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LATTICE-LIKE TOTAL PERFECT CODES

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Abstract

A contribution is made to the classification of lattice-like total perfect codes in integer lattices Λ_n via pairs (G, Φ) formed by abelian groups G and homomorphisms $\Phi : \mathbb{Z}^n \rightarrow G$. A conjecture is posed that the cited contribution covers all possible cases. A related conjecture on the unfinished work on open problems on lattice-like perfect dominating sets in Λ_n with induced components that are parallel paths of length > 1 is posed as well.

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1. INTRODUCTION: LATTICE PERFECT DOMINATING SETS

Motivation for this work is the processing of elements in a supercomputer that communicates through a network with the topology of the Cartesian product of cycles. It is an aim to place the Input/Output devices into the network in such a way that the communication of all elements in the network is optimized and each element is at distance at most $t = 1$ from exactly one I/O device. This is related to the well-known conjecture of Golomb and Welch [8] that a perfect t -error correcting Lee code of length larger than 1 over a large alphabet exists only for $t = 1$. See also [1, 11] for additional notions, results and references.

Let $\Gamma = (V, E)$ be a graph. Let $S \subset V$. The subgraph of Γ induced by S is denoted by $[S]$. The set S is a *perfect dominating set* (PDS) of Γ if each vertex $w \in V \setminus S$ has a unique neighbor $v \in S$ [19]. A PDS of Γ is a *total perfect code* (TPC) (resp. a *perfect code* or *efficient dominating set* (EDS)) of Γ if the components of its induced graph are copies of K_2 [4, 12], (resp. K_1 [2, 5, 13]). For $0 < n \in \mathbb{Z}$, let Λ_n be the graph whose vertex set is the integer lattice \mathbb{Z}^n , with an edge between any two vertices whenever their Euclidean distance is 1. A TPC in Λ_n is a particular case of a distance perfect Lee code [7, 11], namely a DPL($n, 4$). The following result is an extension of one of [19] for n -cubes to integer-lattice graphs Λ_n and is proved in more generality in [1].

Theorem 1. *Given a PDS S in Λ_n , the components of $[S]$ in Λ_n are Cartesian products of connected subgraphs of Λ_1 .*

The components of $[S]$ in the statement above will be called *components* of S . If $t \geq 1$ and $\Gamma = (V, E)$ is a graph, then Araujo, Dejter and Horak [1] define a t -perfect distance-dominating set (t -PDDS) in Γ as a subset $S \subset V$ such that, for each $v \in V$, there is a unique component C_v of $[S]$, so that for the distance $d(v, C_v)$ from v to C_v it is $d(v, C_v) \leq t$, and there is in C_v a unique vertex w with $d(v, w) = d(v, C_v)$. Moreover, Araujo, Dejter and Horak conjecture in [1] that if H is a finite path or a Cartesian product of two finite paths, then a t -PDDS whose induced components are isomorphic to H exists in Λ_n if and only if: either **(i)** $t = 1$, $n \geq 2$, and $H = P_k, k \geq 1$; or **(ii)** $t \geq 1$, $n = 2$, and $H = P_k, k \geq 1$; or **(iii)** $t \geq 1$, $n = 2$, and $H = P_2 \square P_k, k \geq 2$; or **(iv)** $t = 1$, $n = 3r + 2, r \geq 0$, and $H = P_2 \square P_2$; or **(v)** $t = 2, n = 3$, and $H = P_2$. Noting that a PDS in Λ_n is a 1-PDDS in $\Gamma = \Lambda_n$, we have the following related conjecture, settled below as Theorem 6.

Conjecture 2. *Let P_r be a path of length $r - 1$. Then, a PDS in Λ_n whose components are isomorphic to P_r exists if and only if $n \geq 1$ and $r \geq 1$.*

As in [1], all PDSs S in Λ_n treated here are obtained by means of an algebraic construction that makes them *lattice-like*, meaning that there exists a lattice L (namely a subgroup of $(\mathbb{Z}^n, +)$ = component-wise additive group on \mathbb{Z}^n) such that for any two components $\Theta_0 = (V_0, E_0)$ and $\Theta_1 = (V_1, E_1)$ induced by S in Λ_n , there exists $z \in L$ with $\Theta_1 = \Theta_0 + z = (V'_0, E'_0)$, where $V'_0 = V_0 + z = \{w \in \mathbb{Z}^n; \exists v \in V_0, w = v + z\}$, and $uv \in E_0$ if and only if $(u + z)(v + z) \in E'_0$, where we also have that for any $z \in L$, $\Theta_i + z$ is a component of S , both for $i = 0$ and 1. Different variations of the construction can be found throughout the literature, see e.g. [3, 9, 10, 14, 15, 16, 17, 18].

So, following for example the steps of Molnár [14] for a classification of all lattice-like perfect Lee codes in Λ_n ($0 < n \in \mathbb{Z}$) via abelian groups G and homomorphisms $\Phi : \mathbb{Z}^n \rightarrow G$ (see an implementation of a construction due initially to

Stein [17] in Theorem 3 and Corollary 4 below) we proceed likewise in the case of lattice-like TPCs in Λ_n and present a collection of pairs (G, Φ) that produce such TPCs. Conjecture 12 below claims that such a collection covers all cases of lattice-like TPCs. In order to pose such claim, we stream down to a list of cases in Theorem 8 and encode the resulting information via concepts defined in successive subsections of Section 2: *quadruple stacks*; Υ -*sequences* and their *breadths*; a further convenient shorthand notation and finally a generalization of that notation. Conjecture 7 refers to the unfinished work of open problems on lattice-like PDSs in Λ_n for which the induced components are parallel paths of length > 1 .

The above mentioned algebraic construction consists of: **(a)** letting L be a distinguished lattice in $(\mathbb{Z}^n, +)$ generated by elements $u_1, \dots, u_n \in \mathbb{Z}^n$, so that $L = \{\alpha_1 u_1 + \dots + \alpha_n u_n; \alpha_i \in \mathbb{Z}, i = 1, \dots, n\}$; **(b)** letting $T \subseteq \mathbb{Z}^n$ be a set containing one element from each coset of \mathbb{Z}^n/L , so that $\{T + u; u \in L\}$ is a partition of \mathbb{Z}^n into parts of size $|\mathbb{Z}^n/L|$, with all induced subgraphs $[T + u]$ of $T + u$ in Λ_n pairwise isomorphic, where $u \in L$.

For a given lattice L , we can partition the vertex set of Λ_n into parts such that the corresponding induced subgraphs have different shapes depending on the choice of T . For example, set $L = \{\alpha_1(5, 1) + \alpha_2(2, 2); \alpha_i \in \mathbb{Z}, i = 1, 2\}$ in Λ_2 . Then, $(\mathbb{Z}^2, +)/L = \mathbb{Z}_8$ and the graph $[T]$ might be a path of length 7 or the union of a 6-cycle and a path of length 3, as shown in Figure 1.

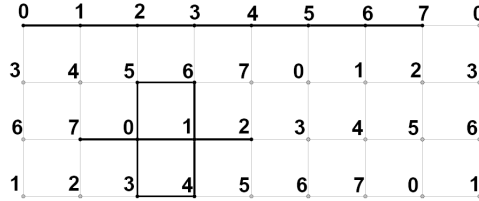


Figure 1. Example with $L = \{\alpha_1(5, 1) + \alpha_2(2, 2); \alpha_i \in \mathbb{Z}, i = 1, 2\}$.

If no confusion arises, n -tuples representing elements of \mathbb{Z}^n will be written without external parentheses or commas, and $00 \dots 0 = O$, $10 \dots 0 = e_1$, $010 \dots 0 = e_2, \dots, 00 \dots 1 = e_n$. A PDS in Λ_n whose components are all isomorphic to a fixed finite graph Θ is denoted by $\text{PDS}[\Theta]$. Our main tool can be stated as follows.

Theorem 3 [1, 11]. *Let $D = (V, E)$ be a subgraph of Λ_n . Then there is a lattice-like tiling of Λ_n by copies of D if and only if there is an abelian group G and a homomorphism $\Phi : \mathbb{Z}^n \rightarrow G$ so that the restriction of Φ to V is a bijection.*

Corollary 4. *In the setting of Theorem 3, if $n = 2^m$, ($0 \leq m \in \mathbb{Z}$), and $u \in \mathbb{U}_{2^{m+2}} \subset \mathbb{Z}_{2^{m+2}}$, then there is one homomorphism $\Phi = \Phi_u$ for which*

$\Phi_u(e_1) = u$ which is unique up to permutations of e_2, \dots, e_n and transpositions $(-e_2 \ e_2), \dots, (-e_n \ e_n)$. Moreover, all TPCs obtained from the resulting homomorphisms Φ_u are pairwise equivalent.

