

## PACKING THE HYPERCUBE

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### Abstract

Let  $G$  be a graph that is a subgraph of some  $n$ -dimensional hypercube  $Q_n$ . For sufficiently large  $n$ , Stout [20] proved that it is possible to pack vertex-disjoint copies of  $G$  in  $Q_n$  so that any proportion  $r < 1$  of the vertices of  $Q_n$  are covered by the packing. We prove an analogous theorem for edge-disjoint packings: For sufficiently large  $n$ , it is possible to pack edge-disjoint copies of  $G$  in  $Q_n$  so that any proportion  $r < 1$  of the edges of  $Q_n$  are covered by the packing.

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### 1. INTRODUCTION

Hypercubes and their subgraphs have been studied for decades (see [12]), with interest derived from both their pure mathematical structure and numerous applications, for example in coding theory (see [21]) and parallel computing (see [14]). A graph is called *cubical* if it is a subgraph of a hypercube. Many researchers have studied which graphs are cubical [5, 6, 7, 9, 10], as well as the complexity of determining if a graph is cubical [4]. Others have relaxed the conditions for embedding, and studied properties of relaxed embeddings of non-cubical graphs in the hypercube [1, 15]. For a given cubical graph  $G$ , it is natural to ask how efficiently  $G$  can pack or cover the hypercube (see [11, 12, 20]). Many people have studied edge-decompositions of the hypercube into either fixed subgraphs [3, 13, 20], or subgraphs that depend on the dimension of the cube [8, 17, 19, 22]. The same question has been asked for vertex-decompositions [20, 18].

In 1990, Stout proved Theorem 1 [20]: Given a cubical graph  $G$ , for sufficiently large  $n$  it is possible to pack vertex-disjoint copies of  $G$  in  $Q_n$  so that any

proportion  $r < 1$  of the vertices of  $Q_n$  are covered by the packing. We prove this in Section 3. Stout made an analogous conjecture for edge-disjoint packings: Given a cubical graph  $G$ , for sufficiently large  $n$  it is possible to pack edge-disjoint copies of  $G$  in  $Q_n$  so that any proportion  $r < 1$  of the edges of  $Q_n$  are covered by the packing. Stout proved this result for some graphs  $G$ , such as trees. We prove the conjecture in general in Theorem 2. The proofs of both theorems are elementary and constructive. The last section contains some conclusions and open problems.

## 2. DEFINITIONS AND NOTATION

For a graph  $G$ , let  $V(G)$  denote the set of vertices in  $G$ ,  $E(G)$  denote the set of edges in  $G$ , and  $|V(G)|$  and  $|E(G)|$  denote the respective cardinalities of these sets. A subgraph of a graph  $H$  isomorphic to another graph  $G$  is called an *embedding* of  $G$  in  $H$ . A *vertex-disjoint packing* (resp. *edge-disjoint packing*) of  $G$  in  $H$  is a set of embeddings of  $G$  in  $H$  such that no two share a vertex (resp. edge). The *cardinality* of a packing  $P$ , denoted  $|P|$ , is the number of embeddings in the packing. A vertex or edge of  $H$  is *covered* by a packing if it is contained in some embedding in the packing. The *vertex density* (resp. *edge density*) of a packing  $P$  of  $G$  in  $H$  is the proportion of the total number of vertices (resp. edges) of  $H$  covered by  $P$ . When it is clear what type of density we are considering, we may refer to either of these concepts just as density. We say the vertices (resp. edges) of a graph are *partitioned* by a set of subgraphs if every vertex (resp. edge) of the graph belongs to exactly one of the subgraphs. A vertex-disjoint packing (resp. edge-disjoint packing) of  $G$  in  $H$  is *perfect* if all vertices (resp. edges) of  $H$  are covered by the packing.

