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ALMOST-RAINBOW EDGE-COLORINGS OF SOME SMALL SUBGRAPHS

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Abstract

Let $f(n, p, q)$ be the minimum number of colors necessary to color the edges of K_n so that every K_p is at least q-colored. We improve current bounds on these nearly "anti-Ramsey" numbers, first studied by Erdős and Gyárfás. We show that $f(n, 5, 9) \geq \frac{7}{4}n - 3$, slightly improving the bound of Axenovich. We make small improvements on bounds of Erdős and Gyárfás by showing $\frac{5}{6}n+1 \leq f(n, 4, 5)$ and for all even $n \not\equiv 1 \pmod 3$, $f(n, 4, 5) \leq n-1$. For a complete bipartite graph $G = K_{n,n}$, we show an n-color construction to color the edges of G so that every $C_4 \subseteq G$ is colored by at least three colors. This improves the best known upper bound of Axenovich, Füredi, and Mubayi.

Keywords: Ramsey theory, generalized Ramsey theory, rainbow-coloring, edge-coloring, Erdős problem.

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1. Introduction

1.1. Definitions

For basic graph theoretic notation and definition see Diestel [3]. All graphs G are undirected with the vertex set V and edge set E. We use $|G|$ for $|V|$ and $\|G\|$ for

$|E|$. K_n denotes the complete graph on n vertices and $K_{n,m}$ the bipartite graph with n vertices and m vertices in the first and second part, respectively. For any edge (u, v), let $\mathcal{C}(u, v)$ be the color on that edge, and for any vertex v, let $\mathcal{C}(v)$ be the set of colors on the edges incident to v. We say that an edge-coloring is *proper* if every pair of incident edges are of different colors. If vertices u, v are adjacent, we write $u \sim v$.

1.2. Coloring edges

Given a graph G of order n and integers p, q so that $2 \leq p \leq n$ and $1 \leq q \leq \binom{p}{2}$, call an edge-coloring (p, q) if every $K_p \subseteq K_n$ receives at least q colors on its edges. Let $f(n, p, q)$ be the minimum colors in a (p, q) coloring of K_n. This generalization of classical Ramsey functions was first mentioned by Erdős in [4] and later studied by Erdős and Gyárfás in [5]. Further, define $\phi(n, p, q)$ to be the minimum colors in a proper (p, q) coloring of K_n.

Extending the definition, for any graph G, call an edge coloring (H, q) if every subgraph $H \subseteq G$ receives at least q colors on its edges. Let $f(G, H, q)$ be the minimum colors in an (H, q) coloring of the edges of G. We say that a coloring of H is *almost-rainbow* if $q = \|H\| - 1$, that is, one color is repeated once.

For an extended survey regarding bounds on rainbow colorings, see [7].

Using the Local Lemma, the authors in [5] were able to produce bounds for $f(n, p, q)$, with several difficult cases unresolved. Among those were $f(n, 4, 3)$, $f(n, 4, 4)$, $f(n, 4, 5)$, and $f(n, 5, 9)$. In these cases they showed that $f(n, 4, 3) \leq c\sqrt{n}$, $c\sqrt{n} \leq f(n, 4, 4) \leq cn^{\frac{2}{3}}$, $\frac{5n-1}{6} \leq f(n, 4, 5) \leq n$, and $\frac{4}{3}n \leq f(n, 5, 9) \leq cn^{\frac{3}{2}}$. The authors further mentioned that in this branch of generalized Ramsey theory, finding the orders of magnitude of $f(n, 4, 4)$ and $f(n, 5, 9)$ are "the most interesting open problems, at least to show that the latter is non-linear". The authors then stated the linearity of said function as Problem 1.

As for $f(n, 4, 5)$, the authors showed that $\frac{5(n-1)}{6} \leq f(n, 4, 5)$ with an upper bound of n for odd n and $n - 1$ for even n if $n - 1$ is prime.

In [9], Mubayi showed that

$$f(n, 4, 3) \leq e^{O(\sqrt{\log n})}$$

and in [8] Kostochka and Mubayi showed that for some constant c,

$$f(n, 4, 3) \geq \frac{c \log n}{\log \log \log n}.$$

Fox and Sudakov in [6], further improved the lower bound to $\frac{\log n}{4000}$.

As for the other case, in [1], Axenovich showed that for some constant c,

$$\frac{1 + \sqrt{5}}{2} n \leq f(n, 5, 9) \leq 2n^{1 + \frac{c}{\sqrt{\log n}}}.$$

In that same paper, she remarked that Tóth had communicated to her that the lower bound can be improved to $2n-6$, however, the result has remained unpublished for over ten years.

In Section 2, we show

$$f(n,5,9) \geq \frac{7}{4}n - 3.$$

In Section 3, we make minimal improvements in the work of [5], showing $\frac{5}{6}(n-1)+1 \leq f(n,4,5) \leq n-1$ for even n not congruent to one mod three.

