Discussiones Mathematicae
Graph Theory 34 (2014) 127–136
doi:10.7151/dmgt.1724

ON THE NUMBERS OF CUT-VERTICES AND END-BLOCKS IN 4-REGULAR GRAPHS

Dingguo Wang[2,3] and Erfang Shan[1,2]

[1]*School of Management, Shanghai University, Shanghai 200444, China*
[2]*Department of Mathematics, Shanghai University, Shanghai 200444, China*
[3]*College of Mathematics Science, Chongqing Normal University*
Chongqing 400047, China

e-mail: efshan@shu.edu.cn

Abstract

A *cut-vertex* in a graph G is a vertex whose removal increases the number of connected components of G. An *end-block* of G is a block with a single cut-vertex. In this paper we establish upper bounds on the numbers of end-blocks and cut-vertices in a 4-regular graph G and claw-free 4-regular graphs. We characterize the extremal graphs achieving these bounds.

Keywords: 4-regular graph, claw-free, cut-vertices, end-blocks.

2010 Mathematics Subject Classification: 05C35.

1. Introduction

It is well known that a connected graph with n vertices contains at most $n-2$ cut-vertices and at most $n-1$ cut-edges. The unique connected graphs with $n-2$ cut-vertices are paths, while trees are unique connected graphs with $n-1$ cut-edges. However, if additional constraints on graphs are given, then the problem of determining the maximum number of cut-vertices or cut-edges becomes nontrivial. Many interesting results were obtained for the case of regular graphs. Rao [6, 7] determined the bounds on the number of cut-vertices and the number of cut-edges in a graph of order n and size m. These problems with additional constraints on the degree such as $\Delta(G) \leq d$ and $\delta(G) \geq d$ were also considered in Rao [7, 8]. For a connected graph G of order n and $\delta(G) \geq d$, the maximum number of cut-vertices was determined in Clark and Entringer [3] for $d \geq 5$ and in Albertson and Berman [1] for $d \geq 2$. Nirmala and Rao [4] obtained the upper bounds on the number of cut-vertices in a d-regular graph with odd $d \geq 5$ and even $d \geq 6$. In [5] and [9], the authors determined the maximum number of cut-edges in a connected d-regular graph of order n.

Although there have been many results on the problem for regular graphs, the upper bounds on the number of cut-vertices have not been considered explicitly for 4-regular graphs. In order to investigate the maximum number of cut-vertices in 4-regular graphs, we need to consider the maximum number of their end-blocks. In this paper we present the upper bounds on the numbers of end-blocks and cut-vertices for 4-regular graphs and claw-free 4-regular graphs, respectively, and we characterize the extremal graphs achieving these bounds.

2. Basic Notation and Terminology

Let $G = (V(G), E(G))$ be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$ of order $|V(G)|$ and size $|E(G)|$. The *open neighborhood* of a vertex v is $N(v) = \{u : uv \in E\}$ and the *closed neighborhood* of v is $N[v] = \{u : uv \in E\} \cup \{v\}$. The degree $d_G(v)$ of a vertex v, or simply $d(v)$, is the number of edges incident to v, that is, $d_G(v) = |N(v)|$. The maximum and minimum degrees of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A graph G is said to be *k-regular* if $d_G(v) = k$ for $v \in V(G)$. For a subset $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$. With K_n and C_n we denote the n-vertex complete graph and n-vertex cycle graph. With $K_{m,n}$ we denote a complete bipartite graph with partitions of size m and n. The graph $K_{1,3}$ is also called a *claw* and K_3 a *triangle*. For a given graph F, we say that a graph G is *F-free* if it does not contain F as an induced subgraph. In particular, if G contains no $K_{1,3}$ as an induced subgraph, we say that G is a $K_{1,3}$-free graph or *claw-free* graph.

