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SYMMETRIC HAMILTON CYCLE DECOMPOSITIONS OF COMPLETE MULTIGRAPHS

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Abstract

Let $n \geq 3$ and $\lambda \geq 1$ be integers. Let λK_n denote the complete multigraph with edge-multiplicity λ. In this paper, we show that there exists a symmetric Hamilton cycle decomposition of λK_{2m} for all even $\lambda \geq 2$ and $m \geq 2$. Also we show that there exists a symmetric Hamilton cycle decomposition of $\lambda K_{2m} - F$ for all odd $\lambda \geq 3$ and $m \geq 2$. In fact, our results together with the earlier results (by Walecki and Brualdi and Schroeder) completely settle the existence of symmetric Hamilton cycle decomposition of λK_n (respectively, $\lambda K_n - F$, where F is a 1-factor of λK_n) which exist if and only if $\lambda(n-1)$ is even (respectively, $\lambda(n-1)$ is odd), except the non-existence cases $n \equiv 0$ or $6 \pmod 8$ when $\lambda = 1$.

Keywords: complete multigraph, 1-factor, symmetric Hamilton cycle, decomposition.

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1. Introduction

Let $n \geq 3$ and $\lambda \geq 1$ be integers. Let λK_n denote the complete multigraph obtained from the complete graph K_n by replacing each edge with λ edges. A partition of λG into edge-disjoint Hamilton cycles is called *Hamilton cycle decomposition* of λG. A Hamilton cycle decomposition $\mathcal{H}$ of G is *cyclic* if $V(G) = \mathbb{Z}_n$, and $(v_0+1, v_1+1, \ldots, v_{n-1}+1) \in \mathcal{H}$ whenever $(v_0, v_1, \ldots, v_{n-1}) \in \mathcal{H}$. It is 1-*rotational* if $V(G) = \mathbb{Z}_{n-1} \cup \{\infty\}$, and $(v_0+1, v_1+1, \ldots, v_{n-1}+1) \in \mathcal{H}$ whenever $(v_0, v_1, \ldots, v_{n-1}) \in \mathcal{H}$, where $\infty + 1 = \infty$ is meaningful. Let the vertex set of λK_n be labeled as follows:

$$V(\lambda K_n) = \begin{cases} \{0,1,2,3,\ldots,m,\overline{1},\overline{2},\overline{3},\ldots,\overline{m}\}, & \text{if } n \text{ is odd, say } n = 2m+1; \\ \{1,2,3,\ldots,m,\overline{1},\overline{2},\overline{3},\ldots,\overline{m}\}, & \text{if } n \text{ is even, say } n = 2m. \end{cases}$$

A Hamilton cycle (or a 2-factor) of λK_n or $\lambda K_n - F$ is said to be *symmetric* if it is invariant under the involution $i \to \overline{i}$, where $\overline{\overline{i}} = i$ and the vertex 0 is a fixed point of this involution. A Hamilton cycle decomposition of λK_{2n+1} (respectively, λK_{2n}) is symmetric if it admits an involutory automorphism fixing all its cycles and fixing exactly one vertex (respectively, fixing no vertices). Also a Hamilton cycle decomposition of $\lambda K_{2n+1} - F$ is symmetric if it admits an involutary automorphism switching all pairs of vertices that are adjacent in F. A symmetric Hamilton cycle (or a 2-factor) in $K_{n,n}$ with bipartition $\{1,2,3,\ldots,n\}$ and $\{\bar{1},\bar{2},\bar{3},\ldots,\bar{n}\}$ containing the edge $i\bar{j}$ should also contain $\bar{i}j$. The *cartesian product*, $G_1 \Box G_2$, of the graphs G_1 and G_2 has the vertex set $V(G_1) \times V(G_2)$ and edge set $E(G_1 \Box G_2) = \{(u_1,v_1)(u_2,v_2) \mid u_1 = u_2 \text{ and } v_1v_2 \in E(G_2) \text{ or } v_1 = v_2 \text{ and } u_1u_2 \in E(G_1)\}$.

Buratti and Del Fra [6] proved that a cyclic Hamilton cycle decomposition of K_n exists if and only if $n \neq 15$ and $n \notin \{p^\alpha \mid p \text{ is an odd prime and } \alpha \geq 2\}$. Jordon and Morris [9] proved that for an even $n \geq 4$, there exists a cyclic Hamilton cycle decomposition of $K_n - F$ if and only if $n \equiv 2, 4 \pmod 8$ and $n \neq 2p^\alpha$ where p is an odd prime and $\alpha \geq 1$. Buratti *et al.* [5] completely solved the existence of cyclic Hamilton cycle decomposition of λK_n and of $\lambda(K_{2n} - F)$ for every λ. In general, finding necessary and sufficient conditions for the existence of cyclic m-cycle decomposition of K_n is an interesting problem and has received much attention in recent days.

Walecki [10] proved the existence of a Hamilton cycle decomposition of K_n (when n is odd) and $K_n - F$ (when n is even), where F is a 1-factor of K_n. Further, it is easy to observe that the addition by $\frac{n-1}{2}$ gives an involutory map fixing every cycle of the decomposition to be symmetric. Akiyama [1] *et al.* also constructed a new symmetric Hamilton cycle decomposition of K_n for odd $n > 7$, but is not isomorphic to Walecki decomposition.

