W. Imrich, S. Klavžar, *Tree-like isometric subgraphs of hypercubes*, Discuss. Math. Graph Theory, 23 (2003), 227–240] as a generalization of median graphs. We present some incorrectnesses from that article. In particular we point to a gap in the proof of the theorem about the dismantlability of the cube graph of a tree-like partial cube and give a new proof of that result, which holds also for a bigger class of graphs, so called tree-like partial Hamming graphs. We investigate these graphs and show some results which imply previously-known results on tree-like partial cubes. For instance, we characterize tree-like partial Hamming graphs and prove that every tree-like partial Hamming graph G contains a Hamming graph that is invariant under every automorphism of G. The latter result is a direct consequence of the result about the dismantlability of the intersection graph of maximal Hamming graphs of a tree-like partial Hamming graph.

Keywords: partial Hamming graph, expansion procedure, dismantlable graph, gated subgraph, intersection graph.

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1. Introduction

Median and quasi-median graphs are well studied classes of graphs, cf. [3, 4, 11, 15, 16, 17, 18, 19, 24]. One of the well-known characterizations of median graphs is that they constitute the class of retracts of hypercubes, see Bandelt [1]. On the other hand, regular median graphs are precisely hypercubes [18]. For a survey of many different aspects of median graphs, the reader is referred to [16]. Quasi-median graphs have been introduced by Mulder [19] as a natural nonbipartite extension of median graphs. Chung, Graham, and Saks [11] and independently Wilkeit [24] proved that they are the weak retracts of Hamming graphs. On the other hand, Hamming graphs are the regular quasi-median graphs [19]. In [3] a survey of characterizations of quasi-median graphs is given including some new ones.

Partial cubes, that is, isometric subgraphs of hypercubes, have been first investigated by Graham and Pollak [12], see also [25]. A nonbipartite extension of this class are isometric subgraphs of Hamming graphs, called partial Hamming graphs, see [10, 13, 23]. Since (weak) retracts are isometric subgraphs, quasi-median graphs are partial Hamming graphs and median graphs are partial cubes.

Median graphs have many interesting properties, cf. [4, 16, 17, 18], but not a lot of them can be extended to partial cubes. Brešar, Imrich and Klavžar [8] introduced a class of tree-like partial cubes which lies between median graphs and partial cubes and possesses many of the properties of median graphs. The authors characterized tree-like partial cubes and listed several properties which are shared with median graphs.

Tree-like partial Hamming graphs which we introduce in this paper are defined with an expansion procedure. There are also many other classes of graphs defined or characterized with a certain type of expansion. The most investigated such classes are median graphs, quasi-median graphs, partial cubes and partial Hamming graphs [10, 19]. But there are also several subclasses of partial cubes and partial Hamming graphs with nice (maybe just partial) results using expansions [7, 14]. Because of those nice results one could ask whether graphs characterized with an expansion procedure have also other interesting properties.

In this paper we consider tree-like partial cubes and their generalizations. In the next section we introduce tree-like partial Hamming graphs and recall some well-known definitions and results. We follow with a section in which we detect a mistake in the proof of the result from [8] about dismantlability of the cube graph of a tree-like partial cube. We also present a counterexample of the assertion from [8] that convex subgraphs of a tree-like partial cubes are tree-like partial cubes. We continue with a section in which we extend some results on tree-like partial cubes to a bigger class of tree-like partial Hamming graphs. In particular we show that Hamming graphs are the only regular tree-like partial Hamming graphs and that any gated subgraph of a graph from this class is also in this class, which implies a characterization of tree-like partial Hamming graphs. Finally we prove a result about dismantlability of the intersection graph of maximal Hamming graph s of a tree-like partial Hamming graph which gives a corrected proof of the result from [8] about dismantlability of the cube graph of a tree-like partial cube.