Let $\Theta_0 = (V, E')$ be a subgraph of Λ_n and let $z \in \mathbb{Z}^n$. Then $\Theta_0 + z$ denotes a graph $\Theta_1 = (V', E'')$, where $V' = V + z = \{v + z; v \in V\}$ and $uv \in E' \iff (u + z)(v + z) \in E''$. A PDS can be constructed via Corollary 4 if and only if it is lattice-like. Moreover, every lattice-like PDS S in Λ_n is *periodic*, that is to say that there are integers p_1, \dots, p_n such that $v \in S$ implies $v \pm p_i e_i \in S$, for all $i = 1, \dots, n$. However, not every periodic PDS in Λ_n is lattice-like, as can be seen in Sections 6–7 of [1].

1.1. Lattice PLC($n, 1$) codes

Every PDS[K_1] in Λ_n can be considered as a perfect 1-error correcting Lee code PLC($n, 1$) [1]. The existence of such codes has been proved independently by several authors. Kárteszi asked whether there exists a PLC(3, 1). Feller, for $n = 3$, and then Korchmáros, and Golomb and Welch [8] showed that there is a PLC($n, 1$) for every $n \geq 2$. The following stronger theorem has been proved by Molnár [14].

Theorem 5. *The number of lattice-like PLC($n, 1$) codes equals the number of abelian groups of order $2n + 1$.*

[1] illustrates the method of Corollary 4 in the setting of Theorem 5 by showing that each abelian group of order $2n + 1$ generates a PLC($n, 1$). Since in this case Θ is an isolated vertex, the graph Θ^* is of order $2n + 1$. Let $\Theta_0^* = (V, E)$ be a copy of Θ^* such that $V = \{\pm e_i; i = 1, \dots, n\} \cup \{O\}$. Let G be an abelian group of order $2n + 1$. Choose a set $K = \{g_1, \dots, g_n\}$ formed by n distinct elements of G such that K contains exactly one element from each pair g, g^{-1} . Since no element of G is of order 2, the restriction $\Phi : V \rightarrow G$ of the homomorphism $\Phi : \mathbb{Z}^n \rightarrow G$ given by $\Phi((a_1, \dots, a_n)) = \Phi(e_1)^{a_1} \circ \dots \circ \Phi(e_n)^{a_n} = g_1^{a_1} \circ \dots \circ g_n^{a_n}$ to V is a bijection. Thus, each abelian group of order $2n + 1$ generates a PLC($n, 1$). In order to give some examples that serve as models for what follows, let x_1, x_2, \dots, x_n be the coordinate directions of $\mathbb{Z}^n \subset \mathbb{R}^n$. As said above, the set V is formed by $O, e_1, -e_1, \dots, e_n, -e_n \in \mathbb{Z}^n$. The middle sections of V in the planes $x_1 x_2; x_1 x_3; \dots; x_1 x_n$, (that is, respectively with $x_3 = \dots = x_n = 0$; $x_2 = x_4 = \dots = x_n = 0$; \dots ; $x_2 = x_3 = \dots = x_{n-1} = 0$), can be taken with its vertices associated bijectively via Φ to the elements in the nonempty boxes strictly between the columns $\overset{n}{G}$ and B of the following table, for $n = 2, 3, 4$.

In this table, elements of a direct sum G are expressed as ordered tuples (without parentheses), and the central horizontal row in each box (as said above) contains the values $\Phi(-e_1), \Phi(O), \Phi(e_1)$. Also, each such box having header $x_1 x_j$

has on its central vertical row the values $\Theta(-e_j)$, $\Theta(O)$, $\Theta(e_j)$. To follow up a notation of [14], notice that Φ is determined by the values $\Phi(e_i)$, for $i = 1, \dots, n$, expressed as the column $B = (\Phi(e_1), \dots, \Phi(e_n))^t = \Upsilon^t$ in the table, where t stands for transpose. For example in the last case of the table: $\Phi(e_1) = 1, 0$, $\Phi(e_2) = 0, 1$, $\Phi(e_3) = 1, 1$ and $\Phi(e_4) = 1, 2$. In general, $Ker(\Phi) \subseteq \mathbb{Z}^n$ is a sublattice of \mathbb{Z}^n as well as a linear 1-perfect code in Λ_n . In the table, $Ker(\Phi)$ is determined in each case by the $n \times n$ -matrix M written on the right, where: **(a)** the rows correspond to e_1, e_2, \dots, e_n ; **(b)** $\Upsilon = B^t$ times each column is null in G ; and **(c)** the columns of M constitute a basis of $Ker(\Phi)$. Thus, the inner product, in each G exemplified in the table, of the column B times each column of M is null in G . Furthermore, the columns of M are linearly independent. The elements of the diagonal of each M on the rows indicated by the elements $\Phi(e_1)$, etc., of a basis of G are shown in boldface, meaning that boldface is used just when the only nonzero element of a column is on the diagonal, for further reference.

| $\begin{matrix} n \\ G \end{matrix}$ | x_1x_2 | x_1x_3 | x_1x_4 | B | M | | | |
|---|----------|---------------------------|----------|---------------------------|--------------------------------|---------------------------|-------------------------|--|
| $\begin{matrix} n=2 \\ \mathbb{Z}_5 \end{matrix}$ | 4 | 3 0 1 | | 1 2 | 5 3 0 1 | | | |
| $\begin{matrix} n=3 \\ \mathbb{Z}_7 \end{matrix}$ | 6 | 5 0 2 | 6 | 4 0 3 | 7 5 4 0 1 0 0 0 1 | | | |
| $\begin{matrix} n=4 \\ \mathbb{Z}_9 \end{matrix}$ | 8 | 7 0 2 | 8 | 6 0 3 | 8 | 5 0 4 | 1 2 3 4 | 9 7 6 5 0 1 0 0 0 0 1 0 0 0 0 1 |
| $\begin{matrix} n=4 \\ \mathbb{Z}_3 \oplus \mathbb{Z}_3 \end{matrix}$ | 2,0 | 0,2 0,0 0,1 | 1,0 | 2,2 0,0 1,1 | 1,0 | 2,1 0,0 1,2 | 1,0 | 3 0 2 2 0 3 2 1 0 0 1 0 0 0 0 1 |

1.2. PDS s in Λ_n with components P_r

The following is a result of [1] that can be established via Corollary 4.

Theorem 6. *A PDS in Λ_n whose components are all isomorphic to P_r exists for every $n \geq 1$ and $r \geq 2$. Thus, Conjecture 2 is true.*

Before concentrating in Section 2 on the characterization of lattice-like TPCs, we give an idea and examples of the proof of Theorem 6. The statement follows from Corollary 4 with $G = \mathbb{Z}_{2nr-r+2}$. Now, $V \subseteq \mathbb{Z}^n$ is composed by the vertices $O, e_1, 2e_1, \dots, (r-1)e_1$ and their $2nr - 2r + 2$ neighbors, (namely $-e_1, re_1$ and $\pm e_i, (e_1 \pm e_i), \dots, ((r-1)e_1 \pm e_i$, for $i = 2, \dots, n$). Initial examples of this case, for $r = 3$, are as follows, in a framework as in Subsection 1.1.

We could add here column vectors $B = B_n^r$ as in Subsection 1.1, namely, in our present case, $B_1^3 = (1)$, $B_2^3 = (1, 4)^t$, $B_3^3 = (1, 4, 7)^t$ and $B_4^3 = (1, 4, 7, 10)^t$, and matrices M , obtained from the identity $r \times r$ matrices I_r by replacing the first rows respectively by (5), (11, 7), (17, 13, 10) and (23, 19, 16, 13), respectively

for $n = 1, 2, 3, 4$. In general, $B_n^r = (1, 1 + r, 1 + 2r, \dots, 1 + (n - 1)r)^t$ and M is obtained from I_r by replacing its first row by the row vector $(m = 2nr - r + 2, m - r - 1, m - 2r - 1, \dots, m - nr - 1)$.

| | | | | | | | | | | | | | | | |
|--|----|----------|----------|----------|---|----|----------|----------|----------|---|----|----------|----------|----------|---|
| $\begin{matrix} n=1 \\ \mathbb{Z}_5 \end{matrix}$ | 4 | 0 | 1 | 2 | 3 | | | | | | | | | | |
| $\begin{matrix} n=2 \\ \mathbb{Z}_{11} \end{matrix}$ | 10 | 0 | 1 | 2 | 3 | | | | | | | | | | |
| $\begin{matrix} n=3 \\ \mathbb{Z}_{17} \end{matrix}$ | 16 | 0 | 1 | 2 | 3 | 16 | 0 | 1 | 2 | 3 | | | | | |
| $\begin{matrix} n=4 \\ \mathbb{Z}_{23} \end{matrix}$ | 22 | 0 | 1 | 2 | 3 | 22 | 0 | 1 | 2 | 3 | 22 | 0 | 1 | 2 | 3 |