For a given graph G, a *cut-vertex* of G is a vertex whose removal increases the number of connected components in G. *Cut-edge* is defined in a similar way. With $c(G)$ we denote the number of cut-vertices in G. A *block* of G is a maximal subgraph without a cut-vertex. A block with a single cut-vertex is called an *end-block* and its cut-vertex is called an *end-vertex*. With $eb(G)$ we denote the number of end-blocks in G. For other graph theoretic notation and terminology, we follow [2].

3. 4-regular Graphs

In this section, we present a sharp upper bound on the number of end-blocks and cut-vertices in a connected 4-regular graph, respectively. Furthermore, we characterize the extremal graphs achieving these bounds.

Before we present the main results, we will need the following lemma.

Lemma 1. *If G is a 4-regular graph, then G has no cut-edge.*

Proof. If G has a cut-edge, deleting it leaves two induced subgraphs whose degree sum is odd. This is impossible, since the degree sum in every graph is even. ∎

For characterizing the extremal graphs achieving these bounds, we need the following constructions.

A *cactus graph* is a connected graph in which any two cycles have at most one vertex in common. Equivalently, every block is an edge or a cycle. Let F_6 be the graph obtained from the complete graph K_5 by subdividing one edge. Let I_{11} denote the graph obtained from the disjoint union of two copies of F_6 by identifying their two end-vertices (i.e., vertices of degree 2). Clearly, I_{11} is a 4-regular graph of order 11.

Construction 1. Let

$$\mathcal{H} = \{H : H \text{ is a cactus graph in which each block is a triangle and } \Delta(H) \leq 4\}.$$

Let $\widetilde{\mathcal{H}}$ be the family of 4-regular graphs obtained from disjoint union of any graph H in $\mathcal{H}$ and copies of F_6 by identifying each degree-2 vertex of H with the end-vertex of an F_6. Further, let

$$\mathcal{G} = \{G : G = I_{11} \text{ or } G \in \widetilde{\mathcal{H}}\}.$$

Construction 2. Let $\mathcal{M}$ be the family of 4-regular graphs obtained from the disjoin union of the cycle C_k ($k \geq 3$) and k copies of F_6 by identifying each vertex in C_k with the end-vertex of a copy of F_6.

It is easy to see that F_6 has the minimum number of vertices among all graphs in which one vertex is of degree 2, while the other vertices are of degree 4. Thus, F_6 is the smallest possible end-block in a 4-regular graph.

Theorem 2. *If G is a connected 4-regular graph of order $n \geq 12$, then $eb(G) \leq n/6$ with equality if and only if $G \in \mathcal{M}$.*

Proof. If G contains no cut-vertex, then $eb(G) = 0$ and the assertion holds. Therefore, we may assume that G contains at least one cut-vertex. This implies that G contains at least two end-blocks. Let G' be the graph obtained from G by contracting each end-block to a single vertex. Obviously, G' is connected. By Lemma 1, each end-block of G is contracted to a vertex of degree 2 in G'. Thus $\delta(G') \geq 2$. Let $n' = |V(G')|$ and $m' = |E(G')|$. Hence $m' \geq n'$. The degree-sum formula implies that

(1) $\quad 4n' - 2eb(G) = 2m' \geq 2n'.$

Thus

(2) $\quad eb(G) \leq n'.$

As mentioned earlier, each end-block has at least 6 vertices, so

(3) $n' \leq n - 5eb(G)$.

Combining the inequalities (2), (3), we have $eb(G) \leq n' \leq n - 5eb(G)$, so $eb(G) \leq n/6$.

We next show that $eb(G) = n/6$ if and only if $G \in \mathcal{M}$ for a connected 4-regular graph of order n.

Suppose $G \in \mathcal{M}$. Then there exists an integer $k \geq 3$ such that G is a 4-regular graph obtained from the disjoint union of the cycle C_k ($k \geq 3$) and k copies of F_6 by identifying each vertex in C_k with the end-vertex of a copy of F_6. Thus $eb(G) = k = n/6$.