In [2], the authors showed that $f(K_{n,n}, C_4, 3) \geq \frac{2}{3}n$, $f(K_{n,n}, C_4, 3) \leq n$ for odd $n \geq 5$, and $f(K_{n,n}, C_4, 3) \leq n+1$ for even $n \geq 5$.

In Section 4, we show

$$f(K_{n,n}, C_4, 3) \leq n, \text{ for all } n \geq 3.$$

We believe that this upper-bound is the best possible.

2. Almost-rainbow Five-cliques

2.1. The main tool

Let $f(G)$ be the minimum number of colors needed to color the edges of G so that every path or cycle with four edges is at least three-colored.

Let $\phi(G)$ be defined as $f(G)$ above, except replace "color" by "properly color". By arguments from [1] it is easy to see that $f(n,5,9) \leq \phi(n,5,9) = \phi(K_n)$.

Lemma 1. $\phi(K_{2,n}) = \lceil \frac{3}{2}n \rceil$.

Proof. Suppose the edges of $G = K_{2,n}$ are properly colored so that every path of length four receives at least three colors. Call the vertices in the first part of G, u and v. Choose a color $a \in \mathcal{C}(u) \cap \mathcal{C}(v)$ so that for some vertices x, y in the second part of G, $a = \mathcal{C}(u,x) = \mathcal{C}(v,y)$. Note that there exist colors b, c so that $b = \mathcal{C}(u,y)$, $c = \mathcal{C}(v,x)$, and $b, c \in (\mathcal{C}(u) \cup \mathcal{C}(v)) \backslash (\mathcal{C}(u) \cap \mathcal{C}(v))$. Since there are two colors for every one in $\mathcal{C}(u) \cap \mathcal{C}(v)$, we can say that

$$|\mathcal{C}(u) \cap \mathcal{C}(v)| \leq \left\lfloor \frac{1}{2} \left|(\mathcal{C}(u) \cup \mathcal{C}(v)) \backslash (\mathcal{C}(u) \cap \mathcal{C}(v))\right| \right\rfloor. \tag{1}$$

Applying this inequality to the principle of inclusion-exclusion, we write

$$|\mathcal{C}(u) \cup \mathcal{C}(v)| = |\mathcal{C}(u)| + |\mathcal{C}(v)| - |\mathcal{C}(u) \cap \mathcal{C}(v)| \geq 2n - \frac{1}{3}\left|\mathcal{C}(u) \cup \mathcal{C}(v)\right|.$$

Solving for the union we get

$$|\mathcal{C}(u) \cup \mathcal{C}(v)| \geq \frac{3}{2}n. \tag{2}$$

For the upper bound, we construct an edge-coloring of $G = K_{2,n}$ with $\lceil \frac{3}{2}n \rceil$ colors. Label the vertices of the first part of G, u, v and the second part $\{v_1, v_2, \dots, v_n\}$. Let $r = \lceil \frac{n}{2} \rceil$. Color the edges $(v_1, u), (v_2, u), \dots, (v_r, u)$ by the colors $1, \dots, r$. If n is even, color the edges $(v_n, v), (v_{n-1}, v), \dots, (v_{n-r+1}, v)$ from the set of colors $\{1, \dots, r\}$. If n is odd, color the edges $(v_n, v), (v_{n-1}, v), \dots, (v_{n-r+2}, v)$ by some of the colors from the set $\{1, \dots, r\}$. Color the remaining edges distinctly by all the colors not previously used. Let i and j be such that $\mathcal{C}(u, v_i) = \mathcal{C}(v, u_j)$. Notice that for any $k \in \{1, \dots, n\}$, $\{\mathcal{C}(u, v_i), \mathcal{C}(u, v_j), \mathcal{C}(v, v_i), \mathcal{C}(u, v_k)\}$ are pairwise distinct. Hence every 4-path receives at least three colors. ■

2.2. A small improvement

Theorem 2. $f(n, 5, 9) \geq \frac{7}{4}n - 3$.

Proof. Consider a $(5, 9)$ edge-coloring of $G = K_n$ using s colors. Using the argument of Axenovich [1], we first assume that the coloring is not proper, so there exist incident edges (v_1, v_2) and (v_1, v_3) of the same color. For the coloring to remain $(5, 9)$, all edges of $G \backslash \{(v_1, v_2), (v_1, v_3)\}$ incident to $\{v_1, v_2, v_3\}$ must be of different colors and not $\mathcal{C}(v_1, v_2)$ or $\mathcal{C}(v_2, v_3)$. Therefore, $s \geq 3n - 7 \geq \frac{7}{4}n - 3$ for $n \geq 5$.

Next we assume the coloring is proper. By the pigeonhole principle there exists a color, call it a, used on at least $\binom{n}{2}/s$ edges. Let A be the set of vertices adjacent to edges colored a and choose vertices $u, v \in A$ so that $c(u, v) = a$.