2. Notation and Preliminary Results

All graphs $G = (V, E)$ occurring in this paper are undirected and without loops or multiple edges. The *distance* $d(u, v) = d_G(u, v)$ between two vertices u and v

is the length of a shortest (u,v)-path, and the *interval* $I(u,v)$ between u and v consists of all vertices on shortest (u,v)-paths, that is, of all vertices (metrically) *between* u and v: $I(u,v) = \{x \in V : d(u,x) + d(x,v) = d(u,v)\}$. For a set U of vertices of a graph G we denote with $\langle U \rangle$ the subgraph of G induced with the vertices of U. A subgraph H of G is called *isometric* if $d_H(u,v) = d_G(u,v)$ for all $u, v \in V(H)$. An induced subgraph of G (or the corresponding vertex set) is called *convex* if it includes the interval of G between any pair of its vertices. An induced subgraph H of a graph G is said to be *gated* if for every vertex x outside H there exists a vertex x' (the *gate* of x) in H such that each vertex y of H is connected with x by a shortest path passing through the gate x' (i.e., $x' \in I(x,y)$). Clearly gated subgraphs are convex and convex subgraphs are isometric.

The *Cartesian product* [15] $G = G_1 \Box \cdots \Box G_n$ of n graphs $G_1, \ldots, G_n$ has the n-tuples $(x_1, \ldots, x_n)$ as its vertices (with vertex x_i from G_i) and an edge between two vertices $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ if and only if, for some i, the vertices x_i and y_i are adjacent in G_i, and $x_j = y_j$ for the remaining $j \neq i$. The subgraph G_i^u induced by all vertices that differ from a given vertex u only in the ith coordinate is isomorphic to G_i and called the G_i-layer through u. The Cartesian product of k copies of K_2 is a *hypercube* or *k-cube* Q_k. If all the factors in a Cartesian product are complete graphs then G is called *Hamming graph.* Isometric subgraphs of hypercubes are called *partial cubes* and isometric subgraphs of Hamming graphs are *partial Hamming graphs.*

A graph G is a *median graph* if there exists a unique vertex x to every triple of vertices u, v, and w such that x lies simultaneously on a shortest u,v-path, a shortest u,w-path, and a shortest w,v-path. Median graphs are partial cubes, cf. [19, 15].

Binary expansion was first defined in [17] and a generalization of binary expansion using more covering sets was first introduced in [19]. We will use the definition of general expansion introduced by Chepoi [10] in the following way.

Let G be a connected graph and let $W_1, W_2, \ldots, W_n$ be subsets of $V(G)$ such that:

1. $W_i \cap W_j \neq \emptyset$ for all $i, j \in \{1, \ldots, n\}$;
2. $\bigcup_{i=1}^{n} W_i = V(G)$;
3. there are no edges between sets $W_i \setminus W_j$ and $W_j \setminus W_i$ for all $i, j \in \{1, \ldots, n\}$;
4. subgraphs $\langle W_i \rangle, \langle W_i \cup W_j \rangle$ are isometric in G for all $i, j \in \{1, \ldots, n\}$.

Then to each vertex $x \in V(G)$ we associate a set $\{i_1, i_2, \ldots, i_t\}$ of all indices i_j, where $x \in W_{i_j}$. A graph G' is called an expansion of G relative to the sets $W_1, W_2, \ldots, W_n$ if it is obtained from G in the following way:

1. replace each vertex x of G with a clique with vertices $x_{i_1}, x_{i_2}, \ldots, x_{i_t}$;
2. if an index i_s belongs to both sets $\{i_1, \ldots, i_t\}, \{i'_1, \ldots, i'_l\}$ corresponding to adjacent vertices x and y in G then let $x_{i_s} y_{i_s} \in E(G')$.

If $U = W_i \cap W_j$ is convex in G for all $i, j \in \{1, \ldots, n\}$, we speak of a *convex expansion* and if the intersection is isometric in G, then the expansion is called *isometric*. Contraction is the operation inverse to the expansion. If $n = 2$ then the expansion is called *binary expansion*.

Let U be an isometric subset of a graph G and $n \geq 2$. If $W_1 = V(G), W_2 = W_3 = \cdots = W_n = U$ then the expansion is called *peripheral expansion* of G along U (see Figure 1). Peripheral expansion was first introduced in [20] under the name *extremal expansion*. In this case G' consists of the union of graphs induced by $V(G), \underbrace{U, \ldots, U}_{n-1}$ where the copies of U (one such copy is contained also in $V(G)$) induce a subgraph isomorphic to $K_n \Box U$. We say that a graph G is a *tree-like partial Hamming graph* if it can be obtained from K_1 by a sequence of peripheral expansions. If $n = 2$ in each step of the expansion procedure then G is called a *tree-like partial cube* introduced in [8]. Thus every tree-like partial cube is also a tree-like partial Hamming graph.