Each display of a middle section of V parallel to one of the planes x_1x_j , ($j = 2, \dots, n$), as exemplified, contains just one row formed by $r + 2$ contiguous elements, and two rows formed by r contiguous elements of $\mathbb{Z}_{2nr-r+2}$. However, r divides $r + 2$ if and only if $r \in \{1, 2\}$. Assuming $r > 2$, it can be seen that r does not divide $r + 2$ and the smallest value of $2nr - r + 2$ divisible by a prime square happens for $n = 4$ and $r = 14$; (however, if $r = 2$ and $n = 2$, then $2nr - r + 2 = 8$ is divisible by 2^2). The elements of the resulting group $\mathbb{Z}_{2nr-r+2} = \mathbb{Z}_{2 \cdot 14 \cdot 4 - 4 + 2} = \mathbb{Z}_{100}$ can be distributed into the entries of the corresponding V by means of $\Upsilon = B^t = (1, 15, 29, 43)$, with M obtained from I_r by replacing its first row by $(100, 85, 71, 57)$. Trying to fit the elements of a non-cyclic abelian group G of order 100 into V would mean having 16 successive elements occupying corresponding entries along x_1 starting say one position previous to O , so that a cyclic summand of such a G should have order larger than $r + 2 = 16$. However, any divisor 20, 25 and 50, of 100, cannot be the order of such a summand, because we would be left with either $20 - 16 = 4$ or $25 - 16 = 9$ or $50 - 16 - 14 - 14 = 6$ elements that do not fit into 14 successive entries along x_1 that start from some e_i . Thus, in this case, only \mathbb{Z}_{100} works out.

Conjecture 7. *In the setting of Theorem 6 for $r > 2$, the group G of Corollary 4 is necessarily $G = \mathbb{Z}_{2nr-r+2}$. Moreover, there exists just one Φ for each such G .*

This conjecture would complete the classification of PDS[P_r]s in Λ_n , for $r > 2$.

2. THE CASE OF LATTICE-LIKE TOTAL PERFECT CODES

As expressed above, in the rest of this paper we contribute to the characterization of lattice-like TPCs. We start by stating and proving the following result.

Theorem 8. *In the setting of Theorem 6 for $r = 2$, there exists at least one group homomorphism $\Phi : \mathbb{Z}^n \rightarrow G$, for each abelian group G decomposable as a*

direct sum of factors of even order, but not of the form $(\oplus_{i=1}^m \mathbb{Z}_2) \oplus \Psi(G)$, where $\Psi(G)$ is maximum factor of G of order twice an odd number. On the other hand, there is no such a Φ whose domain is of the form $G = (\oplus_{i=1}^m \mathbb{Z}_2) \oplus \Psi(G)$.

The proof of Theorem 8 is postponed to Subsection 2.4, so first we provide some examples that further illustrate the tables above and the convenience of encoding notation that will be assumed in Subsections 2.3, 2.5. and 2.7. Let $r = 2$. Then, $V \subset \mathbb{Z}^n$ is formed by the vertices O and e_1 and their $4n - 2$ neighbors in Λ_n , (namely $-e_1, 2e_1, \pm e_2, (e_1 \pm e_2), \dots, \pm e_n, (e_1 \pm e_n)$).

2.1. Subcase $n = 2^m, (0 \leq m \in \mathbb{Z})$

Given a group homomorphism $\Phi : \mathbb{Z}^n \rightarrow G$ as in the statement of Theorem 8, the middle sections of V in the planes $x_1x_2, x_1x_3, \dots, x_1x_r, \dots, x_1x_n$ now, can be taken with its vertices associated bijectively, via Φ , to the following elements of G , for $n = 2^m$ and $m = 0, 1, 2$:

| | | | | | | | | | | | | | | | | | | |
|---|-------|--------------|--------------|--------------|--------------|----------|----------|--------------|--------------|--------------|--------------|-------|-------|--------------|--------------|--------------|--------------|-------|
| $\begin{matrix} n=1 \\ \mathbb{Z}_4 \end{matrix}$ | 3 | 0 | 1 | 2 | | | | | | | | | | | | | | |
| $\begin{matrix} n=2 \\ \mathbb{Z}_8 \end{matrix}$ | 7 | 5 | 6 | 1 | 2 | | | | | | | | | | | | | |
| $\begin{matrix} n=2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_4 \end{matrix}$ | 0,3 | 1,3 | 1,0 | 0,0 | 0,1 | 0,2 | | | | | | | | | | | | |
| $\begin{matrix} n=4 \\ \mathbb{Z}_{16} \end{matrix}$ | f | 0 | 1 | 2 | f | 0 | 1 | 2 | f | 9 | 1 | 2 | | | | | | |
| $\begin{matrix} n=4 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_8 \end{matrix}$ | 0,7 | 1,7 | 1,0 | 0,1 | 0,1 | 0,2 | 0,7 | 0,5 | 0,6 | 0,1 | 0,1 | 0,2 | 0,7 | 1,5 | 1,6 | 0,0 | 0,1 | 0,2 |
| $\begin{matrix} n=4 \\ \mathbb{Z}_4 \oplus \mathbb{Z}_4 \end{matrix}$ | 0,3 | 3,3 | 3,0 | 0,0 | 0,1 | 0,2 | 3,3 | 2,3 | 2,0 | 0,0 | 0,1 | 0,2 | 0,3 | 1,3 | 1,0 | 0,0 | 0,1 | 0,2 |
| $\begin{matrix} n=4 \\ (\mathbb{Z}_2)^2 \oplus \mathbb{Z}_4 \end{matrix}$ | 0,0,3 | 1,0,3 | 1,0,0 | 0,0,0 | 0,0,1 | 0,0,2 | 0,0,3 | 0,1,3 | 0,1,0 | 0,0,0 | 0,0,1 | 0,0,2 | 0,0,3 | 1,1,3 | 1,1,0 | 0,0,0 | 0,0,1 | 0,0,2 |
| | | 1,0,1 | 1,0,2 | | | | | 0,1,1 | 0,1,2 | | | | 0,0,3 | 1,1,1 | 1,1,2 | | | |

where lowercase hexadecimal notation makes $a = 10$, etc., the images via Φ of $\{O, e_1\}$ are in boldface and the central left-to-right horizontal (resp. the first, second, ..., last pair of central top-to-bottom vertical) disposition of elements of G in each case corresponds to their assignation from the vertices $-e_1, O, e_1, 2e_1$, (resp. $-e_2, O, e_2$ and $e_1 - e_2, e_1, e_1 + e_2$; $-e_3, O, e_3$ and $e_1 - e_3, e_1, e_1 + e_3$; ...; $-e_n, O, e_n$ and $e_1 - e_n, e_1, e_1 + e_n$) of Λ_n via Φ . Again, to follow up a notation of [14] (as in the right side of the table in the illustration following the statement of Theorem 5) observe that columns $B = \Upsilon^t$ (with $\Phi(e_1)$ in boldface), and matrices M associated to the four cases of $n = 4$ above are as follows.

$$\left\| \begin{array}{c|ccc} \mathbf{1} & \mathbf{16} & 13 & 11 & 9 \\ \hline 3 & 0 & 1 & 0 & 0 \\ \hline 5 & 0 & 0 & 1 & 0 \\ \hline 7 & 0 & 0 & 0 & 1 \end{array} \right\| \left\| \begin{array}{c|ccc} \mathbf{0,1} & \mathbf{2} & 0 & 5 & 1 \\ \hline 1,1 & 0 & \mathbf{8} & 0 & 1 \\ \hline 0,3 & 0 & 0 & 1 & 1 \\ \hline 1,3 & 0 & 0 & 0 & 1 \end{array} \right\| \left\| \begin{array}{c|ccc} \mathbf{0,1} & \mathbf{4} & 0 & 1 & 2 \\ \hline 1,1 & 0 & \mathbf{4} & 2 & 1 \\ \hline 2,1 & 0 & 0 & 1 & 0 \\ \hline 3,1 & 0 & 0 & 0 & 1 \end{array} \right\| \left\| \begin{array}{c|ccc} \mathbf{0,0,1} & \mathbf{2} & 0 & 0 & 1 \\ \hline 1,0,1 & 0 & \mathbf{2} & 0 & 1 \\ \hline 0,1,1 & 0 & 0 & \mathbf{4} & 1 \\ \hline 1,1,1 & 0 & 0 & 0 & 1 \end{array} \right\|$$

For these cases of $n = 2^m$ and cyclic G , given a unit u of $G = \mathbb{Z}_{2^{m+2}}$, namely an element of the multiplicative group $\mathbb{U}_{2^{m+2}} \subset \mathbb{Z}_{2^{m+2}}$ (realized by an odd integer), there exists only one homomorphism Φ for which $\Phi(e_1) = u$, (because if $\Phi(e_1) \notin \mathbb{U}_{2^{r+2}}$, then the least positive integer realizing $\Phi(e_1)$ divides $2^{r+2} = 4n$, so that $\frac{\Phi(e_1)}{2}$ would appear twice in the image of Φ , which then cannot restrict to a bijection, as required, thus providing a contradiction). The following corollary follows from the fact that, in these cases, the images of the pairs $\{e_1, -e_1\}, \dots, \{e_n, -e_n\}$ via any of these homomorphisms Φ cover all the pairs $\{u, -u\} \subseteq \mathbb{U}_{2^{r+2}}$ and thus yield pairwise equivalent lattice-like tilings.