For  $n \in \mathbb{Z}$ ,  $n \geq 1$ , the  *$n$ -dimensional hypercube*, denoted  $Q_n$ , is the graph with  $V(Q_n) = \{0, 1\}^n$ , and edges between vertices which differ in exactly one coordinate. Suppose  $x = [x_1 x_2 \cdots x_n]$ ,  $y = [y_1 y_2 \cdots y_n] \in V(Q_n)$  and  $j$  is the only coordinate such that  $x_j \neq y_j$ . Then the edge  $\{x, y\} \in E(Q_n)$  can be represented by the  $n$ -coordinate vector obtained by changing coordinate  $j$  of  $x$  (or  $y$ ) to a star. For example, in  $Q_4$ ,  $[010*]$  represents the edge containing vertices  $[0100]$  and  $[0101]$ . If  $d \leq n$ , we represent the subgraph  $Q_d$  of  $Q_n$  by an  $n$ -coordinate vector with stars in  $d$  coordinates. For instance  $[1*00*]$  represents the  $Q_2$  in  $Q_5$  with vertices  $\{[10000], [11000], [10001], [11001]\}$  and edges  $\{[1*000], [1000*], [1*001], [1100*]\}$ . We call edges with a star in the same coordinate *parallel*, and the class of edges with a star in coordinate  $i$  the  *$i$ th parallel class*.

For  $d \leq n$ , consider an embedding of  $Q_d$  in  $Q_n$  represented by a vector of length  $n$  with  $d$  stars. We say the vertex  $v \in V(Q_n)$  *corresponds* to  $x = [x_1 x_2 \cdots x_d] \in V(Q_d)$  if the vector representing  $v$  in  $Q_n$  is obtained by replacing

the  $i$ th star with  $x_i$ . For instance, the vertex  $[11000]$  in  $[1 * 00*]$  corresponds to  $[10] \in V(Q_2)$ . We similarly define corresponding edges, subgraphs, and packings. Given a packing  $P$  in  $Q_d$ , we say that the corresponding packing in an embedding of  $Q_d$  in  $Q_n$  is a *copy* of  $P$ .

Say  $P$  is a packing of the graph  $G$  in  $Q_n$ . Given a permutation  $\sigma$  of  $\{1, \dots, n\}$ , define  $\sigma(P)$  to be the packing obtained by applying  $\sigma$  to the coordinates of all vertices and edges of  $P$ . For instance, if  $P$  is a packing of the edges  $[11*]$ ,  $[0 * 1]$ , and  $[*00]$  in  $Q_3$ , and  $\sigma = (123)$ , then  $\sigma(P) = \{[*11], [10*], [0 * 0]\}$ .

### 3. VERTEX-DISJOINT PACKINGS

Suppose  $G$  is a subgraph of  $Q_n$  for some  $n$ , and let  $\mathcal{P}_v(G, n)$  denote the set of vertex-disjoint packings of  $G$  in  $Q_n$ . Define  $v(G, n)$  to be the maximum vertex density of a vertex-disjoint packing of  $G$  in  $Q_n$ . That is,

$$v(G, n) = \max_{P \in \mathcal{P}_v(G, n)} \frac{|P||V(G)|}{|V(Q_n)|} = \max_{P \in \mathcal{P}_v(G, n)} \frac{|P||V(G)|}{2^n}.$$

Define

$$v(G) = \lim_{n \rightarrow \infty} v(G, n).$$

For any  $n$ , the vertices of  $Q_{n+1}$  can be partitioned into two embeddings of  $Q_n$ , represented by the vectors  $[0**\dots*]$  and  $[1**\dots*]$ . Any vertex-disjoint packing  $P$  of  $Q_n$  can be copied into each of these two embeddings to get a vertex-disjoint packing of  $Q_{n+1}$  with the same density as  $P$ . Thus  $v(G, n)$  is monotonic as  $n$  increases. Since it is also bounded above by 1, the limit  $v(G)$  exists for all  $G$ .

There are many graphs  $G$  which are subgraphs of  $Q_d$  for some  $d$  but for which there is no  $n$  such that  $v(G, n) = 1$ . Indeed, since  $|V(Q_n)| = 2^n$ , any graph  $G$  where  $|V(G)|$  is not a power of two will not have a perfect packing. Nonetheless, there is a simple construction that shows that any graph  $G$  that is a subgraph of the hypercube can be packed perfectly in an asymptotic sense. That is, if we set  $n$  to be large enough, we may make  $v(G, n)$  as close to 1 as we like. This is the content of Theorem 1. Stout [20] proved Theorem 1 but has not published the result. For completeness, we include a proof here.