Conversely, suppose that $eb(G) = n/6$ for a connected 4-regular graph of order n. Then all the inequalities in equations (1)–(3) are equalities. Thus $n - 5eb(G) = n' = eb(G) = m'$. This implies that G' is a cycle, since $\delta(G') \geq 2$. So $G \in \mathcal{M}$. ■

Remark. For a connected 4-regular graph G of order n, if $n \leq 11$ and G has no cut-vertex, then $eb(G) = 0 \leq n/6$. But if G contains cut-vertices, then $G = I_{11}$ and $eb(G) = 2$, so the assertion is not true.

Theorem 3. *If G is a connected 4-regular graph of order $n \geq 8$, then $c(G) \leq (2n - 15)/7$. Equality holds if and only if $G \in \mathcal{G}$.*

Proof. We apply induction on n. For $c(G) = 0$, the assertion is trivial, so let $c(G) \geq 1$. Since F_6 is the smallest possible end-block in G, I_{11} is the smallest possible 4-regular graph with a single cut-vertex and so $n \geq 11$. If $c(G) = 1$, then clearly the assertion holds. Now let G be given with $n > 11$ and $c(G) \geq 2$, and assume the assertion holds for 4-regular graphs with fewer vertices.

Let v be a cut-vertex in G. By Lemma 1, $G - v$ has two connected components, denoted by G_1 and G_2. For $i = 1, 2$ let G'_i be the graph obtained from G by replacing $G[V(G_i) \cup \{v\}]$ with the graph F_6. Now, the cut-vertices of G are the cut-vertices from G_i, $i = 1, 2$, together with vertex v. Since v is a cut-vertex in both G'_1 and G'_2, and F_6 contains no cut-vertex, we have $c(G) = c(G'_1) + c(G'_2) - 1$. With $n_i = |V(G'_i)|$ for $i = 1, 2$, we have $n = n_1 + n_2 - 11$.

If neither $G[V(G_1) \cup \{v\}]$ nor $G[V(G_2) \cup \{v\}]$ is isomorphic to F_6, then G'_1 and G'_2 have fewer vertices than G. By the induction hypothesis, we have

$$c(G) = c(G'_1) + c(G'_2) - 1 \leq (2n_1 - 15)/7 + (2n_2 - 15)/7 - 1 = (2n - 15)/7,$$

and the assertion holds. Otherwise, every cut-vertex of G is an end-vertex of a copy of F_6. So each cut-vertex is the end-vertex of a unique copy of F_6. Since $c(G) \geq 2$, we have $c(G) = eb(G)$. If $c(G) = 2$, then $n \geq 16$ by Lemma 1.

Hence the assertion follows. If $c(G) \geq 3$, then $n \geq 18$. By Theorem 2, we have $c(G) = eb(G) \leq n/6 \leq (2n - 15)/7$, as desired.

We next show that $c(G) = (2n - 15)/7$ if and only if $G \in \mathcal{G}$ for a connected 4-regular graph of order $n \geq 8$.

Suppose $G \in \mathcal{G}$ and G has n vertices. We show that $c(G) = (2n - 15)/7$ by induction on n. Obviously, $n \geq 11$. If $n = 11$, then $G = I_{11}$ and the equality follows. Now let $n > 11$, and assume that the assertion holds for graphs with fewer vertices. Since $G \neq I_{11}$, $G \in \widetilde{\mathcal{H}}$. Let H be the graph obtained from G by contracting each end-block of G to a single vertex. Then $H \in \mathcal{H}$ and there exists a triangle $G[\{x,\ y,\ z\}]$ such that $d_H(x) = d_H(y) = 2$. If H is a triangle, then it is easy to check that the equality holds. Otherwise, let $H' = H - \{x, y\}$. Then $d_{H'}(z) = 2$. Let G' be the graph obtained from H' by attaching F_6 to each vertex of degree 2 of H'. Then $G' \in \mathcal{G}$, $|V(G')| = n - 7$ and $c(G') = c(G) - 2$. By the induction hypothesis, we have $c(G') = (2|V(G')| - 15)/7$. This implies that $c(G) = (2|V(G)| - 15)/7$, as desired.