Corollary 9. *In the setting of Theorem 8, if $n = 2^m$, ($0 \leq m \in \mathbb{Z}$), and $u \in \mathbb{U}_{2^{m+2}} \subset \mathbb{Z}_{2^{m+2}}$, then there is one homomorphism $\Phi = \Phi_u$ for which $\Phi_u(e_1) = u$ which is unique up to permutations of e_2, \dots, e_n and transpositions $(e_2 - e_2), \dots, (e_n - e_n)$. Moreover, all TPCs obtained from the resulting homomorphisms Φ_u by means of Corollary 4 are pairwise equivalent.*

However, we will see after Remark 10 (for $n = 2^3 = 8$ and specific non-cyclic groups G of order $4n = 32$) that there are homomorphisms Φ for cases of $n = 2^m$ in which the direction x_1 along which $\{O, e_1\}$ lies is associated to different non-equivalent summands of $G = \oplus_{i=1}^s \mathbb{Z}_{c_i}$. This is also the case for $n \neq 2^m$, (starting with $n = 24$).

2.2. Subcase $n \neq 2^m$, ($0 \leq m \in \mathbb{Z}$)

We will see now some cases of $n \neq 2^m$, for $0 \leq m \in \mathbb{Z}$, starting with the following three examples, disposed as those above, with corresponding vectors B and matrices M shown to the right:

| | | | | | | | | | | | |
|-------------------|-----|----------|----------|---|-----|----------|----------|----------|-----------|----------|----------|
| $n=3$ | b | 9 | a | 2 | b | 7 | 8 | 1 | 12 | 9 | 7 |
| \mathbb{Z}_{12} | | 0 | 1 | | | 0 | 1 | 3 | 0 | 1 | 0 |
| | | 3 | 4 | | | 5 | 6 | 5 | 0 | 0 | 1 |
| $n=3$ | | b | 2 | | | 7 | a | 1 | 3 | 0 | 1 |
| \mathbb{Z}_{12} | 9 | 0 | 3 | 6 | 9 | 0 | 3 | 3 | 3 | 4 | 2 |
| | | 1 | 4 | | | 5 | 8 | 5 | 0 | 0 | 1 |
| $n=3$ | | a | 1 | | | 8 | b | 2 | 3 | 0 | 4 |
| \mathbb{Z}_{12} | 9 | 0 | 3 | 6 | 9 | 0 | 3 | 3 | 2 | 4 | 0 |
| | | 2 | 5 | | | 4 | 7 | 4 | 0 | 0 | 1 |

where the order of x_1, \dots, x_n differs now, so that e_1, \dots, e_n have their images via Φ as shown in the column B for the second and third cases, for which x_1 and x_2 appear permuted in order for us to obtain all possible patterns. Likewise, for any $n > 1$ and abelian group G of order $4n$, we can partition $G \setminus \Phi(\{-e_1, 0, e_1, 2e_1\})$ into quadruples $\{-d_i, -d_i + \Phi(e_1), d_i, d_i + \Phi(e_1)\}$, where $2 \leq i \leq n$. Then, a group homomorphism Φ as above is determined by setting $\Phi(e_i) = d_i$, for every $1 \leq i \leq n$. What follows simplifies encoding this.

2.3. Quadruple stacks

The table above represents three different homomorphisms $\Phi : \mathbb{Z}^3 \rightarrow \mathbb{Z}_{12}$, with their data further representable by three quadruples each, namely $((\Phi(-e_1), \Phi(O), \Phi(e_1), \Phi(2e_1)))$ and $((\Phi(-e_j), \Phi(e_1 - e_j), \Phi(e_j), \Phi(e_1 + e_j)))$, for $j = 2, 3$. These quadruples can be arranged as on upper-left, upper-center-left and upper-center-right of the following table, for the three homomorphisms, as *stacks* of three quadruples each:

| | | | | | | | | | | | | | | | |
|-----|----------|----------|----------|-----|----------|----------|----------|-----|----------|----------|----------|----|----------|----------|----|
| b | 0 | 1 | 2 | b | 2 | 1 | 4 | a | 1 | 2 | 5 | 18 | 3 | 2 | 7 |
| 9 | a | 3 | 4 | 9 | 0 | 3 | 6 | 9 | 0 | 3 | 6 | 16 | 1 | 4 | 9 |
| 7 | 8 | 5 | 6 | 7 | a | 5 | 8 | 8 | b | 4 | 7 | 15 | 0 | 5 | 10 |
| — | — | — | — | — | — | — | — | — | — | — | — | 14 | 19 | 6 | 11 |
| 19 | 0 | 1 | 2 | 19 | 4 | 1 | 6 | 19 | 4 | 1 | 6 | 17 | 2 | 3 | 8 |
| 17 | 18 | 3 | 4 | 17 | 2 | 3 | 8 | 18 | 3 | 2 | 7 | 16 | 1 | 4 | 9 |
| 15 | 16 | 5 | 6 | 15 | 0 | 5 | 10 | 15 | 0 | 5 | 10 | 15 | 0 | 5 | 10 |
| 13 | 14 | 7 | 8 | 13 | 18 | 7 | 12 | 12 | 17 | 8 | 13 | 14 | 19 | 6 | 11 |
| 11 | 12 | 9 | 10 | 11 | 16 | 9 | 14 | 11 | 16 | 9 | 14 | 13 | 18 | 7 | 12 |
| — | — | — | — | — | — | — | — | — | — | — | — | — | — | — | — |

where, indicated in boldface, are the values of G corresponding to the elements of an associated lattice-like TPC S . On all the bottom and the upper-right parts of the table, there are five similar *quadruple stacks*, for $n = 5$ and $G = \mathbb{Z}_{20}$.

Pairs of images via Φ of adjacent vertices in G along coordinate direction x_1 in each such a quadruple stack, except for the quadruple $\Phi(\{-e_1, 0, e_1, 2e_1\})$, only appear as the first two and as the last two entries in each participant quadruple.

Matrices whose columns generate $Ker(\Phi)$ are given for the quadruple stacks for $n = 5$ and $G = \mathbb{Z}_{20}$ at the bottom and upper-right of the table above are:

| | | | | | | | | | | | | | |
|----------|-----------|----|----|----|----|----------|-------|----------|-------|----------|-------|----------|-------|
| 1 | 20 | 17 | 15 | 13 | 11 | 1 | 52031 | 1 | 53031 | 3 | 52043 | 2 | 51042 |
| 3 | 0 | 1 | 0 | 0 | 0 | 3 | 01000 | 2 | 01000 | 4 | 01000 | 4 | 01000 |
| 5 | 0 | 0 | 1 | 0 | 0 | 5 | 33422 | 5 | 33422 | 5 | 12422 | 5 | 23422 |
| 7 | 0 | 0 | 0 | 1 | 0 | 7 | 00010 | 8 | 00010 | 6 | 00010 | 6 | 00010 |
| 9 | 0 | 0 | 0 | 0 | 1 | 9 | 00001 | 9 | 00001 | 7 | 00001 | 8 | 00001 |

This table contains not only matrices whose columns generate $Ker(\Phi)$ but also the vectors B .

2.4. Proof of Theorem 8

Proof. Clearly, the data of a group homomorphism Φ as in the statement of Theorem 8 can always be arranged as a quadruple stack. This implies that all factors of G must be of even order. However, if the excluded case of $G = (\oplus_{i=1}^m \mathbb{Z}_2) \oplus \Psi(G)$ were to yield such a Φ , then at most by taking a change of coordinates in G produced by replacing Φ by its composition with a group automorphism γ of G allows to assume that each quadruple in the stack has terms with fixed coordinates in the first m factors. (For example, if $G = \mathbb{Z}_2 \oplus \mathbb{Z}_4$ with Φ given by $\Phi(e_1) = 1, 1$ and $\Phi(e_2) = 0, 1$, then composing Φ with the automorphism γ of G given by $\gamma(0, 1) = 1, 1$ and $\gamma(1, 0) = 1, 0$ allows to replace Φ by $\Phi' = \gamma\Phi$ that effectively has each quadruple in the corresponding stack with fixed coordinate in the first factor, since in this case $m = 1$.) This leaves a selection for such fixed coordinates

of twice an odd number, which is not a multiple of 4, as needed for the existence of Φ (or Φ'). Thus, $G \neq (\oplus_{i=1}^m \mathbb{Z}_2) \oplus \Psi(G)$, as in the statement. It is clear that if this inequality holds, then a quadruple stack as required always takes place. ■

The simplest example of the excluded situation is that of $G = \mathbb{Z}_2 \oplus \mathbb{Z}_6$, for $n = 3$. Each element i, j of $\mathbb{Z}_2 \oplus \mathbb{Z}_6$ will be written ij (without the separating comma). In G , there are 3 cyclic orbits of order larger than 4, namely order 6. These orbits are generated by 01, 11 and 12, or their opposites, so only these may be taken as images of e_1 (or $-e_1$) via Φ . For example, take $H = \Phi(\{-e_1, O, e_1, 2e_1\}) = \{05, 00, 01, 02\}$. Then $\Phi(e_2)$ (or $\Phi(e_3)$) can only be selected as an element $ij \in G$ of order at least 3 not in H and such that $ij + 01$ is neither in H . Such an element is any of these: 11, 12, 14, 15. Filling the eight elements of V in the plane x_1x_2 with images via Φ by assuming $\Phi(e_2)$ is one of these four elements excludes either $\{03, 04, 13, 14\}$ or $\{03, 04, 10, 11\}$, neither of which can fill the images $\Phi(e_3)$, $\Phi(e_1 + e_3)$, $\Phi(-e_3)$ and $\Phi(e_1 - e_3)$ via the group epimorphism Φ .