Brualdi and Schroeder [4] proved that $K_n - F$ has a decomposition into Hamilton cycles which are symmetric with respect to the 1-factor F if and only if $n \equiv 2$ or $4 \pmod 8$, and also show that the complete bipartite graph $K_{n,n}$ (respectively $K_{n,n} - F$) has a symmetric Hamilton cycle decomposition if and only if n is even (respectively n is odd). As Hamilton/ symmetric Hamilton cycle decomposition of K_n for even n does not exists, considering the existence of such decomposition in λK_n gets merit (for suitable λ and n), since it covers a wider class of graphs.

Recently, Buratti and Merola [7] observed that every cyclic Hamilton cycle decomposition of λK_{2n} or $\lambda K_{2n} - F$ whose cycles having stabilizer of even order is, in particular symmetric: the required involutory automorphism would be in fact the addition by n, and also pointed that the existence of a symmetric Hamilton cycle decomposition of $K_n - F$ for $n \equiv 4 \pmod 8$ (part of the main result of the paper by Brualdi and Schroeder [4]) implicitly follows from the result of Jordon and Morris [9]. Also, the result of Buratti *et al.* [5] gives, implicitly, the existence of a symmetric Hamilton cycle decomposition of $2K_{4m}$, $m \geq 1$.

In this paper, we show that there exists a symmetric Hamilton cycle decomposition of λK_{2m} for all even $\lambda \geq 2$ and $m \geq 2$. Also we show that there exists a symmetric Hamilton cycle decomposition of $\lambda K_{2m} - F$ for all odd $\lambda \geq 3$ and $m \geq 2$. In fact, our results together with the results of Walecki, Brualdi and Schroeder prove that the complete multigraph λK_n (respectively, $\lambda K_n - F$) has a symmetric Hamilton cycle decomposition if and only if $\lambda(n-1)$ is even (respectively, $\lambda(n-1)$ is odd) except the non-existence cases $n \equiv 0$ or $6 \pmod 8$ when $\lambda = 1$, which were proved by Brualdi and Schroeder.

2. Notation and Preliminaries

Throughout this paper, we use the following notation:

- $V(\lambda K_n) = \begin{cases} \{0,1,2,3,\ldots,r,\overline{1},\overline{2},\overline{3},\ldots,\overline{r}\}, & \text{if } n \text{ is odd, say } n = 2r+1; \\ \{1,2,3,\ldots,r,\overline{1},\overline{2},\overline{3},\ldots,\overline{r}\}, & \text{if } n \text{ is even, say } n = 2r. \end{cases}$
- $\lambda K_r^\star$ is the complete multigraph with the vertex set $\{1,2,\ldots,r\}$.
- $\lambda \overline{K_r^\star}$ is the complete multigraph with the vertex set $\{\overline{1},\overline{2},\ldots,\overline{r}\}$.
- $\lambda K_{2s,2s}$ is the complete bipartite multigraph with bipartition $\{1,2,\ldots,2s\}$ and $\{\overline{1},\overline{2},\ldots,\overline{2s}\}$.
- $(1,2,\ldots,m,\overline{1},\overline{2},\ldots,\overline{m})$ denotes a symmetric cycle of length $2m$.
- For our convenience, we view λK_{2r}, $\lambda K_{2r} - F$ as follows:

  (i) $\lambda K_{2r} = \lambda K_r^\star \oplus \lambda K_{r,r} \oplus \lambda \overline{K_r^\star}$

  (ii) $\lambda K_{2r} - F = \lambda K_r^\star \oplus \lambda K_{r,r} - F \oplus \lambda \overline{K_r^\star}$, where $F = \{i\overline{i} \in E(K_{r,r}) \mid 1 \leq i \leq r\}$.
- F' denotes the 1-factor $\{i(\overline{s+i}), (s+i)\overline{i} \in E(K_{2s,2s}) \mid 1 \leq i \leq 2s\}$ of $K_{2s,2s}$.
- I denotes the 1-factor $\{i(s+i) \in E(K_{2s}^\star) \mid 1 \leq i \leq s\}$ of $K_{2s}^\star$.
- $\overline{I}$ denotes the 1-factor $\{\overline{i}(\overline{s+i}) \in E(\overline{K_{2s}^\star}) \mid 1 \leq i \leq s\}$ of $\overline{K_{2s}^\star}$.

To prove our results we state the following.