Conversely, let $c(G) = (2n - 15)/7$ for a connected 4-regular graph of order $n \geq 8$. We will show that $G \in \mathcal{G}$. Note that $c(G)$ is odd. If $c(G) = 1$, then $n = 11$. Obviously, $G = I_{11} \in \mathcal{G}$. If G contains a cut-vertex that does not lie in a copy of F_6, then the equality holds for both G_1' and G_2'. By the induction hypothesis, $G_1', G_2' \in \mathcal{G}$. This implies that $G \in \mathcal{G}$. If $c(G) = eb(G)$, then $n/6 = (2n - 15)/7$, and so $n = 18$. In this case, the graph G is obtained from the disjoint union of a triangle and three copies of F_6 by identifying each degree-2 vertex of H with the degree-2 vertex (end-vertex) of F_6. Clearly $G \in \mathcal{G}$. ■

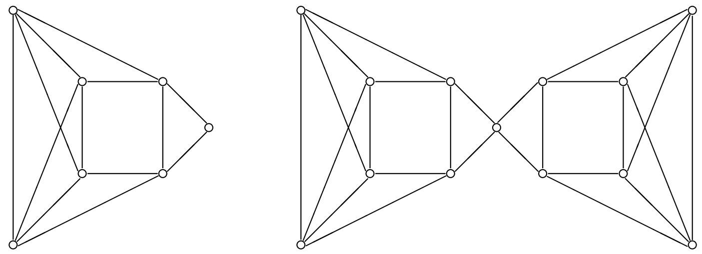
4. Claw-free 4-regular Graphs

In this section we discuss analogous results for a connected claw-free 4-regular graph. We establish an upper bound on the numbers of end-blocks and cut-vertices for a connected claw-free 4-regular graph, respectively. Moreover, we characterize the extremal graphs achieving these bounds.

As before, for characterizing the extremal graphs achieving these bounds, we give the following construction.

Construction 3. The graphs F_7 and I_{13} are exhibited in Figure 1. $\mathcal{H}$ is constructed as described in Construction 1. Let $\widetilde{\mathcal{H}}_1$ be the family of the 4-regular graphs obtained from disjoint union of any graph H in $\mathcal{H}$ and copies of F_7 by identifying each degree-2 vertex of H with the end-vertex of an F_7. Further, let

$$\mathcal{G}' = \{G : G = I_{13} \text{ or } G \in \widetilde{\mathcal{H}}_1\}.$$



(a) F_7 (b) I_{13}

Figure 1. F_7 and I_{13}

Lemma 4. *For a connected claw-free* 4*-regular graph* G *containing at least one cut-vertex,* F_7 *is the end-block of* G *with the smallest number of the vertices.*

Proof. Let B be an end-block of G and v an end-vertex lying in B. By Lemma 1 and the claw-freeness of G, v must be the common vertex of two triangles whose edges are disjoint. This implies that $B-v$ contains two adjacent degree-3 vertices, while every other vertex has degree 4. It is easy to see that $|V(B-v)| \geq 6$. Hence $|V(B)| \geq 7$. When $|V(B)| = 7$, it is not difficult to check that $B = F_7$. ∎

Theorem 5. *If* G *is a connected claw-free* 4*-regular graph of order* n*, then* $eb(G) \leq (n+3)/8$ *with equality if and only if* $G \in \mathcal{G}'$*.*

Proof. Let G' be the graph obtained from G by contracting each end-block of G to a single vertex. Obviously, G' is connected and each end-block of G corresponds to a vertex of degree 2 in G'. Let V_2 denote the set of vertices of degree 2 of G'. Obviously, each vertex of V_2 lies in a unique triangle of G' by the claw-freeness of G. If $G'[V_2]$ is a triangle, then it is easy to verify that $G' = G'[V_2]$. Otherwise, each component of $G'[V_2]$ is either an isolated vertex or K_2.