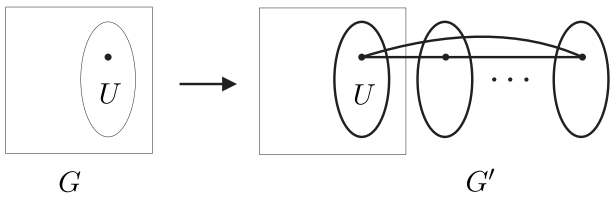

Figure 1. Peripheral expansion of G along U.

Partial cubes were characterized as graphs that can be obtained from K_1 by a sequence of binary expansions [10] and median graphs are graphs that can be obtained from K_1 by a sequence of binary convex expansions [17, 19]. Moreover, by a result of Mulder [20], these expansions can be assumed to be peripheral. Hence, by definition, every median graph is also a tree-like partial cube and every tree-like partial cube is also a partial cube. From the result of Chepoi [10], who proved that partial Hamming graphs are exactly the graphs that can be obtained from K_1 by a sequence of expansions, it follows that every tree-like partial Hamming graph is also a partial Hamming graph.

Let $G = (V, E)$ be a connected graph and ab an edge of G. Then we use the following notation:

$$W_{ab} = \{w \in V \,:\, d_G(a, w) < d_G(b, w)\},$$

$$U_{ab} = \{w \in W_{ab} \,:\, w \text{ has a neighbor in } W_{ba}\},$$

$F_{ab} = \{e \in E : e \text{ is an edge between } W_{ab} \text{ and } W_{ba}\}$.

As in [23] we denote for a subgraph H of a graph G,

$$W(H) = \{x \in V(G) : \text{ for each } a \in H, d(a,x) = d(H,x)\}.$$

Note that in bipartite graphs W_{ab} and W_{ba} are disjoint, $V = W_{ab} \cup W_{ba}$ and $W(\langle\{a,b\}\rangle) = \emptyset$ for any edge ab in G.

A graph G is an *amalgam* of two subgraphs G' and G'' if $G' \cup G'' = G, G' \cap G'' \neq \emptyset$, and there are no edges between $G' \setminus G''$ and $G'' \setminus G'$. We also say that G is obtained by an *amalgamation* along the common subgraph $G' \cap G''$ of G' and G''. The amalgamation is called *isometric* if the intersection $G' \cap G''$ is an isometric subgraph of G' and G''.

A subgraph V' of G is called *peripheral* if there exist graphs G', V, U such that G is an isometric amalgam of G' and V along U, where $V \cong K_n \Box U$ for some $n \geq 2$ and $V' = V \setminus U$. It is clear that $V' \cong K_{n-1} \Box U$. The corresponding vertex set of V' is called *periphery*. Peripheral subgraphs were first introduced in [20] under the name *extremal subgraphs*. A peripheral subgraph was also used by Brešar [6], where the amalgamation was gated instead of isometric. To simplify the notation let U denote also the corresponding vertex set of U and let V' denote also the corresponding vertex set of V'.

Every tree-like partial Hamming graph G can be obtained with an expansion procedure. Therefore we will use the following notation. Let G be obtained by peripheral expansion from a tree-like partial Hamming graph G' along U and let V' be the subgraph of G obtained in this expansion step. Then G is also isometric amalgam of G' and the graph induced with the vertices of $V' \cup U$ along U. Since $V = K_n \Box U$ and $V' = V \setminus U$, V' is peripheral subgraph of G. Thus every tree-like partial Hamming graph contains a periphery. Note also that $U = U_{ab}$ and $\langle U_{ba} \cup W(\langle\{a,b\}\rangle)\rangle = V'$ for any edge ab between G' and V'.

3. Tree-like Partial Cubes

Here is the main characterization of tree-like partial cubes proved in [8].