Proposition 1 [1]. *Let $p \geq 7$ be a prime. There exists a Hamilton cycle decomposition $\mathcal{G}_p$ of K_p which is not isomorphic to the Walecki's decomposition $\mathcal{W}_p$ of K_p.*

Theorem 2 [1]**.** *Let $n > 7$ be an odd integer. There exists a symmetric Hamilton cycle decomposition of K_n which is not isomorphic to the Walecki's Hamilton cycle decomposition $\mathcal{W}_n$. Further, it is not isomorphic to $\mathcal{G}_n$ when n is a prime.*

Theorem 3 [4]**.** *For each integer $m \geq 1$, there exist a symmetric Hamilton cycle decomposition of $K_{2m,2m}$, and $K_{2m+1,2m+1} - F$, where F is a 1-factor.*

Theorem 4 [4]**.** *Let $n > 2$ be an integer. Then $K_n - F$ has a symmetric Hamilton cycle decomposition if and only if $n \equiv 2, 4 \pmod 8$.*

Remark 5 [4]**.** Consider the complete bipartite graph $K_{2m,2m}$ with $V(K_{2m,2m}) = \{1, 2, \ldots, 2m, \bar{1}, \bar{2}, \ldots, \overline{2m}\}$. Let $E_k = \{a\bar{b} \in E(K_{2m,2m}) \mid a + b \equiv k \pmod{2m}\}$. Clearly, each $S_i = E_{2i} \cup E_{2i+1}$ is a symmetric Hamilton cycle of $K_{2m,2m}$ and $\{S_1, S_2, \ldots, S_m\}$ gives a symmetric Hamilton cycle decomposition of $K_{2m,2m}$. Note that each S_i contain the edges $\{i(\overline{i+1}), \bar{i}(i+1), (\overline{m+i})(m+i+1), (m+i)(\overline{m+i+1}), i\bar{i}, (m+i)(\overline{m+i})\}$, $1 \leq i \leq m$ and the additions are taken with modulo $2m$.

Remark 6. Let $V(K^\star_{2m})=\{1, 2, \ldots, 2m\}$. Then $H = (1, 2, 2m, 3, 2m-1, 4, 2m-2, \ldots, m+2, m+1, 1)=\{ab \in E(K^\star_{2m}) \mid a + b \equiv 2 \text{ or } 3 \pmod{2m}\}$ is a Hamilton cycle of $K^\star_{2m}$. Now we define an injective map $f_i : \{1, 2, 3, \ldots, 2m\} \to \{1, 2, 3, \ldots, 2m\}$, $1 \leq i \leq 2m-1$ as follows:

$f_i(1) = 1,$

$$f_i(x) = \begin{cases} x + i - 1, & \text{if } x \in \{2, 3, \ldots, 2m - i + 1\}; \\ x - 2m + i, & \text{if } x \in \{2m - i + 2, 2m - i + 3, \ldots, 2m\}. \end{cases}$$

Let $H_i = f_i(H)$. Then $\{H_1, H_2, \ldots, H_{2m-1}\}$, $\{H_1, H_2, \ldots, H_m\}$ and $\{H_{m+1}, H_{m+2}, \ldots, H_{2m-1}\}$ respectively give a Hamilton cycle decomposition of multigraphs $2K^\star_{2m}$, $K^\star_{2m} \oplus I$ and $K^\star_{2m} - I$, where $I = \{i(m+i) \in E(K^\star_{2m}) \mid 1 \leq i \leq m\}$. Note that each H_i contain the edges $\{i(i+1), (m+i)(m+i+1)\}$, $1 \leq i \leq m$ (see Figure 1).

Also we observe that the Hamilton cycle decompositions given above will imply a 1-rotational Hamilton cycle decomposition of $2K^\star_{2m}$, $K^\star_{2m} \oplus I$ and $K^\star_{2m} - I$ by just replacing the symbols 1 by ∞ and x, $2 \leq x \leq 2m$, by $x - 1$.

3. Complete Multigraphs

In this section, we investigate the existence of a symmetric Hamilton cycle decomposition of complete multigraph λK_n, when $\lambda(n-1)$ is even. Since the symmetric Hamilton cycle decomposition of λK_n, when n odd, exists from the well known Walecki's construction [10], our main focus is to find a symmetric Hamilton cycle decomposition of $2K_{2m}$.

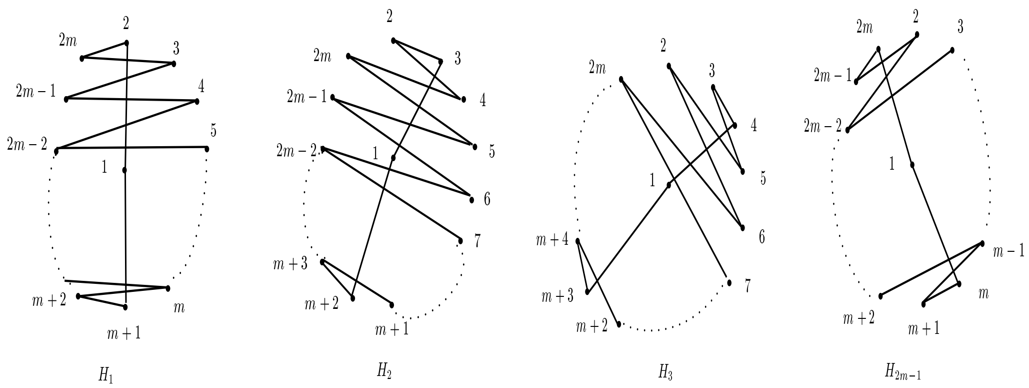

Figure 1. H_1, H_2, H_3, …, H_{2m-1} of K_{2m+1}.