We say that an edge is *in* A if both vertices adjacent to that edge are in A. Notice that the number of colors on the edges in A adjacent to $u \geq 2\binom{n}{2}/s - 1$, the same for v, and $c(u, v)$ is counted both times. Let H be the complete bipartite graph with vertices $\{u, v\}$ in the first part and the vertices of $G \backslash A$ in the second part. Let the edge coloring of H be induced by the edge coloring of G. For any $x \in A$ and $y \in G$, $\mathcal{C}(u, x) \neq \mathcal{C}(v, y)$, else we produce a two-colored four-edge path. The same reasoning holds for $y \in A$ and $x \in G$. This implies that the colors on the edges of H are distinct from the colors previously counted. Hence we apply Lemma 1 to H to obtain

$$s \geq 2\frac{\binom{n}{2}}{s} - 1 + 2\frac{\binom{n}{2}}{s} - 1 - 1 + \frac{3}{2}\left(n - 2\frac{\binom{n}{2}}{s}\right). \tag{3}$$

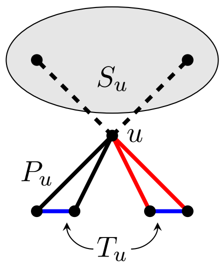
Solving for s we obtain the result. ■

3. Almost-rainbow Four-cliques

We obtain a marginal improvement for the lower bound on $f(n, 4, 5)$ and extend the even case of the upper bound from [5] to all complete graphs with orders not congruent to one modulo three.

Theorem 3. (i) $\frac{5}{6}(n-1)+1 \leq f(n,4,5)$.
(ii) $f(n,4,5) \leq n-1$ *for even* $n \not\equiv 1 \pmod 3$.

Proof. Given a $(4,5)$ coloring of the edges of $G = K_n$, for a fixed vertex u, let P_u denote the set of edges incident to u, whose colors are repeated on other edges incident to u. Let S_u denote the set of edges with non-repeated colors, incident to u. Let T_u denote the set of edges incident to edges from P_u of the same color.



Notice that

1. $\mathcal{C}(P_u) \cap \mathcal{C}(S_u) = \emptyset$ by definition.
2. $\mathcal{C}(P_u) \cap \mathcal{C}(T_u) = \emptyset$ else we obtain an induced four-colored K_4 on the edges $p \in P_u$ and $t \in T_u$ that share the same same color and the edges $p_1, p_2 \in P_u$ that share the same color and are incident to t (p may be equal to p_1, depending on the coloring).
3. $\mathcal{C}(S_u) \cap \mathcal{C}(T_u) = \emptyset$ else we obtain an induced four colored K_4 on the the edge $s \in S_u$ and $t \in T_u$ of the same color and the two edges of P_u with the same color, which are incident to t.
4. For any vertex v distinct from u, if $(u, v) \in P_u$ so that $\mathcal{C}(u, v) = \mathcal{C}(u, w)$ for some w, then $(u, v) \notin P_v$ and $(v, w) \notin P_v$.
5. For any vertex v distinct from u, $T_u \cap T_v = \emptyset$.

Notice that

$$2\sum_{u\in V(G)} |T_u| = \sum_{u\in V(G)} |P_u|,$$

so that

$$\sum\nolimits_u |T_u| + \sum\nolimits_u |P_u| = 3\sum\nolimits_u |T_u| = 3\frac{1}{n}\sum\nolimits_u |T_u| \times n \leq \binom{n}{2}$$

by the above claim 5, and we obtain

$$\frac{1}{n}\sum\nolimits_u |T_u| \leq \frac{n-1}{6}.$$

By the pigeonhole principle, choose a vertex u so that $|T_u| \leq \frac{n-1}{6}$. Notice that $n-1 = \deg u = |S_u| + |P_u| \leq |S_u| + \frac{n-1}{3}$, so that

$$|S_u| \geq \frac{2}{3}(n-1).$$

Summing up the colors of edges incident to u we get

$$|\mathcal{C}(u)| = |S_u| + \frac{1}{2}|P_u| \geq \frac{2}{3}(n-1) + \frac{1}{6}(n-1) = \frac{5}{6}(n-1).$$

However, $\mathcal{C}(T_u)$ must be nonempty and distinct from the colors counted above, hence

$$|\mathcal{C}(u)| \geq \frac{5}{6}(n-1) + 1.$$

For the upper bound we color the edges of K_n by a classical proper coloring (see [10] for example) and show that such a coloring is $(4, 5)$.

For odd n, we n-color the edges of K_n by drawing the vertices in the form of a regular n-gon and coloring the consecutive edges around the boundary in order with colors 1 to n. Next we color every edge parallel to a boundary edge by the same color as that boundary edge. Call the resulting labeled graph G_n. Notice that every $K_4 \subseteq G_n$ with a pair of parallel edges is a non-rectangular trapezoid. Hence the coloring is $(4, 5)$.