**Theorem 1** (Stout). *Let  $G$  be a nonempty graph which is a subgraph of  $Q_n$  for some  $n$ . Then  $v(G) = 1$ .*

**Proof.** Given  $G$ , a subgraph of  $Q_d$ , we construct a sequence of vertex-disjoint packings  $P_i$  of  $G$  in  $Q_{id}$  for  $i \geq 1, i \in \mathbb{Z}$ , whose densities converge to 1. Since  $v(G, n)$  is monotonic, this suffices to prove the theorem.

Start with any vertex-disjoint packing  $P_1$  of  $G$  in  $Q_d$ , and suppose this packing has vertex density  $r > 0$ . If  $r = 1$ , we are done. Otherwise, assume we have a

packing  $P_i$  of  $G$  in  $Q_{id}$  and proceed inductively as follows. Partition the vertices of  $Q_{(i+1)d}$  into a set  $A$  of  $2^d$  vertex-disjoint copies of  $Q_{id}$ , each of the form  $[**\cdots*Y]$ , where there are  $id$  stars, and  $Y$  is a binary string of length  $d$ . By copying  $P_i$  into each element of  $A$ , we obtain a vertex-disjoint packing  $P'_{i+1}$  of the same density as  $P_i$ . The uncovered vertices of  $Q_{(i+1)d}$  can be partitioned into a set  $B$  of vertex-disjoint copies of  $Q_d$ , each of the form  $[x_1x_2\cdots x_{id}**\cdots*]$  where there are  $d$  stars and  $[x_1x_2\cdots x_{id}]$  is a vertex of  $Q_{id}$  not covered by  $P_i$ . By copying  $P_1$  into each element of  $B$ , we obtain a vertex-disjoint packing  $P''_{i+1}$  covering a proportion  $r$  of the uncovered vertices. Let  $P_{i+1}$  be the union of  $P'_{i+1}$  and  $P''_{i+1}$ . Then  $P_{i+1}$  is a vertex-disjoint packing of  $G$  in  $Q_{(i+1)d}$ , and if we let  $\rho(P)$  denote the density of a packing  $P$ , then we obtain for  $i \geq 1$

$$\rho(P_{i+1}) = \rho(P'_{i+1}) + \rho(P''_{i+1}) = \rho(P_i) + (1 - \rho(P_i))r = r + \rho(P_i)(1 - r).$$

By induction, the density of  $P_i$  is  $r + (1 - r)r + (1 - r)^2r + \cdots + (1 - r)^{i-1}r$ , and since

$$\sum_{i=1}^{\infty} (1 - r)^{i-1}r = \frac{r}{1 - (1 - r)} = 1,$$

we can find a packing  $P_i$  with density as close to 1 as we wish.  $\blacksquare$

#### 4. EDGE-DISJOINT PACKINGS

Let  $\mathcal{P}_e(G, n)$  denote the set of edge-disjoint packings of  $G$  in  $Q_n$ , and define  $e(G, n)$  to be the maximum edge density of an edge-disjoint packing of  $G$  in  $Q_n$ :

$$e(G, n) = \max_{P \in \mathcal{P}_e(G, n)} \frac{|P||E(G)|}{|E(Q_n)|} = \max_{P \in \mathcal{P}_e(G, n)} \frac{|P||E(G)|}{n2^{n-1}}.$$

Unlike  $v(G, n)$ ,  $e(G, n)$  is not monotonic. For instance,  $e(Q_2, 2) = 1$ , but  $e(Q_2, 3) = 2/3$ . This is due to the fact that for  $n \geq 2$ , the edges of  $Q_{n+1}$  cannot be partitioned into edge-disjoint copies of  $Q_n$ . However, the edges of  $Q_{mn}$  can be partitioned into edge-disjoint copies of  $Q_m$  or  $Q_n$ , and in particular  $Q_{n^{i+1}}$  can be decomposed into edge-disjoint copies of  $Q_{n^i}$  or  $Q_n$ , and this fact will be helpful in proving Theorem 2. Define

$$e(G) = \lim_{n \rightarrow \infty} e(G, n).$$

We will show that this limit exists in the proof of Theorem 2.