Lemma 7. *For all integers $m \geq 1$, there exists a symmetric Hamilton cycle decomposition of $K_{2m} \Box K_2$.*

Proof. Let $V(K_{2m}) = \{u_1, u_2, \dots, u_{2m}\}$ and $V(K_2) = \{v_1, v_2\}$. For our convenience, we denote $V(K_{2m} \Box K_2) = \bigcup_{s=1}^{2} V_s$, where $V_1 = \{i \mid i = (u_i, v_1), 1 \leq i \leq 2m\}$, $V_2 = \{\bar{i} \mid \bar{i} = (u_i, v_2), 1 \leq i \leq 2m\}$ and $E(K_{2m} \Box K_2) = \{ij, \bar{i}\,\bar{j}, i\bar{i} \mid i \neq j,\ i, j = 1, 2, \dots, 2m\}$. For $1 \leq k \leq 2m$, $1 \leq l \leq m$, we define

$$\begin{aligned} E_k &= \{ij \in E(K_{2m} \Box K_2) \mid i \neq j, i + j \equiv k \pmod{2m}\}, \\ \overline{E}_k &= \{\bar{i}\,\bar{j} \in E(K_{2m} \Box K_2) \mid i \neq j, i + j \equiv k \pmod{2m}\}, \\ J_l &= \{i\bar{i} \in E(K_{2m} \Box K_2) \mid 2i \equiv 2l \pmod{2m}\}. \end{aligned}$$

Note that $E_{2l} \cup E_{2l+1}$ and $\overline{E}_{2l} \cup \overline{E}_{2l+1}$ are Hamilton paths with end vertices l, $m+l$ and $\bar{l}$, $\overline{m+l}$ of $K^{\star}_{2m}$ and $\overline{K}^{\star}_{2m}$ respectively. For each l, $1 \leq l \leq m$, we define $H_l = E_{2l} \cup E_{2l+1} \cup J_l \cup \overline{E}_{2l} \cup \overline{E}_{2l+1}$. Clearly, each H_l is a symmetric Hamilton cycle and $\{H_1, H_2, \dots, H_m\}$ gives a symmetric Hamilton cycle decomposition of $K_{2m} \Box K_2$. ∎

Lemma 8. *For all integers $m \geq 1$, there exists a symmetric Hamilton cycle decomposition of $2(K_{2m+1} \Box K_2)$.*

Proof. Let $V(K_{2m+1}) = \{u_1, u_2, u_3, \dots, u_{2m+1}\}$ and $V(K_2) = \{v_1, v_2\}$. We denote $V(K_{2m+1} \Box K_2) = \bigcup_{s=1}^{2} V_s$ where $V_1 = \{i \mid i = (u_i, v_1), 1 \leq i \leq 2m\}$, $V_2 = \{\bar{i} \mid \bar{i} = (u_i, v_2), 1 \leq i \leq 2m\}$ and $E(K_{2m+1} \Box K_2) = \{ij, \bar{i}\,\bar{j}, i\bar{i} \mid, i \neq j, i, j = 1, 2, \dots, 2m + 1\}$.

For all k, $1 \le k \le 2m+1$, we define

$$\begin{aligned} E_k &= \{ij \in E(K_{2m+1}\Box K_2) \mid i \neq j, i+j \equiv k \,(\mathrm{mod}\ 2m+1)\},\\ \overline{E}_k &= \{\bar{i}\,\bar{j} \in E(K_{2m+1}\Box K_2) \mid i \neq j, i+j \equiv k \,(\mathrm{mod}\ 2m+1)\}.\end{aligned}$$

Note that $E_{2l} \cup E_{2l+1}$, $E_{2l-1} \cup E_{2l}$ and $E_1 \cup E_{2m+1}$ are Hamilton paths of $K^\star_{2m}$ with end vertices l, $m+1+l$; l, $m+l$; and $m+1$, $2m+l$ respectively. Similarly, $\overline{E}_{2l} \cup \overline{E}_{2l+1}$, $\overline{E}_{2l-1} \cup \overline{E}_{2l}$ and $\overline{E}_1 \cup \overline{E}_{2m+1}$ are Hamilton paths of $\overline{K^\star_{2m}}$ with end vertices $\bar{l}$, $\overline{m+1+l}$; $\bar{l}$, $\overline{m+l}$; and $\overline{m+1}$, $\overline{2m+l}$ respectively.

For each l, $1 \le l \le m$, we define

$$\begin{aligned} H_l &= E_{2l} \cup E_{2l+1} \cup \{l\bar{l}, (m+1+l)(\overline{m+1+l})\} \cup \overline{E}_{2l} \cup \overline{E}_{2l+1},\\ H'_l &= E_{2l-1} \cup E_{2l} \cup \{l\bar{l}, (m+l)(\overline{m+l})\} \cup \overline{E}_{2l-1} \cup \overline{E}_{2l},\\ H_{2m+1} &= E_1 \cup E_{2m+1} \cup \{(2m+1)(\overline{2m+1}), (m+1)(\overline{m+1})\} \cup \overline{E}_1 \cup \overline{E}_{2m+1}.\end{aligned}$$

Clearly, each H_l, H'_l are symmetric Hamilton cycles and $\{H_1, H_2, \ldots, H_m, H'_1, H'_2, \ldots, H'_m, H_{2m+1}\}$ gives a symmetric Hamilton cycle decomposition of $2(K_{2m+1} \Box K_2)$. ■

Remark 9. Note that the symmetric Hamilton cycles H_l and H'_l, $1 \le l \le m$ obtained in Lemma 8 contain the edges $\{l(l+1), \bar{l}(\overline{l+1})\}$ and $\{(2m+l+1)(2m+1+l+1), (\overline{2m+l+1})(\overline{2m+1+l+1})\}$ respectively.