2.5. Υ -sequences

To continue, we return to the two tables of Subsection 2.3: note that there are no more cases for $n = 3, 5$, than those presented there. We say that the x_1 -increment $\Upsilon_1 = \Phi(e_1)$ of each resulting lattice-like TPC is the element immediately to the right of $\mathbf{0}$ (the null element of G) in each quadruple stack, namely for $n = 3$: $\Upsilon_1 = \mathbf{1}, \mathbf{3}, \mathbf{3}$; for $n = 5$: $\Upsilon_1 = \mathbf{1}, \mathbf{5}, \mathbf{5}, \mathbf{5}, \mathbf{5}$. In general, we specify an x_j -increment $\Upsilon_j = \Phi(e_j)$, for each $j = 1, \dots, n$, as the third entry of a corresponding tabulated quadruple, and we do this for all such quadruples, preserving boldface just for $\mathbf{0}$ and Υ_1 . To concentrate and encode the introduced notion of a quadruple stack, from now on, the row vectors $\Upsilon = B^t$ of x_j -increments Υ_j will be rearranged as Υ -sequences of the form $(\Upsilon_j, \dots, \Upsilon_n, \Upsilon_1, \Upsilon_2, \dots, \Upsilon_{j-1})$, starting each at the “smallest” Υ_j and, after Υ_n is written, continuing with Υ_1 , where: **(a)** positions Υ_1 appear as boldface entries; **(b)** the subindex j in each case advances one unit stepwise to the right, then back to the first entry if necessary, and so on. For example for $n = 3$ and 5 of Subsection 2.3, and now also for 6, the Υ -sequences are:

$$\begin{array}{ll}
 (1, 3, 5); & (1, 3, 5, 7, 9, 11); \\
 (1, \mathbf{3}, 5); & (1, \mathbf{3}, 5, 7, 9, 11); \\
 (2, \mathbf{3}, 4); & (2, \mathbf{3}, 4, 8, 9, 10); \\
 (\mathbf{1}, 3, 5, 7, 9); & (\mathbf{01}, 03, 05, 11, 13, 15); \\
 (1, 3, \mathbf{5}, 7, 9); & (01, \mathbf{03}, 05, 11, 13, 15); \\
 (1, 2, \mathbf{5}, 8, 9); & (01, \mathbf{03}, 05, 12, 13, 14); \\
 (3, 4, \mathbf{5}, 6, 7); & (02, \mathbf{03}, 04, 11, 13, 15); \\
 (2, 4, \mathbf{5}, 6, 8); & (02, \mathbf{03}, 04, 12, 13, 14);
 \end{array}$$

where the groups G involved in the eight sequences on the right are equal to \mathbb{Z}_{24} (first three sequences) and to $\mathbb{Z}_2 \oplus \mathbb{Z}_{12}$ (final five sequences) and we agree

that commas that separate components of elements of direct-sum groups are eliminated in expressing Υ -sequences, for simplicity of notation. The first two Υ -sequences with $G = \mathbb{Z}_2 \oplus \mathbb{Z}_{12}$ above have associated matrices, (where $a = 10$, $b = 11$, $\mathbf{c} = \mathbf{12}$ and the agreement above is taken into account).

$$\left\| \begin{array}{c|c} \mathbf{01} & \mathbf{c97a0b} \\ \mathbf{03} & \mathbf{0100b0} \\ \mathbf{05} & \mathbf{0010bb} \\ \mathbf{11} & \mathbf{000201} \\ \mathbf{13} & \mathbf{000010} \\ \mathbf{15} & \mathbf{000011} \end{array} \right\| \left\| \begin{array}{c|c} \mathbf{01} & \mathbf{301a0b} \\ \mathbf{03} & \mathbf{3420b0} \\ \mathbf{05} & \mathbf{0010bb} \\ \mathbf{11} & \mathbf{002201} \\ \mathbf{13} & \mathbf{000010} \\ \mathbf{15} & \mathbf{000011} \end{array} \right\| \left\| \begin{array}{c|c} \mathbf{01} & \mathbf{301822} \\ \mathbf{03} & \mathbf{342010} \\ \mathbf{05} & \mathbf{001000} \\ \mathbf{12} & \mathbf{000203} \\ \mathbf{13} & \mathbf{000010} \\ \mathbf{14} & \mathbf{000011} \end{array} \right\| \left\| \begin{array}{c|c} \mathbf{02} & \mathbf{301500} \\ \mathbf{03} & \mathbf{242002} \\ \mathbf{04} & \mathbf{001010} \\ \mathbf{11} & \mathbf{000201} \\ \mathbf{13} & \mathbf{000010} \\ \mathbf{15} & \mathbf{000011} \end{array} \right\| \left\| \begin{array}{c|c} \mathbf{02} & \mathbf{301a0b} \\ \mathbf{03} & \mathbf{2420b0} \\ \mathbf{04} & \mathbf{0010bb} \\ \mathbf{12} & \mathbf{000201} \\ \mathbf{13} & \mathbf{000010} \\ \mathbf{14} & \mathbf{000011} \end{array} \right\|$$

2.6. Characterizing lattice like TPC s and their Υ -sequences

Remark 10. In order to help characterizing lattice-like PDS s as in Theorem 8, notice that a minimal list of expressions of abelian groups G of order $4n$ leading to different lattice-like TPC s of Λ_n by means of the theorem is composed by \mathbb{Z}_{4n} , if G is cyclic, and by the direct sums $G = \oplus_{i=1}^s \mathbb{Z}_{c_i}$ of maximal cyclic subgroups \mathbb{Z}_{c_i} of G , if G is non-cyclic, where $\gcd(c_1, \dots, c_s) > 1$, with the numbers $c_i = 2^{b_i} a_i$, for $i = 1, \dots, s$, having: **(a)** b_i maximal; **(b)** a_i , (resp. b_i), taken as large as possible, successively for the descending indices $i = s, s - 1, \dots, 2, 1$, (resp. $i = s - 1, \dots, 2, 1$); **(c)** the summand \mathbb{Z}_{c_s} in $G = \oplus_{i=1}^s \mathbb{Z}_{c_i}$ associated to the direction x_1 of \mathbb{Z}^n along which $\{O, e_1\}$ lies.

For example, if $n = 6$, then we consider $G = \mathbb{Z}_2 \oplus \mathbb{Z}_{12}$ instead of its equivalent form $G = \mathbb{Z}_4 \oplus \mathbb{Z}_6$. If $n = 8$, then we consider two expressions of the same group, namely $G = \mathbb{Z}_4 \oplus \mathbb{Z}_8$ and $G = \mathbb{Z}_8 \oplus \mathbb{Z}_4$, leading to two different lattice-like TPC s of Λ_8 . We will suggest below a procedure to determine how many homomorphisms Φ exist for each expression of a G in the list claimed in Remark 10, to be stated as Conjecture 12.