Theorem 1 [8]**.** *A partial cube G is tree-like if and only if every gated subgraph of G contains a periphery.*

The authors of [8] remarked that Theorem 1 implies that convex subgraphs of tree-like partial cubes are tree-like partial cubes. We claim that this is not always true. Indeed if H is convex subgraph of G then a gated subgraph of H is not necessary gated in G. We reject the result also with the counterexample depicted on Figure 2, where the outer six-cycle (periphery U_{ab}) is convex but it is not a tree-like partial cube.

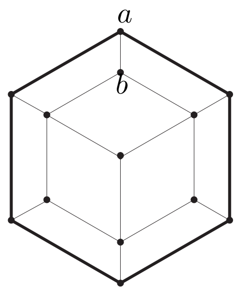

Figure 2. Tree-like partial cube with convex C_6.

We continue with pointing to an error in the proof of the following theorem from [8].

Theorem 2 [8]. *Every weak retract of a tree-like partial cube is a tree-like partial cube.*

In the proof of this theorem the authors used that a periphery U of a tree-like partial cube G is a tree-like partial cube which is not always true. Furthermore let u and x be two adjacent vertices of a periphery U of a tree-like partial cube G and let v and y be their unique neighbors in $G \setminus U$, respectively. In the proof of Theorem 2 the authors also claimed that the subgraph of G induced by $G \setminus (W_{vu} \cap W_{vy})$ is a tree-like partial cube which is again not necessarily true. A counterexample is depicted on Figure 3. Therefore the question is whether weak retracts of tree-like partial cubes are tree-like partial cubes?

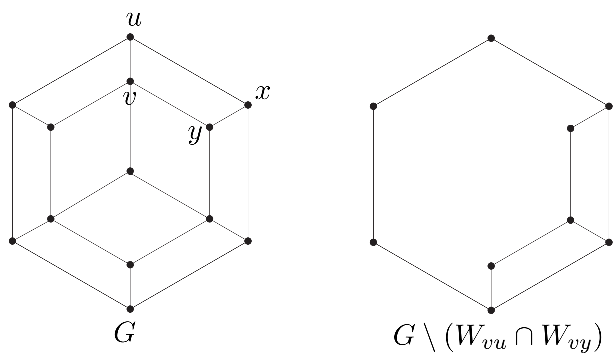

Figure 3. Tree-like partial cube G, two adjacent vertices u and x of a periphery U of G (the outer C_6) and a subgraph of G induced by $G \setminus (W_{vu} \cap W_{vy})$ which is not a tree-like partial cube (it contains a gated subgraph (the outer C_6) without a periphery).

On the other hand, if H is a gated subgraph of G and H' is a gated subgraph of H then H' is gated in G. Thus Theorem 1 directly implies that gated subgraphs of tree-like partial cubes are tree-like partial cubes.

Cube graphs are the intersection graphs of maximal hypercubes. The intersection graph of maximal Hamming graphs of G is a graph H, in symbols $H = Q(G)$, in which the vertices are the maximal Hamming subgraphs of G and two vertices in H are adjacent whenever the corresponding Hamming graphs in G intersect. Note that the only Hamming graphs in partial cubes are hypercubes. Thus the intersection graph of maximal Hamming graphs of a partial cube G is exactly the cube graph of G. Furthermore, for a partial cube G, let G^{Δ} denote the graph obtained from a graph G that has the same vertex set as G and in that two vertices are adjacent whenever they are in the same hypercube of G [5]. The *clique graph* of a graph G is the intersection graph of maximal cliques in G.

Dismantlable graphs are defined by an elimination procedure, that is a generalization of the elimination of simplicial vertices in chordal graphs. We say that a vertex u in a graph G is *dominated* by its neighbor v if all neighbors of u except v are also neighbors of v. If G can be reduced to the one-vertex graph by successive removal of dominated vertices then G is called a *dismantlable graph.* Dismantlable graphs were investigated in [2, 21, 9].

The authors of [8] proved that the cube graph $Q(G)$ of a tree-like partial cube G is dismantlable. They used the argument that the cube graph of a tree-like partial cube G coincides with the clique graph of G^{Δ}, which is not true. For example, let $G = Q_3^-$, which is a graph obtained from Q_3 with the removal of one vertex. Then the cube graph of Q_3^- is isomorphic to K_3 and the clique graph of $(Q_3^-)^{\Delta}$ is isomorphic to K_4 (see Figure 4). We give a new proof of this result using different accession in Section 4.