For even n, choose a K_{n-1} subgraph and color it as above, obtaining G_{n-1}. Next construct the graph $w \times G_{n-1}$, joining the above graph to a vertex w. Since for any vertex u of G_{n-1}, there are only $n-2$ incident edges, some color is missing. Apply this color to the edge (u, w) and continue likewise for all vertices of G_{n-1}. Call the resulting labeled graph G_n^*.

For vertices $x, y, z \in G_n^*$ with so that (x, y) and (y, z) are boundary edges, we say that y is *opposite* an edge e if the line bisecting angle uvw is the perpendicular bisector of e. Notice that the edges opposite to y share the same color, which is not used on any edge incident to y. By the above observation, $G_{n-1} \subseteq G_n^*$ is $(4, 5)$-colored, hence it is enough to show that for w as chosen above in the definition of G_n^* and any other distinct vertices x, y, z of G_n^*, the induced subgraph receives at most one repeated color. Choose any vertex $v \in G_n^*$. For $i = 1, \ldots, n-2$ label the vertices with counterclockwise distance i from v, u_i, where arithmetic of label indices is performed modulo $n-1$. Notice that the only edges that share the color $\mathcal{C}(w, v)$ are $(u_1, u_{-1}), (u_2, u_{-2}), \ldots, (u_{n-2}, u_{-(n-2)})$. For $i = 1, \ldots, \frac{n-2}{2}$, if $\mathcal{C}(u_i, w) = \mathcal{C}(u_{-i}, v)$, then for any edge e opposite u_i, $\mathcal{C}(e) = \mathcal{C}(u_{-i}, v)$. However, this means that

$$\mathcal{C}(u_{i-1}, u_{i+1}) = \mathcal{C}(e) = \mathcal{C}(u_{-i}, v) \Leftrightarrow vu_{2k} = vu_{-k}$$

$$\Leftrightarrow 3k \equiv 0 \,(\mathrm{mod}\ (n-1)) \Leftrightarrow n \equiv 1 \,(\mathrm{mod}\ 3). \qquad \blacksquare$$

4. Almost-rainbow Four-cycles

We show the improved upper bound for the bipartite problem, when the two parts of G are of equal size.

Theorem 4.

$$f(K_{n,n}, G_4, 3) \leq n, \text{ for all } n \geq 3.$$

4.1. The coloring

We will explore the matrix

$$G = \begin{pmatrix} 1 & 2 & 3 & \ldots & r & \ldots & c+1 & \ldots & n \\ 3 & 1 & 2 & \ldots & r-1 & \ldots & c & \ldots & n-1 \\ v_3 & n-1 & 1 & \ldots & r-2 & \ldots & c-1 & \ldots & n-2 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{n+1-r} & r+1 & r+2 & \ldots & 1 & \ldots & r+c & \ldots & r \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{n-1} & 3 & 4 & \ldots & r+1 & \ldots & c+2 & \ldots & 2 \\ n-2 & u_2 & u_3 & \ldots & u_r & \ldots & u_{c+1} & \ldots & 1 \end{pmatrix}.$$

The values of v_i and u_i will be defined shortly.

Let permutation σ be the $n-1$ cycle $(1\ 2\ \cdots\ n-1)$. That is, σ sends i to $i+1 \,(\mathrm{mod}\ n-1)$. For a natural number m we shall write $m \,(\mathrm{mod}\ (n-1))$ for its representative in $\{1, 2, \ldots, n-1\}$. For each r we defined $\sigma^{(r)}$ by the rule $\sigma^{(r)}(c) \equiv r + c \,(\mathrm{mod}\ (n-1))$. Let us start with the matrix

$$C = \begin{pmatrix} 2 & 3 & \dots & c+1 & \dots & n \\ \sigma^0(1) & \sigma^0(2) & \dots & \sigma^0(c) & \dots & \sigma^0(n-1) \\ \sigma^{n-2}(1) & \sigma^{n-2}(2) & \dots & \sigma^{n-2}(c) & \dots & \sigma^{n-2}(n-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma^r(1) & \sigma^r(2) & \dots & \sigma^r(c) & \dots & \sigma^r(n-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sigma^2(1) & \sigma^2(2) & \dots & \sigma^2(c) & \dots & \sigma^2(n-1) \end{pmatrix}.$$

We define the matrix G by adding the first column $V = \{v_1, \dots, v_{n-1}, vu\}$ and the last row $U = \{vu, u_2, \dots, u_n\}$ to the matrix C.

$$G = \begin{pmatrix} v_1 & 2 & 3 & \dots & c+1 & \dots & n \\ v_2 & \sigma^0(1) & \sigma^0(2) & \dots & \sigma^0(c) & \dots & \sigma^0(n-1) \\ v_3 & \sigma^{n-2}(1) & \sigma^{n-2}(2) & \dots & \sigma^{n-2}(c) & \dots & \sigma^{n-2}(n-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{n+1-r} & \sigma^r(1) & \sigma^r(2) & . & \sigma^r(c) & . & \sigma^r(n-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{n-1} & \sigma^2(1) & \sigma^2(2) & \dots & \sigma^2(c) & \dots & \sigma^2(n-1) \\ vu & u_2 & u_3 & \dots & u_{c+1} & \dots & u_n \end{pmatrix}.$$