Note 10. It is observed that for every Hamilton path decomposition of K_{2m} we can find a symmetric Hamilton cycle decomposition of $K_{2m,2m}$ and $K_{2m}\Box K_2$, also to every Hamilton path decomposition of $2K_{2m+1}$ we can find a symmetric Hamilton cycle decomposition of $2(K_{2m+1}\Box K_2)$.

Theorem 11. *For all integers $m \ge 1$, there exists a symmetric Hamilton cycle decomposition of $2K_{4m+2}$.*

Proof. Let $V(2K_{4m+2}) = \{1, 2, \ldots, 2m+1, \bar{1}, \bar{2}, \ldots, \overline{2m+1}\}$. Now the complete multigraph $2K_{4m+2}$ can be viewed as follows: $2K_{4m+2} = 2(K_{2m+1}\Box K_2) \oplus 2(K_{2m+1,2m+1} - F)$, where $F = \{i\bar{i} \in E(K_{2m+1,2m+1}) \mid 1 \le i \le 2m+1\}$ is a 1-factor of $K_{2m+1,2m+1}$. We know that $2(K_{2m+1}\Box K_2)$ and $(K_{2m+1,2m+1} - F)$ have symmetric Hamilton cycle decompositions by Lemma 8 and Theorem 3, respectively. ■

We recall that Buratti and Merola [7] observed that every cyclic Hamilton cycle decomposition of λK_{2n} or $\lambda K_{2n} - F$ whose cycles have stabilizer of even order is, in particular symmetric: the required involutory automorphism would be in fact the addition by n. So the result of Buratti *et al.* [5] deduce the existence of a symmetric Hamilton cycle decomposition of $2K_{4m}$, $m \ge 1$.

The next construction provides an alternative proof for the existence of a symmetric Hamilton cycle decomposition of $2K_{4m}$, $m \geq 1$ which is implicitly contained in Buratti *et al.* ([[5], Lemma 3.5]).

Theorem 12. *For all integers $m \geq 1$, there exists a symmetric Hamilton cycle decomposition of $2K_{4m}$.*

Proof. Let $V(2K_{4m}) = \{1, 2, \ldots, 2m, \bar{1}, \bar{2}, \ldots, \overline{2m}\}$. For $m = 1$ the graph is $2K_4$. Clearly, $\{(1, \bar{2}, 2, \bar{1}), (1, 2, \bar{1}, \bar{2}), (1, \bar{1}, \bar{2}, 2)\}$ gives a symmetric Hamilton cycle decomposition of $2K_4$.

For $m \geq 2$, we write $2K_{4m} = 2K^{\star}_{2m} \oplus K_{2m,2m} \oplus K'_{2m,2m} \oplus 2\overline{K}^{\star}_{2m}$. Now the idea of decomposing $2K_{4m}$ into symmetric Hamilton cycles is as follows: First we decompose $K_{2m,2m}$ and $K'_{2m,2m}$ into symmetric Hamilton cycles $S_1, S_2, \ldots, S_m$ and $S'_1, S'_2, \ldots, S'_m$, and $2K^{\star}_{2m}$, $2\overline{K}^{\star}_{2m}$ into Hamilton cycles $\{H_1, H_2, \ldots, H_{2m-1}\}$, $\{H'_1, H'_2, \ldots, H'_{2m-1}\}$ respectively. Then by decomposing each $H_i \oplus S_i \oplus H'_i$, $1 \leq i \leq m$ and $H_{m+j} \oplus S'_j \oplus H'_{m+j}$, $1 \leq j \leq m-1$ into symmetric Hamilton cycles C^i_1, C^i_2 and D^i_1, D^i_2 respectively, we get the symmetric Hamilton cycle decomposition $\{C^1_1, C^2_1, \ldots, C^m_1, C^1_2, C^2_2, \ldots, C^m_2, D^1_1, D^2_1, \ldots, D^{m-1}_1, D^1_2, D^2_2, \ldots, D^{m-1}_2, S'_m\}$ of $2K_{4m}$.