**Theorem 2.** *Let  $G$  be a graph with at least one edge which is a subgraph of  $Q_n$  for some  $n$ . Then  $e(G) = 1$ .*

**Proof.** Given  $G$ , a subgraph of  $Q_{d_0}$ , we first construct an edge-disjoint packing  $P_1$  of  $G$  in  $Q_d$ , where  $d = 2d_0$ , with the property that  $P_1$  covers the same number of edges in each parallel class. This step is not strictly necessary, but starting with a packing with this property simplifies subsequent calculations.

Let  $P_0$  be any edge-disjoint packing of  $G$  in  $Q_{d_0}$  with positive density, let  $d = 2d_0$ , and let  $\sigma$  be the cyclic permutation  $(12 \cdots d_0)$ . Consider the set  $A$  of embeddings of  $Q_{d_0}$  in  $Q_d$  of the form  $[** \cdots * Y]$ , where there are  $d_0$  stars, and  $Y$  is a binary string of length  $d_0$ . The embeddings in  $A$  partition the edges in parallel classes 1 to  $d_0$  into  $2^{d_0}$  edge-disjoint copies of  $Q_{d_0}$ . Since  $|A| = 2^{d_0} \geq d_0$ , we can copy the packings  $P_0, \sigma(P_0), \sigma^2(P_0), \dots, \sigma^{d_0-1}(P_0)$  into  $d_0$  of the embeddings in  $A$ , leaving the others empty. We do the same for the set  $B$  of embeddings of the form  $[X ** \cdots *]$ , where there are  $d_0$  stars, and  $X$  is a binary string of length  $d_0$ . Call the resulting edge-disjoint packing  $P_1$ . Since the image of a given edge of  $P_0$  is in a different parallel class under each of the permutations  $\sigma, \sigma^2, \dots, \sigma^{d_0-1}$ , and the identity permutation,  $P_1$  contains exactly one copy of each edge in  $P_0$  in each parallel class of  $Q_d$ . Thus  $P_1$  covers the same number of edges in each parallel class.

Next we construct a sequence of edge-disjoint packings  $P_i$  of  $G$  in  $Q_{d^i}$  for  $i \geq 1, i \in \mathbb{Z}$ , whose edge densities converge to 1. Assume  $P_1$  has density  $r_1 > 0$ , and proceed inductively, as follows. Suppose we have a packing  $P_i$  of  $G$  in  $Q_{d^i}$  with density  $r_i$  that covers the same number of edges in each parallel class. For  $k \in \mathbb{Z}, 0 \leq k \leq d - 1$ , let  $A_k$  be the set of  $2^{d^i(d-1)}$  edge-disjoint embeddings of  $Q_{d^i}$  in  $Q_{d^{i+1}}$  corresponding to strings of the form  $[X ** \cdots * Y]$  where there are  $d^i$  stars,  $X$  is a binary string of length  $kd^i$ , and  $Y$  is a binary string of length  $(d - (k + 1))d^i$ . Let  $A = \bigcup A_k$ , and note that  $A$  partitions the edges of  $Q_{d^{i+1}}$ . Thus we can copy  $P_i$  into each cube in  $A$  to obtain an edge-disjoint packing  $P'_{i+1}$  of  $G$  in  $Q_{d^{i+1}}$  with the same density as  $P_i$ . Next, we augment this packing. For each integer  $j, 1 \leq j \leq d^i$ , let  $B_j$  be the set of  $2^{d^{i+1}-d}$  edge-disjoint embeddings of  $Q_d$  in  $Q_{d^{i+1}}$  with stars in the  $(kd^i + j)$ th positions,  $k \in \mathbb{Z}, 0 \leq k \leq d - 1$ , and all other coordinates fixed. Let  $B = \bigcup B_j$ , and note that  $B$  partitions the edges of  $Q_{d^{i+1}}$ . Let  $B' \subseteq B$  be the set of elements of  $B$  that contain no edges of  $P'_{i+1}$ . Copy the packing  $P_1$  into each cube in  $B'$  to obtain the edge-disjoint packing  $P''_{i+1}$ . Let  $P_{i+1}$  be the union of  $P'_{i+1}$  and  $P''_{i+1}$ .