For $n = 8$, there are two different homomorphisms $\Phi : \mathbb{Z}^n \rightarrow G$ with a common target group G of order $4n = 32$ [1]. In the setting of Remark 10, we may write $G = \mathbb{Z}_8 \oplus \mathbb{Z}_4$ or $G = \mathbb{Z}_4 \oplus \mathbb{Z}_8$, so we can associate the last summand of G , in each of these two cases, to the direction x_1 of \mathbb{Z}^8 along which $\{O, e_1\}$ lies. Corresponding quadruple stacks, column vector B and matrices M for both cases are respectively on the left and right halves of the following table.

$$\left\| \begin{array}{c|c|c|c} \mathbf{0,3} & \mathbf{0,0} & \mathbf{0,1} & \mathbf{0,2} \\ \mathbf{1,3} & \mathbf{1,0} & \mathbf{1,1} & \mathbf{1,2} \\ \mathbf{2,3} & \mathbf{2,0} & \mathbf{2,1} & \mathbf{2,2} \\ \mathbf{3,3} & \mathbf{3,0} & \mathbf{3,1} & \mathbf{3,2} \\ \mathbf{4,3} & \mathbf{4,0} & \mathbf{4,1} & \mathbf{4,2} \\ \mathbf{5,3} & \mathbf{5,0} & \mathbf{5,1} & \mathbf{5,2} \\ \mathbf{6,3} & \mathbf{6,0} & \mathbf{6,1} & \mathbf{6,2} \\ \mathbf{7,3} & \mathbf{7,0} & \mathbf{6,1} & \mathbf{7,2} \end{array} \right\| \left\| \begin{array}{c|c} \mathbf{0,1} & \mathbf{40123456} \\ \mathbf{1,1} & \mathbf{08654321} \\ \mathbf{2,1} & \mathbf{00100000} \\ \mathbf{3,1} & \mathbf{00010000} \\ \mathbf{4,1} & \mathbf{00001000} \\ \mathbf{5,1} & \mathbf{00000100} \\ \mathbf{6,1} & \mathbf{00000010} \\ \mathbf{7,1} & \mathbf{00000001} \end{array} \right\| \left\| \begin{array}{c|c|c|c} \mathbf{0,7} & \mathbf{0,0} & \mathbf{0,1} & \mathbf{0,2} \\ \mathbf{0,5} & \mathbf{0,6} & \mathbf{0,3} & \mathbf{0,4} \\ \mathbf{1,7} & \mathbf{1,0} & \mathbf{1,1} & \mathbf{1,2} \\ \mathbf{1,5} & \mathbf{1,6} & \mathbf{1,3} & \mathbf{1,4} \\ \mathbf{2,7} & \mathbf{2,0} & \mathbf{2,1} & \mathbf{2,2} \\ \mathbf{2,5} & \mathbf{2,6} & \mathbf{2,3} & \mathbf{2,4} \\ \mathbf{3,7} & \mathbf{3,0} & \mathbf{3,1} & \mathbf{3,2} \\ \mathbf{3,5} & \mathbf{3,6} & \mathbf{3,3} & \mathbf{3,4} \end{array} \right\| \left\| \begin{array}{c|c} \mathbf{0,1} & \mathbf{85423130} \\ \mathbf{0,3} & \mathbf{01000000} \\ \mathbf{1,1} & \mathbf{00457575} \\ \mathbf{1,3} & \mathbf{00010000} \\ \mathbf{2,1} & \mathbf{00001000} \\ \mathbf{2,3} & \mathbf{00000100} \\ \mathbf{3,1} & \mathbf{00000010} \\ \mathbf{3,3} & \mathbf{00000001} \end{array} \right\|$$

It is clear now that the list mentioned in Remark 10 has at least a representative from each group G of order $4n$ leading to a lattice-like TPC of Λ_n . We keep on this track in order for us to be able to count which homomorphisms Φ lead effectively to non-equivalent lattice-like TPC s of Λ_n .

We present now Υ -sequences for $n = 9$, where the boldface entries yield the positions Υ_1 :

$$\begin{array}{cccccccc} (1, & \mathbf{3}, & 5, & 7, & \mathbf{9}, & 11, & 13, & 15, & 17); \\ (1, & 2, & \mathbf{3}, & 4, & \mathbf{9}, & 14, & 15, & 16, & 17); \\ (5, & 6, & 7, & 8, & \mathbf{9}, & 10, & 11, & 12, & 13); \\ (2, & 4, & 6, & 8, & \mathbf{9}, & 10, & 12, & 14, & 16); \end{array}$$

where $G = \mathbb{Z}_{4n} = \mathbb{Z}_{4(2k+1)} = \mathbb{Z}_{36}$. In fact, we consider the generic concatenation

$$(1) \quad \Upsilon = (y_1, \dots, y_k, \mathbf{n}, 2n - y_k, \dots, 2n - y_1) = Y' | \mathbf{n} | (2n - Y'),$$

where, for each $Y' = (y_1, \dots, y_k)$ such that $\{y_1, \dots, y_k\} \subseteq \{1, 2, \dots, 2k\}$ and $y_1 < y_2 < \dots < y_k$, only one entry in $\{y_i, n - y_i\}$ is present in Y' , for each $i = 1, \dots, k$. Each of these concatenations corresponds to an Υ -sequence with n , as indicated, taken boldface, that is as Υ_1 . If Y' is composed by the odd numbers in $\{1, \dots, k\}$, then any divisor of n could also be taken boldface. We denote the sequences (1) in each case orderly as $Y^1, \dots, Y^i, \dots, Y^j, \dots, Y^{2^k}$, setting a super-index to each term, as in $Y^i = (y_1^i, y_2^i, \dots, 2n - y_1^i)$. Their order is given by majorization: if the terms of $Y^i = (y_1^i, \dots, 2n - y_1^i)$ are less than or equal to the corresponding terms of $Y^j = (y_1^j, \dots, 2n - y_1^j)$, then $i < j$.

Because of (1), the four-line table above can be extended to $t = 16$ lines, of which in the table are represented those corresponding to $Y' = (1, 3, 5, 7)$, (leading to three lattice-like TPCs, as observed), $Y' = (1, 2, 3, 4)$, $Y' = (5, 6, 7, 8)$ and $Y' = (2, 4, 6, 8)$, (each leading to a single lattice-like TPC). Thus, there are eighteen homomorphisms $\Phi : \mathbb{Z}^9 \rightarrow \mathbb{Z}_{36}$: 16 with $\Upsilon_1 = \mathbf{9}$ and one each for $\Upsilon_1 = \mathbf{1}$ and $\Upsilon_1 = \mathbf{3}$.

2.7. Breadths of Υ -sequences

For $n = 9$ and $G = \mathbb{Z}_3 \oplus \mathbb{Z}_{12}$, the following constitutes a table corresponding to the one previous to the concatenation (1):

$$\begin{array}{l} A=(\mathbf{01}, \mathbf{03}, 05, 10, 12, 14, 16, 18, 1a); \\ B=(02, \mathbf{03}, 04, 10, 12, 14, 16, 18, 1a); \\ C=(\mathbf{01}, \mathbf{03}, 05, \mathbf{11}, \mathbf{13}, 15, 21, 23, 25); \\ D=(02, \mathbf{03}, 04, 11, 13, 15, 21, 23, 25); \\ E=(02, \mathbf{03}, 04, 12, \mathbf{13}, 14, 22, 23, 24); \end{array}$$

This allows to see that there is a total of ten homomorphisms $\Phi : \mathbb{Z}^9 \rightarrow \mathbb{Z}_3 \oplus \mathbb{Z}_{12}$, one per distinguished boldface entry. Succinctly, some initial columns of the transpose matrices M^t associated to such homomorphisms are:

| $A(\mathbf{01})$ | $A(\mathbf{03})$ | $B(\mathbf{03})$ | $C(\mathbf{01})$ | $C(\mathbf{03})$ | $C(\mathbf{11})$ | $C(\mathbf{13})$ | $D(\mathbf{03})$ | $E(\mathbf{03})$ | $E(\mathbf{13})$ |
|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| c 000 | 3 300 | 3 300 | c 000 | 3 300 | 3 009 | 3 0003 | 3 200 | 3 200 | 3 2000 |
| 9100 | 0400 | 0400 | 9100 | 0400 | 9100 | 91000 | 0400 | 0400 | 01003 |
| 7010 | 7010 | 4010 | 7010 | 7010 | 7010 | 70100 | 4010 | 4010 | 40100 |
| 000 3 | 000 3 | 000 3 | 900 3 | 900 3 | 000 c | 50012 | 030 3 | 020 3 | 20012 |
| <i>a</i> 002 | <i>a</i> 002 | 5002 | 7002 | 7002 | 7002 | 0000 c | 2102 | 1102 | 0000 c |
| 8002 | 8002 | 4002 | 5002 | 5002 | 5002 | 10002 | 1102 | 2002 | 10002 |
| 6002 | 6002 | 3002 | <i>a</i> 001 | <i>a</i> 001 | <i>a</i> 001 | 80001 | 5001 | 4001 | 50004 |
| 4002 | 4002 | 2002 | 8001 | 8001 | 8001 | 60001 | 4001 | 0111 | 30001 |
| 2002 | 2002 | 1002 | 6001 | 6001 | 6001 | 40001 | 3001 | 3001 | 40004 |

with the final remaining columns coinciding with those of the identity 9×9 matrix in the same positions, and where each of the ten cases is indicated with the corresponding capital letter in $\{A, B, C, D, E\}$ of the first display in this subsection, followed by a valid (two-digit) boldface Υ_1 expressed between parenthesis. Here, we also highlighted in boldface those diagonal entries > 1 .