Let $n' = |V(G')|$ and $m' = |E(G')|$. First, we have the following claim.

Claim 1. $m' \geq n' + eb(G) - 3$.

Proof. We apply induction on n'. If $n' = 1$, then $m' = 0$ and $eb(G) \leq 2$, the assertion holds. Next let $n' > 1$ and assume the assertion holds for smaller n'. We distinguish the following three cases depending on $G'[V_2]$.

Case 1. $G'[V_2]$ is a triangle. As mentioned above, we know that $G' = G'[V_2]$ is a triangle. In this case, G is the graph obtained from disjoint union of a triangle and three end-blocks by identifying each vertex of the triangle with the end-vertex of an end-block. Then $m' = 3 = n' + eb(G) - 3$, as claimed.

Case 2. $G'[V_2]$ contains a component that is isomorphic to K_2. Let $K_2 = G'[\{u,\ v\}]$ and let u, v lie in the triangle $G'[\{u,\ v,\ w\}]$ of G'. Then $d_{G'}(w) = 4$.

Let $G_1' = G' - \{u, v\}$ and let G_1 be the 4-regular graph obtained from G_1' by attaching F_7 to each vertex of degree 2 of G_1'. Let $n_1' = |V(G_1')|$ and $m_1' = |E(G_1')|$. Then $n_1' = n' - 2 < n'$, $m_1' = m' - 3$, $eb(G) = eb(G_1) + 1$. By the induction hypothesis, we have $m_1' \geq n_1' + eb(G_1) - 3$. Thus $m' \geq n' + eb(G) - 3$, as claimed.

Case 3. Each component of $G'[V_2]$ is an isolated vertex, i.e., V_2 is an independent set of G'. Choose any isolated vertex u in V_2 and let it be the end-vertex of an end-block B. Then u lies in a unique triangle, say $G'[\{u, v, w\}]$, in G'. Obviously, $d_{G'}(v) = d_{G'}(w) = 4$. We consider the following three subcases.

Subcase 3.1. $N_{G'}(v) \cap N_{G'}(w) - \{u\} = \emptyset$. Let G_1 be the graph obtained from G by removing the vertices of B, the edge vw and identifying v and w. Let G_1' be the graph obtained from G_1 by contracting each end-block to a single vertex and let $n_1' = |V(G_1')|$ and $m_1' = |E(G_1')|$. Then $n_1' = n' - 2 < n'$, $m_1' = m' - 3$, $eb(G) = eb(G_1) + 1$. By the induction hypothesis, we have $m_1' \geq n_1' + eb(G_1) - 3$. Thus $m' \geq n' + eb(G) - 3$, as claimed.

Subcase 3.2. $|N_{G'}(v) \cap N_{G'}(w) - \{u\}| = 1$. Let $G_1' = G' - u - vw$. Then v and w have degree 2 in G_1'. Now let G_1 be the 4-regular graph obtained from G_1' by attaching F_7 to each vertex of degree 2 of G_1'. Let $n_1' = |V(G_1')|$ and $m_1' = |E(G_1')|$. Then $n_1' = n' - 1 < n'$, $m_1' = m' - 3$, $eb(G) = eb(G_1) - 1$. By the induction hypothesis, we have $m_1' \geq n_1' + eb(G_1) - 3$. Thus $m' > n' + eb(G) - 3$, as claimed.

Subcase 3.3. $|(N_{G'}(v) \cap N_{G'}(w)) - \{u\}| = 2$. Let $N_{G'}(v) - \{u, w\} = N_{G'}(w) - \{u, v\} = \{x, y\}$. Then $xy \in E(G)$ by claw-freeness of G.