The entries of G will be defined as follows: for every 4-tuple $(i, j; l, m)$ with $1 \le i < j \le n$ and $1 \le l < m \le n$ the (2×2) matrix

$$G(i, j; l, m) = \begin{pmatrix} a_{il} & a_{im} \\ a_{jl} & a_{jm} \end{pmatrix}.$$

We consider the colorings for the edges V and U in three types of even $n \,(\mathrm{mod}\ 6)$.

Type 1: Matrix $G_1 = G$ for $n \equiv 2 \,(\mathrm{mod}\ 6)$; $[\, n = 2 + 6k,\ k \ge 1 \,]$

$$a_{i,1} = \begin{cases} 1, & i = 1, \\ 3, & i = 2, \\ n, & 3 \le i \le \frac{n}{2} + 1, \\ 2(i-1) - n, & \frac{n}{2} + 2 \le i \le n - 1, \\ n - 2, & i = n. \end{cases}$$

$$a_{n,l} = \begin{cases} n - 2l, & 1 \le l \le \frac{n}{2} - 1, \\ n, & \frac{n}{2} \le l \le n - 2, \\ n - 1, & l = n - 1, \\ 1, & l = n. \end{cases}$$

Type 2: Matrix $G_2 = G$ for $n \equiv 6 \,(\mathrm{mod}\ 6)$; $[\, n = 6 + 6k,\ k \ge 1 \,]$

We define Y as $\frac{n}{2}-2$ for even k, and as $\frac{n}{2}+1$ for odd k.

$$a_{i,1} = \begin{cases} 1, & i=1, \\ 3, & i=2, \\ n, & 3 \le i \le \frac{n}{2}+1, \\ Y, & i=\frac{n}{2}+2, \\ 2(i-2)-n, & \frac{n}{2}+3 \le i \le n-1, \\ n-2, & i=n. \end{cases}$$

$$a_{n,l} = \begin{cases} n-2, & l=1, \\ n-2(l+1), & 2 \le l \le \frac{n}{2}-2, \\ Y, & l=\frac{n}{2}-1, \\ n, & \frac{n}{2} \le l \le n-2, \\ n-1, & l=n-1, \\ 1, & l=n. \end{cases}$$

Exception for $n = 6$; $[k = 0]$ the first row $V = \{1, 5, 6, 6, 4\}$, the last column $U = \{3, 6, 6, 6, 5, 1\}$.

Type 3: Matrix $G_3 = G$ for $n \equiv 4 \pmod 6$; $[\, n = 4 + 6k,\ k \ge 4\,]$

The regularity starts with $n > 22$.

$$a_{i,1} = \begin{cases} 1, & i=1, \\ 3, & i=2, \\ n, & 3 \le i \le \frac{n}{2}+1, \\ n-9, & i=\frac{n}{2}+2, \\ 2(i-2)-n, & \frac{n}{2}+3 \le i \le \frac{5n+4}{6}, \\ 2(i-1)-n, & \frac{5n+10}{6} \le i \le n-1, \\ n-2, & i=n. \end{cases}$$

$$a_{n,l} = \begin{cases} n-2l, & 1 \le l \le \frac{n-4}{6}, \\ n-2(l+1), & \frac{n+2}{6} \le l \le \frac{n}{2}-2, \\ n-9, & l=\frac{n}{2}-1, \\ n, & \frac{n}{2} \le l \le n-2, \\ n-1, & l=n-1, \\ 1, & l=n. \end{cases}$$

Exceptions:

For $n = 10$ we replace $(n-9)$ with $(n-8)$.

For $n = 16$ we replace $(n-9)$ with $(n-11)$.

For $n = 22$ we replace $(n-9)$ with $(n-5)$ and the definitions:

$$a_{i,1} = \begin{cases} 2(i-2)-n, & \frac{n}{2}+3 \le i \le \frac{5n-2}{6}, \\ 2(i-1)-n, & \frac{5n+4}{6} \le i \le n-1. \end{cases}$$

$$a_{n,l} = \begin{cases} n-2l, & 1 \le l \le \frac{n-10}{6}, \\ n-2(l+1), & \frac{n-4}{6} \le l \le \frac{n}{2}-2. \end{cases}$$

4.2. Sketch of proof

First, we show that every 4-cycle defined in the *basic coloring* (matrix entries a_{ij} where $1 < i \leq n, 1 \leq j < n$) is *almost rainbow*. That is, given $i < j$ and $l < m$ we show that $a_{i,l}, a_{j,l}, a_{i,m}, a_{j,m}$ contains at least three distinct elements in the *basic coloring*.