Given an Υ -sequence with x_1 -increment Υ_1 , we define the *breadth* $\beta(\Upsilon, \Upsilon_1)$ as the distance between the two vertices $O(S')$ and $O(S'')$ that correspond to the vertex $\Phi(0) = O = O(\Theta_0^*)$ of Θ_0^* (via adequate translations induced by the lattice $L = \Phi^{-1}(0)$) in two components S' and S'' of the lattice-like TPC determined by Υ and Υ_1 , where S' and S'' are consecutive components along the x_1 axis. In the table above for example, $\beta(C, \mathbf{01}) = \beta(C, \mathbf{11}) = \beta(C, \mathbf{13}) = 12$, while $\beta(C, \mathbf{03}) = 4$, determining just two different breadths for Υ -sequence C . Considering the other six cases for A, B, D, E , we see that there are between 8 and 10 distinct homomorphisms $\Phi : \mathbb{Z}^9 \rightarrow \mathbb{Z}_3 \oplus \mathbb{Z}_{12}$. It rests to answer the following.

Conjecture 11. *Any two TPCs with the same breadth are equivalent.*

In general, $\det(M) = |T| = |G| = 4n$, the volume of a fundamental domain of L . Also, if the last summand \mathbb{Z}_{c_s} in $G = \bigoplus_{i=1}^s \mathbb{Z}_{c_i}$ (as in (b) of Remark 10) corresponds with the first coordinate x_1 in \mathbb{Z}^n along which $\{O, e_1\}$ lies, then the diagonal entry of M corresponding to Υ_1 is $\frac{c_s}{\pi_s(\Upsilon_1)}$, where $\pi_s : G \rightarrow \mathbb{Z}_{c_s}$ is the projection of G onto that last summand, since $\pi_s(\Upsilon_1)$ divides c_s .

Corresponding Υ -sequences for $n = 12$, with G not involving \mathbb{Z}_2 as a summand are:

$$\begin{aligned}
& (\mathbf{1}, \mathbf{3}, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23); \\
& (\mathbf{01}, \mathbf{03}, 05, 10, 12, 14, 16, 18, 1a, 21, 23, 25); \\
& (02, \mathbf{03}, 04, 10, 12, 14, 16, 18, 1a, 22, 23, 24); \\
& (\mathbf{01}, \mathbf{03}, 05, \mathbf{11}, \mathbf{13}, 15, 21, 23, 25, 31, 33, 35); \\
& (02, \mathbf{03}, 04, 12, \mathbf{13}, 14, 22, 23, 24, 32, 33, 34); \\
& (01, \mathbf{03}, 05, 13, 13, 14, 21, 23, 25, 32, 33, 34); \\
& (02, \mathbf{03}, (04, 11, 13, 15, 22, 23, 24, 31, 33, 35);
\end{aligned}$$

where the first line provides two Υ -sequences for $G = \mathbb{Z}_{48}$ and the remaining lines provide eleven Υ -sequences for $G = \mathbb{Z}_4 \oplus \mathbb{Z}_{12}$, of which the fourth line has $\beta(\Upsilon, \mathbf{01}) = \beta(\Upsilon, \mathbf{11}) = 12$ and $\beta(\Upsilon, \mathbf{03}) = \beta(\Upsilon, \mathbf{13}) = 4$ and the fifth line has $\beta(\Upsilon, \mathbf{01}) = \beta(\Upsilon, \mathbf{13}) = 4$, so that there are eight distinct resulting lattice-like TPCs. There are two other Υ -sequences comprisable in one line for $G =$

$\mathbb{Z}_2 \oplus \mathbb{Z}_{24}$, one for each of $\Upsilon_1 = \mathbf{01}$ and $\Upsilon_1 = \mathbf{03}$, and still nine other Υ -sequences comprising in eight lines for $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{12}$, of which one has $\Upsilon_1 = \mathbf{001}$ and eight have $\Upsilon_1 = \mathbf{003}$. All of these remaining Υ -sequences provide distinct lattice-like TPCs.

For $n = 15$ and $G = \mathbb{Z}_{60}$, the Υ -sequences are:

(**1**, **3**, **5**, 7, 9, 11, 13, **15**, 17, 19, 21, 23, 25, 27, 29);
 (1, **2**, **3**, 4, 5, 6, 7, **15**, 23, 24, 25, 26, 27, 28, 29);
 (8, 9, 10, 11, 12, 13, 14, **15**, 16, 17, 18, 19, 20, 21, 22);
 (2, 4, 6, 8, 10, 12, 14, **15**, 16, 18, 20, 22, 24, 28, 30);

so that by (1) there are $2^7 = 128$ homomorphisms Φ with boldface **15** and three additional arising from the first line, with boldface entries **1**, **3** and **5**, yielding a total of 131 homomorphisms $\Phi : \mathbb{Z}^{15} \rightarrow \mathbb{Z}_{60}$ that lead to different lattice-like TPCs.

2.8. Adoption of a shorthand notation

For $n = 18$ and $G = \mathbb{Z}_6 \oplus \mathbb{Z}_{12}$, we adopt the following shorthand for the different Υ -sequences. Let $Y_i^1 = 135$, $Y_i^2 = 234$, (resp. for $Y' = \{1\}$, $Y' = \{2\}$; see (1) above), $P_i = 024$ and $Q_i = 246$, where $i \in \mathbb{Z}_6$. Let $X_i = x_{i1}x_{i2}x_{i3}$ stand for either Y_i^1, Y_i^2, P_i, Q_i . A 6-tuple $(X_0X_1X_2X_3X_4X_5)y$ stands for an Υ -sequence formed by the concatenation of the triples X_i , from $i = 0$ to $i = 5$, with an element y among the composing elements of G taken as the boldface element. (This Υ -sequence must be understood with the subindex i of each X_i taken as the \mathbb{Z}_6 entry of each considered element of $G = \mathbb{Z}_6 \oplus \mathbb{Z}_{12}$, while the corresponding \mathbb{Z}_{12} entry corresponds to the components of X_i). This way for example $(Y_0^1Y_1^1Y_2^1Y_3^1Y_4^1Y_5^1)01 = (\mathbf{01}, 03, 05, 11, 13, 15, 21, 23, 25, 31, 33, 35, 41, 43, 45, 51, 53, 55)$.

Then, a set of representatives of different Υ -sequences for homomorphisms $\Phi : \mathbb{Z}^{18} \rightarrow \mathbb{Z}_6 \oplus \mathbb{Z}_{12}$ is given by:

- (a) $(Y_0^1X_1X_2Y_3^1X_4X_5)01$, where $(X_i, X_{6-i}) \in \{(Y_i^1, Y_{6-i}^1), (P_i, Q_{6-i})\}$, if $x_i \in \{1, 2\}$, (4 cases);
- (b) $(X_0X_1X_2X_3X_4X_5)03$, where $X_i \in \{Y_i^1, Y_i^2\}$ if $i \in \{0, 3\}$ and $(X_i, X_{6-i}) \in \{(Y_i^1, Y_{6-i}^1), (Y_i^2, Y_{6-i}^2), (P_i, Q_{6-i})\}$, if $x_i \in \{1, 2\}$, (36 cases);
- (c) $(Y_0^1Y_1^1Y_2^1Y_3^1Y_4^1Y_5^1)j$, where $j = 11, 13, 21, 23, 31, 33$, (6 cases, with their breadths $\beta(\Upsilon, \Upsilon_1)$ already present in items (a) and (b)).

This yields a lower bound of 40 distinct lattice-like TPCs, in this case. We have to add to this the numbers of Υ -sequences for \mathbb{Z}_{72} , $\mathbb{Z}_3 \oplus \mathbb{Z}_{24}$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_{36}$ in order to obtain a better estimation for the homomorphisms $\Phi : \mathbb{Z}^{18} \rightarrow G$ yielding lattice-like TPCs by means of Corollary 4.

2.9. Generalization via the shorthand notation

If $G = \bigoplus_{i=1}^s \mathbb{Z}_{c_i}$ as in Remark 10, then the last summand \mathbb{Z}_{c_s} yields the coordinate direction x_1 along which the component $\{O, e_1\}$ lies, where $c_s = 2^{b_s} a_s$ with $1 \leq a_s$ odd (=maximal a_i , for $i = 1, \dots, s$) and $b_s \geq 2$. Let $G' = \bigoplus_{i=1}^{s-1} \mathbb{Z}_{c_i}$. We generalize the setting of the last example as in the following items **(A)**–**(B)**, where G' is assumed with $s = 2$.