The packings  $P'_{i+1}$  and  $P''_{i+1}$  are each edge-disjoint, and  $P'_{i+1}$  does not cover any edges in  $B'$ , so  $P_{i+1}$  is an edge-disjoint packing of  $Q_{d^{i+1}}$ . Since  $P_1$  and  $P_i$  each cover the same number of edges in each parallel class,  $P_{i+1}$  inherits this property as well. To show this, let  $0 \leq k < d, 1 \leq j \leq d$ , and let  $S$  denote the set of  $2^{d^{i+1}-1}$  edges in the  $(kd^i + j)$ th parallel class of  $Q_{d^{i+1}}$ . Our aim is to show that the number of edges in  $S$  covered by  $P_{i+1}$  does not depend on  $k$  or  $j$ , i.e.  $P_{i+1}$  covers the same number of edges in each parallel class.

The parallel class  $S$  can be partitioned into  $2^{d^i(d-1)}$  subsets of size  $2^{d^i-1}$ , each

of which is the  $j$ th parallel class of some  $Q_{d^i}$  in  $A_k$ . Since the packing  $P_i$  is copied into each cube in  $A_k$  to form the packing  $P'_{i+1}$ , and  $P_i$  covers the same proportion  $r_i$  of the edges in each parallel class of  $Q_{d^i}$ ,  $P'_{i+1}$  will cover the proportion  $r_i$  of the edges in  $S$ .

The edges in  $S$  can also be partitioned into  $2^{d^{i+1}-d}$  subsets of size  $2^{d-1}$ , each of which is the  $(k+1)$ th parallel class of some  $Q_d$  in  $B_j \subseteq B$ . An element of  $B$  is in  $B'$  if and only if for each  $k$  from 0 to  $d-1$ , the coordinates from  $kd^i + 1$  to  $kd^i + d^i$  correspond to an edge in  $Q_{d^i}$  not covered by  $P_i$ . Since  $P_i$  has density  $r_i$  in each parallel class, the proportion  $(1-r_i)^d$  of the elements of  $B_j$  will be in  $B'$ , and the edges in  $S$  contained in  $B'$  can be partitioned into  $(1-r_i)^d 2^{d^{i+1}-d}$  subsets of size  $2^{d-1}$ , each of which is the  $(k+1)$ th parallel class of some  $Q_d$  in  $B'$ . Since the packing  $P_1$  is copied into each cube in  $B'$  to form the packing  $P''_{i+1}$ , and  $P_1$  covers the same proportion  $r_1$  of the edges in each parallel class of  $Q_d$ ,  $P''_{i+1}$  will cover the proportion  $(1-r_i)^d r_1$  of the edges in  $S$ .

Since none of the calculations above depended on the particular choices of  $k$  and  $j$ ,  $P_{i+1}$  covers the proportion  $r_i + (1-r_i)^d r_1$  of the edges in each parallel class, and so it has density  $r_{i+1} = r_i + (1-r_i)^d r_1$ .

To see that the values of  $r_i$  converge to 1, let  $s_i = 1 - r_i$  to obtain the recurrence  $s_{i+1} = s_i - s_i^d r_1$  from  $r_{i+1} = r_i + (1-r_i)^d r_1$ . Since  $s_i^d > 0$  for all  $i$ , the values of  $s_i$  decrease as  $i$  increases, and we aim to show that they converge to 0. For the sake of contradiction, suppose there is some  $\alpha > 0$  such that  $s_i > \alpha$  for all  $i$ . If  $s_i > \alpha$ , then  $s_{i+1} < s_i - \alpha^d r_1$ . So, if  $i > \frac{s_1 - \alpha}{\alpha^d r_1}$  then  $s_{i+1} < s_1 - \frac{s_1 - \alpha}{\alpha^d r_1} \alpha^d r_1 = \alpha$ , a contradiction. Thus  $r_i$  converges to 1 as  $i$  goes to  $\infty$ , and for any  $G$  there is an infinite sequence of packings whose densities converge to 1.