Step 1. We start with the matrix C and look at two occurrences, which are identical for each of the types of even $n \,(\text{mod } 6)$ specified above.

Case 1. We take the submatrix of $G(i,j;l,m)$ with $2 \leq l < m \leq n$, $2 \leq i < j < n$, and let $s = (n+1) - i$, $t = (n+1) - j$. A typical (2×2) submatrix in this case has the form:

$$\begin{pmatrix} \sigma^s(l-1) & \sigma^s(m-1) \\ \sigma^t(l-1) & \sigma^t(m-1) \end{pmatrix}.$$

We wish to show there are three distinct elements: $\sigma^s(l-1) \neq \sigma^t(l-1)$, $\sigma^t(l-1) \neq \sigma^t(m-1)$, $\sigma^s(l-1) \neq \sigma^t(m-1)$.
Suppose $\sigma^{(s)}(l-1) \equiv \sigma^{(t)}(l-1) \Rightarrow s \equiv t \,(\text{mod } n-1)$, which is a contradiction. Suppose $\sigma^{(t)}(l-1) \equiv \sigma^{(t)}(m-1) \Rightarrow l \equiv m \,(\text{mod } n-1)$, which is a contradiction. Suppose $\sigma^{(s)}(l-1) \equiv \sigma^{(t)}(m-1) \Rightarrow s+l \equiv t+m \,(\text{mod } n-1)$, and assume there are three distinct elements: $\sigma^t(l-1) \neq \sigma^t(m-1)$, $\sigma^s(m-1) \neq \sigma^t(m-1)$, $\sigma^s(m-1) \neq \sigma^t(l-1)$. Follow the argument above the first two inequalities are correct. Suppose $\sigma^s(m-1) \equiv \sigma^t(l-1) \Rightarrow s+m \equiv t+l \,(\text{mod } n-1)$. Subtracting equations $s+l \equiv t+m$ and $s+m \equiv t+l \Rightarrow l \equiv m \,(\text{mod } n-1)$, which is a contradiction. One of the following two sets has three distinct elements: $\{\sigma^s(l-1),\ \sigma^t(l-1),\ \sigma^t(m-1)\}$ or $\{\sigma^t(l-1),\ \sigma^t(m-1),\ \sigma^s(m-1)\}$.

Case 2. We take the submatrix of $G(i,j;l,m)$ with $2 \leq l < m \leq n$, $i = 1$, $1 < j < n$, and let $r = (n+1) - j$. A typical (2×2) submatrix has the form:

$$\begin{pmatrix} l & m \\ \sigma^r(l-1) & \sigma^r(m-1) \end{pmatrix}.$$

We wish to show there are three distinct elements: $l \neq m$, $m \neq \sigma^r(m-1)$, $l \neq \sigma^r(m-1)$. Suppose $m \equiv \sigma^{(r)}(m-1) \Rightarrow r \equiv 1 \,(\text{mod } n-1)$, which is a contradiction. Suppose $l \equiv \sigma^{(r)}(m-1) \Rightarrow l \equiv r+m-1 \,(\text{mod } n-1)$, and assume there are three distinct elements: $\sigma^r(l-1) \neq \sigma^r(m-1)$, $m \neq \sigma^r(m-1)$, $m \neq \sigma^r(l-1)$. The first two inequalities are correct. Suppose $m \equiv \sigma^r(l-1) \Rightarrow m \equiv r+l-1 \,(\text{mod } n-1)$. Subtracting equations $l \equiv r+m-1$ and $m \equiv r+l-1 \Rightarrow r = 1$ $(\text{mod } n-1)$, which is a contradiction. One of the following two sets has three distinct elements: $\{l,\ m,\ \sigma^r(m-1)\}$ or $\{\sigma^r(l-1),\ \sigma^r(m-1),\ m\}$.

Step 2. For matrix $G(i,j;l,m)$ with $i = 1$, $j = n$ and $2 \leq l < m \leq n$ we look at five cases and consider every matrix type defined above of even $n \,(\text{mod } 6)$.

Case 1. We take $G(i,j;l,m)$ with $2 \le l < m \le \frac{n}{2} - 1$, $i = 1$, $j = n$.

Subcase 1.1. Consider G_1,

$$\begin{pmatrix} l & m \\ n-2l & n-2m \end{pmatrix}.$$

We wish to show there are three distinct entries: $l \neq n-2l$, $n-2l \neq n-2m$, $l \neq n-2m$. Suppose $l = n-2l \Rightarrow 3l = n$ and since $n = 2+6k$ this is a contradiction. Suppose $n-2l = n-2m \Rightarrow l = m$, which is a contradiction. Suppose $l = n-2m$ and we wish to show there are three distinct elements: $l \neq m$, $l \neq n-2l$, $m \neq n-2l$. As shown above the first inequality is correct. Suppose $m = n-2l$ and since $l = n-2m \Rightarrow l = m$, which is a contradiction. One of the following two sets has three distinct elements: $\{l, n-2l, n-2m\}$ or $\{l, m, n-2l\}$.