Suppose that $N(x) - \{v, w, y\} = N(y) - \{v, w, x\} = \{z\}$. Let G_1 be the graph obtained from G by deleting the vertices v, w, x, y of G and identifying u and z. Let G_1' be the graph obtained from G_1 by contracting each end-block of G_1 to a single vertex. Let $n_1' = |V(G_1')|$ and $m_1' = |E(G_1')|$. Then $n_1' = n' - 5 < n'$, $m_1' = m' - 10$, $eb(G) = eb(G_1)$. By the induction hypothesis, we have $m_1' \geq n_1' + eb(G_1) - 3$. Thus $m' > n' + eb(G) - 3$, as claimed.

Otherwise, we have $N(x) - \{v, w, y\} \neq N(y) - \{v, w, x\}$. Let $N(x) - \{v, w, y\} = \{z_1\}$ and $N(y) - \{v, w, x\} = \{z_2\}$. By Lemma 1, none of x, y, z_1 and z_2 is a cut-vertex of G.

If $z_1z_2 \notin E(G)$, let G_1 be the graph obtained from G by deleting the vertices of $V(B) \cup \{v, w, x, y\}$ and adding edge z_1z_2. Then G_1 is a claw-free 4-regular graph. Let G_1' be the graph obtained from G_1 by contracting each end-block of G_1 to a single vertex and let $n_1' = |V(G_1')|$ and $m_1' = |E(G_1')|$. Then $n_1' = n' - 5 < n'$, $m_1' = m' - 9$, $eb(G) = eb(G_1) + 1$. By the induction hypothesis, we have $m_1' \geq n_1' + eb(G_1) - 3$. Thus $m' \geq n' + eb(G) - 3 + 3 > n' + eb(G) - 3$, the desired claim follows.

If $z_1z_2 \in E(G)$, let G_1 be the graph obtained from G by deleting the vertices

v, w, x, y and adding edges uz_1, uz_2. Then G_1 is a claw-free 4-regular graph. Let G_1' be the graph obtained from G_1 by contracting each end-block of G_1 to a single vertex and let $n_1' = |V(G_1')|$ and $m_1' = |E(G_1')|$. Obviously, $n_1' = n' - 4 < n'$, $m_1' = m' - 8$, $eb(G) = eb(G_1)$. By applying the induction hypothesis to G_1', we have $m_1' \geq n_1' + eb(G_1) - 3$. Thus $m' > n' + eb(G) - 3$, and the claim follows immediately. □

Claim 1 and the degree-sum formula yields that

(4) $\quad 4n' - 2eb(G) = 2m' \geq 2(n' + eb(G) - 3).$

Thus

(5) $\quad 2eb(G) - 3 \leq n'.$

Since each end-block of G has at least 7 vertices, we have

(6) $\quad n' \leq n - 6eb(G).$

Combining the inequalities (5), (6), we obtain $2eb(G) - 3 \leq n' \leq n - 6eb(G)$, so $eb(G) \leq (n + 3)/8$.

We next show that if G is a connected claw-free 4-regular graph of order n, then $eb(G) = (n + 3)/8$ if and only if $G \in \mathcal{G}'$.

Suppose that $G \in \mathcal{G}'$ and G has n vertices. We show that $eb(G) = (n + 3)/8$ by induction on n. Obviously, $n \geq 13$. If $n = 13$, then $G = I_{13}$ and the equality follows. Now let $n > 13$, and assume that the assertion holds for graphs in $\mathcal{G}'$ with fewer vertices. Since $G \neq I_{13}$, $G \in \widetilde{\mathcal{H}}_1$. Let H be the graph obtained from G by contracting each end-block of G to a single vertex. Then $H \in \mathcal{H}$ and there exists a triangle $G[\{x,\ y,\ z\}]$ such that $d_H(x) = d_H(y) = 2$. If H is a triangle, then clearly the equality holds. Otherwise, let $H' = H - \{x, y\}$. Then $d_{H'}(z) = 2$. Let G' be the graph obtained from H' by attaching F_7 to each vertex of degree 2 of H'. It is easy to see that $G' \in \mathcal{G}$, $|V(G')| = n - 8$ and $eb(G') = eb(G) - 1$. By the induction hypothesis, we have $eb(G') = (|V(G')| + 3)/8$. This implies that $eb(G) = (|V(G)| + 3)/8$, as desired.