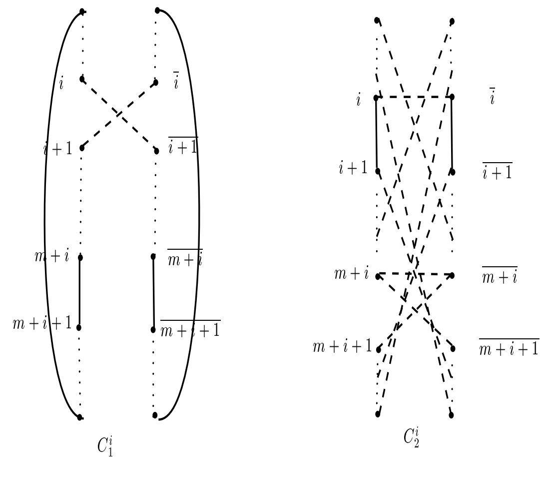

Figure 2. Symmetric Hamilton cycles C^i_1 and C^i_2 from $H_i \oplus S_i \oplus \overline{H}_i$.

We know by Remark 5 that $2K_{2m,2m}$ has a symmetric Hamilton cycle decomposition $\{S_1, S_2, \ldots, S_m, S'_1, S'_2, \ldots, S'_m\}$ such that both S_i and S'_i contain the edges $\{i(\overline{i+1}), \bar{i}(i+1), (\overline{m+i})(m+i+1), (m+i)(\overline{m+i+1}), i\bar{i}, (m+i)(\overline{m+i})\}$. Furthermore, by Remark 6, $2K^{\star}_{2m}$ has a Hamilton cycle decomposition $\{H_1, H_2, \ldots, H_{2m-1}\}$ such that each H_i contain the edges $\{i(i+1), (m+i)(m+i+1)\}$. Similarly, let $\{\overline{H}_1, \overline{H}_2, \ldots, \overline{H}_{2m-1}\}$ be a Hamilton cycle decomposition of $2\overline{K}^{\star}_{2m}$ such that each $\overline{H}_i$ contain the edges $\{\bar{i}(\overline{i+1}), (\overline{m+i})(\overline{m+i+1})\}$.

Now we define C_1^i, C_2^i from $H_i \oplus S_i \oplus \overline{H}_i$, $1 \le i \le m$ as follows:

$$\begin{aligned} C_1^i &= (H_i \setminus \{i(i+1)\}) \cup (\overline{H}_i \setminus \{\bar{i}(\overline{i+1})\}) \oplus \{i(\overline{i+1}), \bar{i}(i+1)\}, \\ C_2^i &= (S_i \setminus \{i(\overline{i+1}), \bar{i}(i+1)\}) \oplus \{i(i+1), \bar{i}(\overline{i+1})\}. \end{aligned}$$

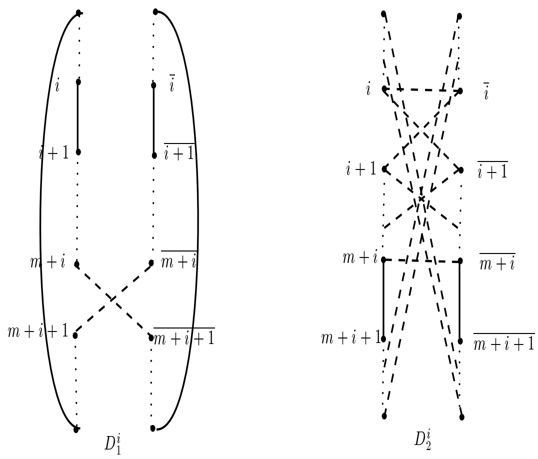

Figure 3. Symmetric Hamilton cycles D_1^i and D_2^i from $H_i \oplus S_j' \oplus \overline{H}_i$.

Now we define D_1^j, D_2^j from $H_{m+j} \oplus S_j' \oplus \overline{H}_{m+j}$, $1 \le j \le m-1$ as follows:

$$\begin{aligned} D_1^j &= (H_{m+j} \setminus \{(m+j)(m+j+1)\}) \cup (\overline{H}_{m+j} \setminus \{\overline{(m+j)}(\overline{m+j+1})\}) \\ &\quad \oplus \{(m+j)(\overline{m+j+1}), \overline{(m+j)}(m+j+1)\}, \\ D_2^j &= (S_j' \setminus \{(m+j)(\overline{m+j+1}), \overline{(m+j)}(m+j+1)\}) \\ &\quad \oplus \{(m+j)(m+j+1), \overline{m+j}(\overline{m+j+1})\}. \end{aligned}$$

It is easy to check that C_1^i, C_2^i, D_1^j and D_2^j are edge-disjoint symmetric Hamilton cycles of $2K_{4m}$, (see Figures 2 and 3). Hence $\{C_1^i, C_2^i, D_1^j, D_2^j, S_m' \mid 1 \le i \le m, 1 \le j \le m-1\}$ gives a symmetric Hamilton cycle decomposition of $2K_{4m}$. ■

Theorem 13. *For all $\lambda \equiv 0\,(\mathrm{mod}\ 2)$ and $n \equiv 0\,(\mathrm{mod}\ 2) \ge 4$, there exists a symmetric Hamilton cycle decomposition of λK_n.*

Proof. Follows from Theorems 11 and 12. ■

4. Complete Multigraph Minus a 1-factor

In this section, we investigate the existence of symmetric Hamilton cycle decomposition of $\lambda K_n - F$, when λK_n has odd regularity.