Subcase 1.2. Consider G_2.

1. G_2 with $2 \le l < m \le \frac{n}{2} - 2$, $i = 1$, $j = n$,

$$\begin{pmatrix} l & m \\ n-2(l+1) & n-2(m+1) \end{pmatrix}.$$

We wish to show there are three distinct entries: $l \neq n-2(m+1)$, $l \neq n-2(l+1)$, $n-2(l+1) \neq n-2(m+1)$. Suppose $l = n-2l-2 \Rightarrow 3l = n-2$ and since $n = 6+6k$ this is a contradiction. Suppose $n-2l = n-2m \Rightarrow l = m$, which is a contradiction. Suppose $l = n-2(m+1)$ and we wish to show there are three distinct elements: $l \neq m$, $l \neq n-2(l+1)$, $m \neq n-2(l+1)$. As shown above the first inequality is correct. Suppose $m = n-2(l+1)$ and since $l = n-2(m+1) \Rightarrow l = m$, which is a contradiction. One of the following two sets has three distinct elements: $\{l, n-2(m+1), n-2(l+1)\}$ or $\{l, m, n-2(l+1)\}$.

2. G_2 with $2 \le l \le \frac{n}{2} - 2$, $m = \frac{n}{2} - 1$, $i = 1$, $j = n$,

$$\begin{pmatrix} l & m \\ n-2(l+1) & Y \end{pmatrix}.$$

If K is *even* $\Rightarrow m = \frac{n}{2} - 1$, $Y = \frac{n}{2} - 2 \Rightarrow Y = m-1$.

We wish to show there are three distinct entries: $l \neq m$, $m \neq m-1$, $l \neq m-1$. Assume $l = m-1$ and we wish to show there are three distinct entries: $m \neq m-1$, $m \neq n-2(l+1)$, $m-1 \neq n-2(l+1)$. Suppose $m = n-2(l+1)$ and since $l = m-1$ and $m = \frac{n}{2} - 1 \Rightarrow n = 6$, which is a contradiction. Suppose $m-1 = n-2(l+1)$ and since $m = \frac{n}{2} - 1$ and $l = m-1 \Rightarrow n = 8$, which is a contradiction. One of the following two sets has three distinct elements: $\{l, m, Y\}$ or $\{m, Y, n-2(l+1)\}$.

If K is *odd* $\Rightarrow m = \frac{n}{2} - 1$, $Y = \frac{n}{2} + 1 \Rightarrow Y = m+2$. Three distinct elements are $\{l, m, Y\}$.

Subcase 1.3. Consider G_3.

1. G_3 with $i = 1$, $j = n$ and $(2 \leq l < m \leq \frac{n-4}{6}$ or $\frac{n+2}{6} \leq l < m \leq \frac{n}{2} - 2)$. The argument is similar to above one with G_1. One of the following two sets has three distinct elements: $\{l, n-2l, n-2m\}$ or $\{l, m, n-2l\}$.

2. G_3 with $2 \leq l \leq \frac{n-4}{6}$, $\frac{n+2}{6} \leq m \leq \frac{n}{2} - 2$, $i = 1$, $j = n$,

$$\begin{pmatrix} l & m \\ n-2l & n-2(m+1) \end{pmatrix}.$$

We wish to show there are three distinct entries: $l \neq n-2l$, $n-2l \neq n-2(m+1)$, $l \neq n-2(m+1)$. Suppose $l = n-2l \Rightarrow 3l = n$ and since $n = 4+6k$ this is a contradiction. Suppose $n-2l = n-2(m+1) \Rightarrow l = m+1$, which is a contradiction. Suppose $l = n-2(m+1)$ and we wish to show there are three distinct elements: $l \neq m$, $l \neq n-2l$, $m \neq n-2l$. The first two inequalities are correct. Suppose $m = n-2l$ and since $l = n-2(m+1) \Rightarrow m = l-2$, which is a contradiction. One of the following two sets has three distinct elements: $\{l, n-2l, n-2(m+1)\}$ or $\{l, m, n-2l\}$.

3. G_3 with $2 \leq l \leq \frac{n-4}{6}$, $m = \frac{n}{2} - 1$, $i = 1$, $j = n$,

$$\begin{pmatrix} l & m \\ n-2l & n-9 \end{pmatrix}.$$

We wish to show there are three distinct entries: $l \neq m$, $m \neq n-9$, $l \neq n-9$. Suppose $l = n-9$ and since $l < \frac{n-4}{6} \Rightarrow n-9 < \frac{n-4}{6} \Rightarrow n < 10$, which is a contradiction. Suppose $m = n-9 \Rightarrow \frac{n}{2} - 1 = n-9 \Rightarrow n = 16$, which is a contradiction. There are three distinct elements $\{l, m, n-9\}$.