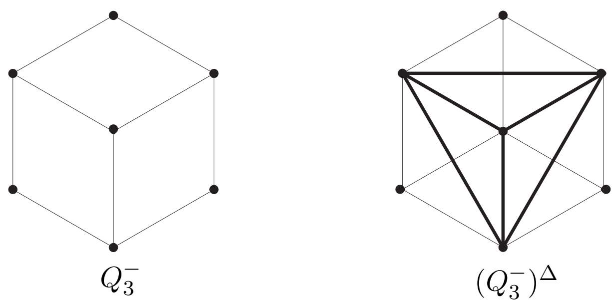

Figure 4. Graphs Q_3^- and $(Q_3^-)^{\Delta}$.

4. Tree-like Partial Hamming Graphs

In this section we list some properties of tree-like partial Hamming graphs which generalize the results on tree-like partial cubes. In particular we characterize tree-like partial Hamming graphs and show that the intersection graph of maximal Hamming graphs of a tree-like partial Hamming graph is dismantlable, which corrects and generalizes the proof about dismantlability of the cube graph of a tree-like partial cube from [8]. From the latter result we deduce that every tree-like partial Hamming graph G contains a Hamming graph that is invariant under every automorphism of G.

There are many properties of tree-like partial Hamming graphs that can be extended from tree-like partial cubes. Here is the characterization of regular tree-like partial Hamming graphs.

Theorem 3. *Regular tree-like partial Hamming graphs are precisely Hamming graphs.*

Proof. Let G be a regular tree-like partial Hamming graph and let G be obtained by peripheral expansion along U from a tree-like partial Hamming graph G'. Let V' be the subgraph of G obtained in the last expansion step, that is, V' a is peripheral subgraph of G, and let V be the subgraph of G induced with $U \cup V'$. Furthermore, let n be the number of copies of U in G, that is $V = K_n \Box U$. Since G is regular and there is no edge from V' to $G' \setminus U$ every vertex x from V' has the same degree as its unique neighbor x' in U. Therefore all vertices from U have degrees $k + n - 1$, where $k = deg_U(x')$. Thus the vertices from U have no neighbors in $G' \setminus U$ which means that $G = K_n \Box U$. Since G is regular and every vertex from U has $n - 1$ neighbors in $G \setminus U$, also U is a regular tree-like partial Hamming graph. Using induction assumption we get that U is a Hamming graph and thus so is G. ■

This result clearly implies the previously-known result for tree-like partial cubes.

Corollary 4. *Regular tree-like partial cubes are hypercubes.*

For the next theorem we need the following well-known result.

Lemma 5 [22]**.** *Let $G = G_1 \Box G_2$ be a Cartesian product of connected graphs. Then H is gated in G if and only if $H = H_1 \Box H_2$, where H_1 (resp. H_2) is gated in G_1 (resp. G_2).*

We already mentioned that gated subgraph of a tree-like partial cube is a tree-like partial cube. The extension of this result to the tree-like partial Hamming graphs gives a useful characterization of these graphs.

Theorem 6. *Every gated subgraph of a tree-like partial Hamming graph is a tree-like partial Hamming graph.*

Proof. The proof is by induction on the number of vertices of a tree-like partial Hamming graph. Let G_1 be a gated subgraph of a tree-like partial Hamming graph G and let G be obtained by a peripheral expansion along U from a tree-like partial Hamming graph G'. Let V' be the subgraph of G obtained in the last expansion step and let V be the subgraph of G induced with the vertices of $U \cup V'$, that is $V \cong K_n \Box U$. If G_1 is contained in G' we infer from the isometry of G' that G_1 is also gated in G', which is a smaller tree-like partial Hamming graph. By induction assumption G_1 is a tree-like partial Hamming graph.

Assume now that $G_1 \cap G' \neq \emptyset$ and $G_1 \cap V' \neq \emptyset$. To complete the proof of this case we need the following two claims.