To complete the proof, we use the sequence of packings  $P_i$  on  $Q_{d^i}$  to define an edge-disjoint packing  $P$  for any  $Q_n$ , where the density of  $P$  is arbitrarily close to 1 for sufficiently large  $n$ . Fix  $n$ , and let  $k$  be the integer such that  $d^k \leq n < d^{k+1}$ . Let  $a_0, a_1, \dots, a_k$  be the unique integers such that  $n = a_k d^k + a_{k-1} d^{k-1} + \dots + a_1 d + a_0$  and  $0 \leq a_i < d$  for each  $i$ . For  $0 \leq i \leq k$ ,  $0 \leq j < a_i$ , let  $C_{i,j}$  be the set of embeddings of  $Q_{d^i}$  in  $Q_n$  corresponding to strings of the form  $[X * \dots * Y]$ , where there are  $d^i$  stars,  $X$  is a binary string of length  $a_0 + a_1 d + \dots + j d^i$ , and  $Y$  is a binary string of length  $(a_i - (j+1))d^i + a_{i+1} d^{i+1} + \dots + a_k d^k$ . For each  $i$ , let  $C_i = \bigcup C_{i,j}$ , and let  $C = \bigcup C_i$ . The elements of  $C$  partition the edges of  $Q_n$ , and  $|C_i| = a_i 2^{n-d^i}$ . Thus we can get an edge-disjoint packing  $P$  of  $Q_n$  by copying  $P_i$  into each element of  $C_i$ . Since the number of edges in each packing  $P_i$  is  $r_i d^i 2^{d^i-1}$ , the density of  $P$  is

$$\frac{1}{|E(Q_n)|} \sum_{i=0}^k |C_i| r_i d^i 2^{d^i-1} = \frac{1}{n 2^{n-1}} \sum_{i=0}^k a_i 2^{n-d^i} r_i d^i 2^{d^i-1} = \frac{1}{n} \sum_{i=0}^k a_i d^i r_i.$$

Since  $a_0 + a_1 d + \dots + a_k d^k = n$ , this density can be interpreted as a weighted average of the  $r_i$ 's. The  $r_i$ 's converge monotonically to 1, and almost all of the

weight is distributed to the highest values of  $i$ , so for large  $n$  this density gets arbitrarily close to 1. More precisely, fix any small  $\epsilon > 0$ . Since the values of  $r_i$  monotonically increase toward 1, we can choose  $n$  large enough so that  $r_{\lfloor k/2 \rfloor} > 1 - \epsilon/2$  and  $\frac{d^{\lfloor k/2 \rfloor + 1}}{n} < \epsilon/2$ . We use the fact that  $a_0 + a_1d + \dots + a_kd^k = n$  and  $a_0 + a_1d + \dots + a_{\lfloor k/2 \rfloor}d^{\lfloor k/2 \rfloor} \leq d^{\lfloor k/2 \rfloor + 1}$  to bound the density of  $P$  from below as follows:

$$\begin{aligned} \frac{1}{n} \sum_{i=0}^k a_i d^i r_i &\geq \frac{1}{n} \sum_{i=\lfloor k/2 \rfloor + 1}^k a_i d^i r_i \geq \frac{1}{n} (n - d^{\lfloor k/2 \rfloor + 1}) r_{\lfloor k/2 \rfloor + 1} \\ &\geq \left(1 - \frac{d^{\lfloor k/2 \rfloor + 1}}{n}\right) (1 - \epsilon/2) \geq 1 - \epsilon. \end{aligned}$$

■

## 5. CONCLUSIONS AND OPEN PROBLEMS

Both constructions in this paper have the property that they begin with an arbitrary nonempty packing and use it to generate a sequence of packings, each in some sense an augmentation of the previous. But the vertex densities of the vertex-disjoint packings in the proof of Theorem 1 converge to 1 much faster than the edge densities of the edge-disjoint packings given in the proof of Theorem 2. Are these rates of convergence optimal, or can one obtain faster convergence by choosing an initial packing more deliberately or by using a more sensitive induction? Is it possible to identify which graphs of a given size have packings whose densities have the slowest convergence to one?

As mentioned in the introduction, it can be shown that some subgraphs have perfect packings. In [23], Wilson gave necessary and sufficient conditions for a subgraph of a sufficiently large complete graph to have a perfect edge-disjoint packing. Is it possible to find such conditions for the hypercube, either for vertex-disjoint packings or edge-disjoint packings? For instance, if  $G$  is cubical and  $|V(G)| = 2^k$  for some natural number  $k$ , does a perfect vertex packing always exist in a sufficiently large hypercube?

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