(A) First, assume $b_2 > 2$. Let $k = \frac{a_2-1}{2}$. Because of (1), there are 2^k sequences of length a_2 with terms in \mathbb{Z}_{2a_2} . We denote these sequences by Y^1, Y^2, \dots, Y^{2^k} , where $Y^1 = \{1, 3, \dots, a_2 - 2, a_2, a_2 + 2, \dots, 2a_2 - 1\}$. These sequences can be used, as in the examples above, to express the different possible lattice-like TPCs. Let $P = \{2j; 0 \leq 2j < 2^{b_2} a_2\}$ and $Q = \{2j; 0 < 2j \leq 2^{b_2} a_2\}$. Then we have x_1 -increments Υ_1 and functions $X : G' \rightarrow \{Y^1, \dots, Y^{2^k}, P, Q\}$ inspired in the previous example (where $G' = \mathbb{Z}_6$ provided the subindexes i of the sequences $X_i \in \Phi(G)$ in each of the 40 cases), with the following characteristics:

- (a) $\Upsilon_1 = (\mathbf{0}, \dots, \mathbf{0}, \mathbf{1})$ and: **(a1)** $X(i) = Y^1$, if $2i = 0$ in G' , and either **(a2)** $X(i) = X(-i) = Y^1$ or **(a3)** $(X(i), X(-i)) = (P, Q)$, otherwise;
- (b) $\Upsilon_1 = (\mathbf{0}, \dots, \mathbf{0}, \mathbf{a}_2)$ and: **(b1)** $X(i) \in \{Y^1, \dots, Y^{2^k}\}$, if $2i = 0$ in G' , and either **(b2)** $X(i) = X(-i) \in \{Y^1, \dots, Y^{2^k}\}$ or **(b3)** $(X(i), X(-i)) = (P, Q)$, otherwise;
- (c) $\Upsilon_1 = (\mathbf{x}_1, \dots, \mathbf{x}_2)$, where $1 < x_2 | a_2$ and $0 \leq x_j \leq \frac{c_j}{2}$, and $X(i) = Y^1$, for every $i \in G'$.

The Υ -sequences arising in item **(c)** have breadths already present in items **(a)** and **(b)**.

(B) Second, if $b_2 = 2$, then the sequences Y^j in item **(b)** above are reduced just to Y^1 . In fact, we dispose only of Y^1 as a sequence of length a_2 with terms in \mathbb{Z}_{2a_2} that conducts to a lattice-like TPC for this case.

To the cases above, we will need to add just the lattice-like TPCs resulting from the following consideration, in order to obtain all of those mentioned in Theorem 8, below.

2.10. General pattern for quadruple stacks

In the examples above, we observe the following pattern for the quadruple stacks. If we can express $n = qm = q(2k + 1)$ for nonnegative integers q, k , then the following sequence yields a set of x_j -increments Υ_j for a quadruple stack leading to a corresponding homomorphism $\Phi : \mathbb{Z}^n \rightarrow \mathbb{Z}_{4n}$, where Υ_1 is a boldface entry and we use the sequence (1) above but with $n = 2k + 1$ replaced by $m = 2k + 1$:

$$(2) \quad \begin{matrix} (y_1, \dots, y_k, & \mathbf{m}, & 2m-y_k, & \dots, & 2m-y_1, & 2m+y_1, & \dots, & 3m-y_k, \\ & 3m, & 4m-y_k, & \dots, & 4m-y_1, & 4m+y_1, & \dots, & 5m-y_k, \\ & \dots, & \dots, & \dots, & \dots, & \dots, & \dots, & \dots, \\ (q-3)m, & (q-2)m-y_k, & \dots, & (q-2)m-y_1, & (q-2)m+y_1, & \dots, & (q-1)m-y_k, \\ (q-1)m, & qm-y_k, & \dots, & qm-y_1. \end{matrix}$$

A particular case of this sequence is:

$$\begin{matrix} (1, 2, \dots, k, & \mathbf{m}, & 2m-k, & \dots, & 2m-1, & 2m+1, & \dots, & 2m+k, \\ & 3m, & 4m-k, & \dots, & 4m-1, & 4m+1, & \dots, & 4m+k, \\ & \dots, & \dots, & \dots, & \dots, & \dots, & \dots, & \dots, \\ (q-3)m, & (q-2)m-k, & \dots, & (q-2)m-1, & (q-2)m+1, & \dots, & (q-2)m+k, \\ (q-1)m, & qm-k, & \dots, & qm-1. \end{matrix}$$

The sequence Y_1 happens here when $q = 1$ and $m = n$. With $m = 9$ we have for example:

$$\begin{aligned} q=1 & \quad (1, 2, 3, 4, \mathbf{9}, 14, 15, 16, 17); \\ q=2 & \quad (1, 2, 3, 4, \mathbf{9}, 14, 15, 16, 17, 19, 20, 21, 22, 27, 32, 33, 34, 35); \\ q=3 & \quad (1, 2, 3, 4, \mathbf{9}, 14, 15, 16, 17, 19, 20, 21, 22, 27, 32, 33, 34, 35, 37, 38, 39, 40, 45, 50, 51, 52, 53). \end{aligned}$$

If $q = 1$, we already have the contributions to Υ -sequences given by the functions X in items **(a)**-**(c)** above, and their corresponding homomorphisms Φ . If $q > 1$, however, there are just 2^k homomorphism Φ with $\Upsilon_1 = \mathbf{m}$ to be counted in each case, for example: 16 for $n = 18$, $G = \mathbb{Z}_{72}$ and $\mathbf{m} = \Upsilon_1 = \mathbf{9}$, so $q = 2$; two for $n = 18$, $G = \mathbb{Z}_{72}$ and $\mathbf{m} = \Upsilon_1 = \mathbf{3}$, so $q = 1$; etc.

In more detail, for example, the quadruple stacks in the above case $q = 2$ are as follows:

| | | | | | | | |
|----|----------|----------|----|----|----|----|----|
| 71 | 8 | 1 | 10 | 53 | 62 | 19 | 28 |
| 70 | 7 | 2 | 11 | 52 | 61 | 20 | 29 |
| 69 | 6 | 3 | 12 | 51 | 60 | 21 | 30 |
| 68 | 5 | 4 | 13 | 50 | 59 | 22 | 31 |
| 63 | 0 | 9 | 18 | 45 | 54 | 27 | 36 |
| 58 | 67 | 14 | 23 | 40 | 49 | 32 | 41 |
| 57 | 66 | 15 | 24 | 39 | 48 | 33 | 42 |
| 56 | 65 | 16 | 25 | 38 | 47 | 34 | 43 |
| 55 | 64 | 17 | 26 | 37 | 46 | 35 | 44 |

that can be further displayed as in the third table in the proof of Theorem 8, above:

| | | | | | | | | | | | | | | | | | | | |
|----|----------|----------|----|----|----------|----------|----|----|----------|----------|----|----|----------|----------|----|----|----------|----------|----|
| 63 | 71 | 8 | 18 | 63 | 58 | 67 | 18 | 63 | 53 | 62 | 18 | 63 | 40 | 49 | 18 | 63 | 45 | 54 | 18 |
| | 0 | 9 | | | 0 | 9 | | | 0 | 9 | | | 0 | 9 | | | 0 | 9 | |
| | 1 | 10 | | | 14 | 23 | | | 19 | 28 | | | 32 | 41 | | | 27 | 36 | |
| 63 | 70 | 7 | 18 | 63 | 57 | 66 | 18 | 63 | 52 | 61 | 18 | 63 | 39 | 48 | | | | | |
| | 0 | 9 | | | 0 | 9 | | | 0 | 9 | | | 0 | 9 | | | | | |
| | 2 | 11 | | | 15 | 24 | | | 20 | 29 | | | 33 | 42 | | | | | |
| 63 | 69 | 6 | 18 | 63 | 56 | 65 | 18 | 63 | 51 | 60 | 18 | 63 | 38 | 47 | | | | | |
| | 0 | 9 | | | 0 | 9 | | | 0 | 9 | | | 0 | 9 | | | | | |
| | 3 | 12 | | | 16 | 25 | | | 21 | 30 | | | 34 | 43 | | | | | |
| 63 | 68 | 5 | 18 | 63 | 55 | 64 | 18 | 63 | 50 | 59 | 18 | 63 | 37 | 46 | | | | | |
| | 0 | 9 | | | 0 | 9 | | | 0 | 9 | | | 0 | 9 | | | | | |
| | 4 | 13 | | | 17 | 26 | | | 22 | 31 | | | 35 | 44 | | | | | |

2.11. Final Conjecture

We can summarize the developments above, including Corollary 9 and Subsections 2.2–10, as the following suggestion of a procedure to determine how many homomorphisms Φ exist for each expression of a G in the list claimed in Remark 10, as follows.

Conjecture 12. *Assume the setting of Theorem 8. If $n = 2^k$ then for each abelian group G of order $4n$ as in Remark 10 there exists just one homomorphism $\Phi : \mathbb{Z}^n \rightarrow G$ leading to a lattice-like TPC of Λ_n . Otherwise, let $n = qm = q(2k+1)$, where $0 \leq q, k \in \mathbb{Z}$. If $q = 1$, then the homomorphisms Φ from \mathbb{Z}^n onto the groups G listed in Remark 10 are as indicated in items **(A)(a)–(c)** and **(B)** of Subsection 2.9, taking into account the possible elimination of repeated cases provided by the concept of breadth of an Υ -sequence in Subsection 2.7. If $q > 1$, then the existing Υ -sequences for such homomorphisms are as indicated in (2) above.*

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