Claim 7. *$G_1 \cap V$ is gated in V.*

Proof. Since V is an isometric subgraph of G it is enough to see that the gate of $v \in V$ in G_1 is from V. Let v be an arbitrary vertex from $V \setminus G_1$ and suppose that the gate g of v in G_1 is from $G \setminus V$. First let $v \in V'$. Since $G_1 \cap V' \neq \emptyset$ there exists $x \in G_1 \cap V'$. Clearly g cannot lie on the interval between v and x, which gives a contradiction. Thus we may assume that $v \in U$. Since there are at least two U-layers of V that have nonempty intersection with G_1, there exists a U-layer U_1 of V different from U which has nonempty intersection with G_1. Let $y'' \in G_1 \cap U_1$ and let y be the copy of v in U_1. Since $g \in G \setminus V$ is the gate of v in G_1, $y \notin G_1$. Let y' be the gate of y in G_1. Then $y' \in U_1$, otherwise y' cannot lie on a shortest y, y''-path of G. Now let v' be the copy of y' in U and note that $v' \in G_1$. Indeed if $v' \notin G_1$ then y' is the gate of v' in G_1 which implies that $G_1 \cap U = \emptyset$, a contradiction. Thus $v' \in G_1$. Since g is the gate of v in G_1, $d(y, y') = d(v, v') = d(v, g) + d(g, v')$ and since y' is the gate of y in G_1, $d(y, g) = d(y, y') + d(y', g) > d(v, v')$. On the other hand we can find y, g-path in G of length $1 + d(v, g) \leq d(v, v')$, a contradiction. □

Claim 8. *$G_1 \cap G'$ is gated in G.*

Proof. Note that if y is a gate for $x \in G \setminus G'$ in G_1 then the unique neighbor y' of y in U is a gate for x in $G_1 \cap G'$. □

G' is an isometric subgraph of G and $G_1 \cap G'$ is gated in G, $G_1 \cap G'$ is also gated in G' and hence it is a tree-like partial Hamming graph by the induction assumption. From the structure of $G_1 \cap V$ it follows that G_1 is obtained from $G_1 \cap G'$ by peripheral expansion along $G_1 \cap U$ which implies that G_1 is a tree-like partial Hamming graph.

Finally let G_1 be contained in V', that is G_1 is contained in a Cartesian product $K_{n-1} \Box U$. Using Lemma 5 we get that $G_1 = H \Box H'$, where H is gated in K_{n-1} and H' is isomorphic to gated subgraph of U. Therefore H is either K_1 or K_{n-1}. Since G_1 is gated in G it is clear that $H \neq K_{n-1}$ (unless $n = 2$), otherwise there is no gate of a vertex $x \in U \cap H'$ in G_1. Thus $G_1 = K_1 \Box H'$ is contained in one U-layer of V'. Now we consider the subgraph G_2 of $U \subseteq G'$ isomorphic to G_1, induced by vertices that correspond to the copy of G_1 in U. We claim that G_2 is gated in G'. Indeed, the distance from any vertex x of G' to a vertex of G_1

is exactly 1 plus the distance from x to the corresponding vertex of G_2. Hence the gatedness of G_2 clearly follows from the gatedness of G_1. Using the induction assumption we get that G_2 is a tree-like partial Hamming graph and therefore so is G_1. ∎

Theorem 9. *A partial Hamming graph G is tree-like if and only if every gated subgraph of G contains a periphery.*

Proof. Let G be a tree-like partial Hamming graph and let G_1 be an arbitrary gated subgraph of G. Then it follows from Theorem 6 that G_1 is a tree-like partial Hamming graph and hence it contains a periphery.

For the converse suppose that G is a partial Hamming graph in which every gated subgraph contains a periphery. Since G is gated in G it contains a periphery and thus one can obtain G by a peripheral expansion from a graph G'. If G' would contain a gated subgraph G_1 without periphery then G_1 would be gated also in G. By induction on the number of vertices we get that G' is a tree-like partial Hamming graph, and thus so is G. ∎

Corollary 10. *For any periphery U of a tree-like partial Hamming graph G, $G \setminus U$ is a tree-like partial Hamming graph.*

Our next goal is to prove that the intersection graph of maximal Hamming graphs of any tree-like partial Hamming graph is dismantlable.