Theorem 14. *For all* $\lambda \equiv 1 \pmod 2$ *and* $n \equiv 2$ *or* $4 \pmod 8$*, there exists a symmetric Hamilton cycle decomposition of* $\lambda K_n - F$.

Proof. We can write $\lambda K_n - F = (\lambda-1)K_n \oplus K_n - F$. Since both n and $\lambda - 1$ are even, $(\lambda - 1)K_n$ and $(K_n - F)$ have a symmetric Hamilton cycle decomposition by Theorems 13 and 4 respectively. ■

Theorem 15. *For all* $n \equiv 6 \pmod 8$*, there exists a symmetric Hamilton cycle decomposition of* $3K_n - F$.

Proof. Let $n = 8m + 6$ and $V(3K_{8m+6}) = \{1, 2, \ldots, 4m+3, \overline{1}, \overline{2}, \ldots, \overline{4m+3}\}$. For $m = 0$, the graph is $3K_6 - F$. Clearly $\{(1, \overline{2}, 3, \overline{1}, 2, \overline{3}), (1, \overline{2}, 3, \overline{1}, 2, \overline{3}), (1, 2, 3, \overline{1}, \overline{2}, \overline{3}), (1, \overline{2}, \overline{3}, \overline{1}, 2, 3), (1, 3, 2, \overline{2}, \overline{3}, \overline{1}), (1, 2, \overline{2}, \overline{1}, \overline{3}, 3), (1, 2, \overline{3}, 3, \overline{2}, \overline{1})\}$ gives a symmetric Hamilton cycle decomposition of $3K_6 - F$, where $F = \{1\overline{1}, 2\overline{2}, 3\overline{3}\}$ is a 1-factor.

Now we construct a symmetric Hamilton cycle decomposition of $3K_n - F$ for $n \geq 14$ as follows: For $1 \leq k \leq 4m+3$, $1 \leq i \leq 2m+1$, we define

$$\begin{aligned}
H_i &= F_{2i} \cup F_{2i+1} \cup \{(4m+3)i, (\overline{4m+3})\overline{i}, (\overline{4m+3})(2m+1+i), \\
&\quad (4m+3)(\overline{2m+1+i})\} \cup F'_{2i} \cup F'_{2i+1}, \\
S_i &= E_{2i} \cup E_{2i+1} \cup \{(4m+3)\overline{i}, (\overline{4m+3})i, (4m+3)(2m+1+i), \\
&\quad (\overline{4m+3})(\overline{2m+1+i})\},
\end{aligned}$$

where

$$\begin{aligned}
E_k &= \{a\overline{b} \in E(K_{4m+2,4m+2}) \mid a \neq b, a+b \equiv k \pmod{4m+2}\}, \\
F_k &= \{ab \in E(K^{\star}_{4m+2}) \mid a+b \equiv k \pmod{4m+2}\}, \\
F'_k &= \{\overline{a}\overline{b} \in E(\overline{K}^{\star}_{4m+2}) \mid a+b \equiv k \pmod{4m+2}\}.
\end{aligned}$$

It is easy to check that each H_i is a symmetric Hamilton cycle of $K_{8m+6} - F$ and each S_i is a symmetric 2-factor of $K_{8m+6} - F$ containing the edges $\{i(\overline{i+1}), \overline{i}(i+1)\}$, where $F = \{i\overline{i} \in E(K_{4m+3,4m+3}) \mid 1 \leq i \leq 4m+3\}$ is a 1-factor. So we write $K_{8m+6} - F = (\oplus_{i=1}^{2m+1} H_i) \oplus (\oplus_{i=1}^{2m+1} S_i)$. Furthermore, by Lemma 8, $2(K_{4m+3} \Box K_2)$ has a symmetric Hamilton cycle decomposition $\{C_1, C_2, \ldots, C_{2m+1}, C'_1, C'_2, \ldots, C'_{2m+1}, C_{4m+3}\}$. Now we can write

$$\begin{aligned}
3K_{8m+6} - F &= 2K_{8m+6} \oplus (K_{8m+6} - F) \\
&= 2(K_{4m+3} \Box K_2) \oplus 2(K_{4m+3,4m+3} - F) \oplus (K_{8m+6} - F) \\
&= ((\oplus_{i=1}^{2m+1} C_i) \oplus (\oplus_{i=1}^{2m+1} C'_i) \oplus C_{4m+3}) \oplus 2(K_{4m+3,4m+3} - F) \\
&\quad \oplus (\oplus_{i=1}^{2m+1} H_i) \oplus (\oplus_{i=1}^{2m+1} S_i).
\end{aligned}$$

We now construct the remaining symmetric Hamilton cycles D_1^i, D_2^i from $C_i \oplus S_i$, $1 \le i \le 2m+1$ as follows:

$$\begin{aligned} D_1^i &= (S_i \setminus \{i(\overline{i+1}), \bar{i}(i+1)\}) \oplus \{i(i+1), \bar{i}(\overline{i+1})\}, \\ D_2^i &= (C_i \setminus \{i(i+1), \bar{i}(\overline{i+1})\} \oplus \{i(\overline{i+1}), \bar{i}(i+1)\}. \end{aligned}$$