Conversely, supposing that $eb(G) = (n + 3)/8$ for a connected claw-free 4-regular graph of order n, we show that $G \in \mathcal{G}'$. By the above proof, we know that all the inequalities in equations (4)–(6) are equalities. That is, $m' = n' + eb(G) - 3$, $2eb(G) = n' + 3$ and $n' = n - 6eb(G)$. The second equality implies that n' is odd, and the last equality implies that every end-block of G is F_7. If $n' = 1$, then clearly $eb(G) = 2$, and since $G = I_{13} \in \mathcal{G}'$, we are done. It suffices to show that $G' \in \mathcal{H}$ for $n' \geq 3$.

Next we show that $G' \in \mathcal{H}$ by induction on n' for $n' \geq 3$. For $n' = 3$, we have $eb(G) = 3$ and so G' is a triangle, the assertion holds. Let $n' > 3$ and assume the assertion holds for smaller n'. Noting that $V_2 \neq \emptyset$, we let $u \in V_2$. We distinguish the following two cases.

Case 1. u is an isolated vertex of $G'[V_2]$. Then u lies in a unique triangle, say $G'[\{u,\ v,\ w\}]$, in G'. Obviously, $d_{G'}(v) = d_{G'}(w) = 4$. Note that $m' = n' + eb(G) - 3$. By Case 3 of the proof of Claim 1, it follows that $N_{G'}(v) \cap N_{G'}(w) - \{u\} = \emptyset$. Let G_1 be the graph obtained from G by removing the vertices of F_7, the edge vw and identifying v and w. Let G_1' be the graph obtained from G_1 by contracting each end-block to a single vertex and let $n_1' = |V(G_1')|$, $m_1' = |E(G_1')|$. Then $n_1' = n' - 2 < n'$, $eb(G) = eb(G_1) + 1$. Obviously, $m_1' = n_1' + eb(G_1) - 3$ and $2eb(G_1) = n_1' + 3$. By the induction hypothesis, we have $G_1' \in \mathcal{H}$. Thus $G \in \mathcal{H}$.

Case 2. u is in a component K_2 of $G'[V_2]$. Let $K_2 = G'[\{u,\ v\}]$ and let u, v lie in the triangle uvw of G'. Then $d_{G'}(w) = 4$. Let $G_1' = G' - \{u, v\}$ and let G_1 be the 4-regular graph obtained from G_1' by attaching an F_7 to each vertex of degree 2 of G_1'. Let $n_1' = |V(G_1')|$, $m_1' = |E(G_1')|$. Then $n_1' = n' - 2 < n'$, $eb(G) = eb(G_1) + 1$. Obviously, $m_1' = n_1' + eb(G_1) - 3$ and $2eb(G_1) = n_1' + 3$. By the induction hypothesis, we have $G_1' \in \mathcal{H}$. Thus $G \in \mathcal{H}$. ■

The proof of the following theorem is analogous to that of Theorem 3 and is omitted.

Theorem 6. *If G is a connected claw-free 4-regular graph of order $n \geq 9$, then $c(G) \leq (n-9)/4$ with equality if and only if $G \in \mathcal{G}'$.*

Acknowledgements

The authors would like to thank the referees for their valuable comments.

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Received 21 May 2012
Revised 8 January 2013
Accepted 8 January 2013