In the rest of the paper we will use the following notation. Let G be obtained by peripheral expansion from a tree-like partial Hamming graph G' along U. Then the graph obtained in this expansion step is a peripheral subgraph V' of G isomorphic to $K_{n-1}\Box U$ for some $n \geq 2$. We denote with V the subgraph of G induced with the vertices of $U \cup V'$, that is $V \cong K_n \Box U$.

Note that in Cartesian products complete graphs lie in layers. Therefore the proof of the following result is obvious and thus we skip it.

Lemma 11. *Let H be a Hamming subgraph of a Cartesian product $G = U \Box K_n$. Then there exists a Hamming graph H' in U such that $H \cong K_m \Box H'$ for some $m \leq n$ and every U-layer of G has either empty intersection with H or the intersection is isomorphic to H'.*

Lemma 12. *Let G be a tree-like partial Hamming graph and let H be a Hamming subgraph of G having nonempty intersection with $G \setminus G'$. Then H is contained in V.*

Proof. Let $H = K_{n_1} \Box \cdots \Box K_{n_k}$ and suppose that $H \nsubseteq G \setminus G'$, which means that H intersects $G \setminus G'$ and G'. Let $H_1 = K_{n_1} \Box \cdots \Box K_{n_{k-1}}$, that is $H = H_1 \Box K_{n_k}$ and let H' be one H_1-layer of H, which means that H' is a subgraph of H. First note that every vertex $x \in G \setminus G'$ has exactly one neighbor x' in G' and x' is contained in U.

First let $H' \subseteq G'$. Since $H \cap (G \setminus G') \neq \emptyset$ there exists $x \in H \cap (G \setminus G')$ which is contained in $H \setminus H'$, since $H' \subseteq G'$. Let x' be the unique neighbor of x in $H' \subseteq G'$. Since x has just one neighbor in G', all the neighbors of x in $H \setminus H'$ are from $G \setminus G'$. Every such vertex has a unique neighbor in $H' \cap G'$. We conclude that $H \setminus H'$ is contained in $G \setminus G'$ and thus H' is contained in U which implies that H is contained in V.

Finally let H' intersect $G \setminus G'$. Then using the induction, we get $H' \subseteq V$. If $H' \subseteq G \setminus G'$, then it is clear that H is contained in V. Thus we may assume that $H' \cap G', H' \cap (G \setminus G') \neq \emptyset$. From Lemma 11 it follows that there exists a Hamming graph H'' in U such that $H' \cong K_m \square H''$ for some $m \leq n$ and every U-layer of V has either empty intersection with H' or the intersection is isomorphic to H''. For the purpose of contradiction suppose that there exists $z \in H \setminus H'$ such that $z \notin V$. Let H_i' be the H_1-layer of $H = H_1 \square K_{n_k}$ that contains z and let z' be the unique neighbor of z in H'. Since z is adjacent to $z' \in H' \subset V$ and $z \notin V$ it is clear that $z' \in U$. From the structure of H' ($H' \cong K_m \square H''$) and since $H' \cap (G \setminus G') \neq \emptyset$, there exists $y' \in H' \cap (G \setminus G')$ that lies in the same K_n-layer of V as z'. Let y be the neighbor of y' in $H_i' \subseteq H \setminus H'$ and thus y is adjacent to z. Since every vertex from $G \setminus G'$ has just one neighbor in G', we get that $y \in G \setminus G'$, which contradicts the fact that y is adjacent to $z \in G \setminus V$. Thus H is contained in V. ■

From Lemma 11 and Lemma 12 we get the following result.

Corollary 13. *Let G be a tree-like partial Hamming graph and let H be a Hamming subgraph of G having nonempty intersection with $G \setminus G'$. Then there exists Hamming graph H' in U such that $H \cong K_m \square H'$ for some $m \leq n$ and every U-layer of V has either empty intersection with H or the intersection is isomorphic to H'.*

Let $\mathcal{Q}^{(G)}$ be the function which maps maximal Hamming subgraphs of G to vertices of $Q(G)$, such that two vertices x and y of $Q(G)$ are adjacent if and only if the Hamming graphs $(\mathcal{Q}^{(G)})^{-1}(x)$ and $(\mathcal{Q}^{(G)})^{-1}(y)$ have nonempty intersection in G. For every Hamming graph H of G we denote the image of H with respect to $\mathcal{Q}^{(G)}$ with $x_H^{(G)}$, that is $\mathcal{Q}^{(G)}(H) = x_H^{(G)}$.