4. G_3 with $\frac{n+2}{6} \leq l \leq \frac{n}{2} - 2$, $m = \frac{n}{2} - 1$, $i = 1$, $j = n$,

$$\begin{pmatrix} l & m \\ n-2(l+1) & n-9 \end{pmatrix}.$$

We wish to show there are three distinct entries: $l \neq m$, $m \neq n-9$, $l \neq n-9$. Suppose $l = n-9$ and since $l < \frac{n}{2} - 2 \Rightarrow n-9 < \frac{n}{2} - 2 \Rightarrow n < 14$, which is a contradiction. Suppose $m = n-9 \Rightarrow \frac{n}{2} - 1 = n-9 \Rightarrow n = 16$, which is a contradiction. There are three distinct elements $\{l, m, n-9\}$.

Case 2. For the submatrix $G(i, j; l, m)$ with $i = 1$, $j = n$ and $(\frac{n}{2} \leq l < m \leq n-2$ or $2 \leq l \leq \frac{n}{2} - 1$, $\frac{n}{2} \leq m \leq n-2)$ three distinct elements are $\{l, m, n\}$.

Case 3. We take the submatrix $G(i, j; l, m)$ with $2 \leq l \leq \frac{n}{2} - 1$, $m = n-1$, $i = 1$, $j = n$.

Subcase 3.1. Consider G_1,

$$\begin{pmatrix} l & m \\ n-2l & n-1 \end{pmatrix}.$$

We wish to show there are three distinct entries: $l \neq n-1$, $n-2l \neq n-1$, $l \neq n-2l$. Suppose $n-2l = n-1 \Rightarrow l = \frac{1}{2}$, which is a contradiction. Suppose $l = n-2l \Rightarrow 3l = n$ and since $n = 2+6k$ this is a contradiction. There are three distinct elements $\{l, n-1, n-2l\}$.

Subcase 3.2. Consider G_2.

1. G_2 with $2 \leq l \leq \frac{n}{2}-2$, $m = n-1$, $i = 1$, $j = n$,

$$\begin{pmatrix} l & m \\ n-2(l+1) & n-1 \end{pmatrix}.$$

We wish to show there are three distinct entries: $l \neq n-1$, $n-2(l+1) \neq n-1$, $l \neq n-2(l+1)$. Suppose $n-2(l+1) = n-1 \Rightarrow l = -\frac{1}{2}$, which is a contradiction. Suppose $l = n-2(l+1) \Rightarrow 3l = n-2$ and since $n = 6+6k$ this is a contradiction. There are three distinct elements $\{l, n-1, n-2(l+1)\}$.

2. G_2 with $l = \frac{n}{2}-1$, $m = n-1$, $i = 1$, $j = n$,

$$\begin{pmatrix} l & m \\ Y & n-1 \end{pmatrix}.$$

If K is *even* $\Rightarrow Y = \frac{n}{2}-2$. If $n-1 = Y \Rightarrow n-1 = \frac{n}{2}-2 \Rightarrow n = -2$, which is a contradiction. Three distinct entries are $\{l, Y, n-1\}$.

If K is *odd* $\Rightarrow Y = \frac{n}{2}+1$, and three distinct entries are $\{l, Y, n-1\}$.

Subcase 3.3. Consider G_3.

1. For G_3 with $l \leq \frac{n-4}{6}$, $m = n-1$, $i = 1$, $j = n$ three distinct elements are $\{l, n-1, n-2l\}$ (similar to G_1.)

2. G_3 with $\frac{n+2}{6} \leq l < n-1$, $m = n-1$, $i = 1$, $j = n$,

$$\begin{pmatrix} l & m \\ n-2(l+1) & n-1 \end{pmatrix}.$$

We wish to show there are three distinct entries: $l \neq n-1$, $n-2(l+1) \neq n-1$, $l \neq n-2(l+1)$. Suppose $n-2(l+1) = n-1 \Rightarrow l = -\frac{1}{2}$, which is a contradiction. Suppose $l = n-2(l+1) \Rightarrow 3l = n-2$ and since $n = 4+6k$ this is a contradiction. Three distinct elements are $\{l, n-1, n-2(l+1)\}$.

3. For G_3 with $l = \frac{n}{2}-1$, $m = n-1$, $i = 1$, $j = n$ three distinct entries are $\{l, n-9, n-1)\}$.

Case 4. For the submatrix $G(i, j; l, m)$ with $\frac{n}{2} \leq l \leq n-2$, $m = n-1$, $i = 1$, $j = n$ three distinct elements are $\{l, n, n-1\}$.

Case 5. For the submatrix $G(i, j; l, m)$ with $l = n - 1$, $m = n$, $i = 1$, $j = n$ three distinct elements are $\{n - 1, n, 1\}$.
The argument for other steps is similar. To see the details please view the appendix to this article on ArXiv at http://arxiv.org/ or contact the first author.

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