Theorem 14. *For any tree-like partial Hamming graph G, the graph $Q(G)$ is dismantlable.*

Proof. The proof is by induction on the number of vertices of a tree-like partial Hamming graph. Let U, V, G' be the subgraphs of G defined above. Then G' is a tree-like partial Hamming graph and hence $Q(G')$ is dismantlable by induction assumption. Clearly $Q(G')$ is induced subgraph of $Q(G)$. Therefore, to complete the proof, it is enough to see that the vertices of $Q(G) \setminus Q(G')$ are dominated in $Q(G)$. Note that every vertex of $Q(G) \setminus Q(G')$ corresponds to the maximal Hamming subgraph H of G such that $H \cap G'$ ($= H \cap U$, using Lemma 12) is not a maximal Hamming graph of G' and $H \cap (G \setminus G') \neq \emptyset$. We will prove that every such vertex of $Q(G)$ is dominated in $Q(G)$. Therefore let H be such maximal Hamming graph of G, that is $x_H^{(G)} \in Q(G) \setminus Q(G')$. Since H is a maximal, $H \cap U \neq \emptyset$ and it follows from Corollary 13 that $H' = H \cap G'$ is a Hamming graph such that $H \cong K_n \Box H'$. Since $x_H^{(G)} \in Q(G) \setminus Q(G')$, H' is not a maximal Hamming graph of G'. Let K be a maximal Hamming graph in G', containing H'. Then, using Lemma 12, we get that K is also maximal Hamming graph of G. Let $H_1, \ldots, H_n$ be maximal Hamming subgraphs of G which have nonempty intersection with $(G \setminus G') \cap H$ and let $H_i' = H_i \cap U = H_i \cap G'$, where the last equality holds because of Lemma 12. Corollary 13 implies that $H_i' \cap H' \neq \emptyset$ for every $i \in \{1, \ldots, n\}$. Furthermore let $K_1, \ldots, K_m$ be maximal Hamming graphs in G' having nonempty intersection with H', such that $K_j \neq H_i'$ for all $j \in \{1, \ldots, m\}$, $i \in \{1, \ldots, n\}$ and $K_j \neq K$ for all $j \in \{1, \ldots, m\}$. Clearly these Hamming graphs are also maximal in G and the neighbors of the vertex $x_H^{(G)}$ in $Q(G)$ are exactly the vertices $x_{H_1}^{(G)}, \ldots, x_{H_n}^{(G)}, x_{K_1}^{(G)}, \ldots, x_{K_m}^{(G)}$ and $x_K^{(G)}$, where the last vertex is also adjacent to all previous neighbors of $x_H^{(G)}$. Therefore the vertex $x_H^{(G)}$ is dominated by its neighbor $x_K^{(G)}$ in $Q(G)$, which completes the proof. ■

Corollary 15. *Let G be a tree-like partial cube. Then the cube graph of G is dismantlable.*

Dismantlability of the intersection graph of maximal Hamming graphs of a tree-like partial Hamming graph implies the following result.

Corollary 16. *Let G be a tree-like partial Hamming graph. Then G contains a Hamming graph that is invariant under every automorphism of G.*

Proof. First note that every automorphism of a tree-like partial Hamming graph G induces an automorphism of $Q(G)$. Observe also that automorphisms of dismantlable graphs always fix a complete subgraph (see also [8]) and thus it follows from Theorem 14 that $Q(G)$ contains a complete subgraph K that is invariant under every automorphism of $Q(G)$. Since vertices of K are pairwise intersecting Hamming graphs of G, their intersection is a Hamming graph which is invariant under all automorphisms of G. ■

5. Concluding Remarks

This paper has much in common with [8]. We explain that convex subgraphs of tree-like partial cubes are not necessary tree-like partial cubes as the authors from [8] asserted. Moreover we correct the proof of the theorem from [8] which says that the cube graph of a tree-like partial cube is dismantlable. Beside that we also generalize the mentioned result. Finally we point to a gap in the proof of the theorem from [8] that weak retracts of tree-like partial cubes are tree-like partial cubes, but it remains open whether the result holds or not.

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