One can check that D_1^i, D_2^i are symmetric Hamilton cycles of $2(K_{4m+3}\Box K_2) \oplus K_{8m+6} - F$. Hence $\{D_1^i, D_2^i, C_i', C_{4m+3}, H_i \mid 1 \le i \le 2m+1\}$ together with the symmetric Hamilton cycle decomposition of $2(K_{4m+3,4m+3} - F)$ which exists by Theorem 3, gives a symmetric Hamilton cycle decomposition of $3K_{8m+6} - F$. ■

Lemma 16. *The graph* $(K_{2m}^\star \oplus I) \oplus K_{2m,2m} \oplus (\overline{K}_{2m}^\star \oplus \overline{I})$, *where* $I = \{i(m+i) \in E(K_{2m}^\star) \mid 1 \le i \le m\}$, $\overline{I} = \{\bar{i}(\overline{m+i}) \in E(\overline{K}_{2m}^\star) \mid 1 \le i \le m\}$ *admits a symmetric Hamilton cycle decomposition for all* $m \ge 1$.

Proof. We know by Remark 5 that $K_{2m,2m}$ has a symmetric Hamilton cycle decomposition $\{S_1, S_2, \ldots, S_m\}$ such that each S_i contain the edges $\{i(\overline{i+1}), \bar{i}(i+1), (\overline{m+i})(m+i+1), (m+i)(\overline{m+i+1}), i\bar{i}, (m+i)(\overline{m+i})\}$. Further by Remark 6, $K_{2m}^\star \oplus I$ has a Hamilton cycle decomposition $\{H_1, H_2, \ldots, H_m\}$ such that each H_i contain the edges $\{i(i+1), (m+i)(m+i+1)\}$. Similarly, let $\{\overline{H}_1, \overline{H}_2, \ldots, \overline{H}_m\}$ be a Hamilton cycle decomposition of $\overline{K}_{2m}^\star \oplus \overline{I}$ such that each $\overline{H}_i$ contain the edges $\{\bar{i}(\overline{i+1}), (\overline{m+i})(\overline{m+i+1})\}$.

For each integer i, $1 \le i \le m$, we construct C_1^i, C_2^i as follows:

$$\begin{aligned} C_1^i &= (H_i \setminus \{i(i+1)\}) \cup (\overline{H}_i \setminus \{\bar{i}(\overline{i+1})\}) \oplus \{i(\overline{i+1}), \bar{i}(i+1)\}, \\ C_2^i &= (S_i \setminus \{i(\overline{i+1}), \bar{i}(i+1)\}) \oplus \{i(i+1), \bar{i}(\overline{i+1})\}. \end{aligned}$$

Clearly, $\{C_1^i, C_2^i \mid 1 \le i \le m\}$ gives a symmetric Hamilton cycle decomposition of $(K_{2m}^\star \oplus I) \oplus K_{2m,2m} \oplus (\overline{K}_{2m}^\star \oplus \overline{I})$. ■

Lemma 17. *The graph* $K_{2m}^\star \oplus F' \oplus \overline{K}_{2m}^\star$, *where* $F' = \{i(\overline{m+i}), \bar{i}(m+i) \in E(K_{2m,2m}) \mid 1 \le i \le m\}$ *admits a symmetric Hamilton cycle decomposition for all* $m \ge 1$.

Proof. For $1 \le l \le m$, we define $H_l = E_{2l} \cup E_{2l+1} \cup \{l(\overline{m+l}), \bar{l}(m+l)\} \cup \overline{E}_{2l} \cup \overline{E}_{2l+1}$, where

$$\begin{aligned} E_k &= \{ij \in E(K_{2m}^\star) \mid i \ne j, i+j \equiv k \pmod{2m}\}, \\ \overline{E}_k &= \{\bar{i}\,\bar{j} \in E(\overline{K}_{2m}^\star) \mid i \ne j, i+j \equiv k \pmod{2m}\}. \end{aligned}$$

Clearly, each H_l is a symmetric Hamilton cycle and $\{H_1, H_2, \ldots, H_m\}$ gives a symmetric Hamilton cycle decomposition of $K_{2m}^\star \oplus F' \oplus \overline{K}_{2m}^\star$. ■

Lemma 18. *The graph $K_{2m,2m} - \{F, F'\}$, where $F = \{i\bar{i} \in E(K_{2m,2m}) \mid 1 \leq i \leq 2m\}$, $F' = \{i(\overline{m+i}), \bar{i}(m+i) \in E(K_{2m,2m}) \mid 1 \leq i \leq m\}$ admits a C_{2m}-factorization for all $m \geq 2$.*

Proof. Let $V(K_{2m,2m}) = \{1, 2, \ldots, 2m, \bar{1}, \bar{2}, \ldots, \overline{2m}\}$. By Remark 6, let $\mathcal{H} = \{H_{m+1}, H_{m+2}, \ldots, H_{2m-1}\}$ be a Hamilton cycle decomposition of $K^\star_{2m} - I$, where $I = \{i(m+i) \in E(K^\star_{2m}) \mid 1 \leq i \leq m\}$. Let $H \in \mathcal{H}$ and if $H = (1, 2, \ldots, 2m)$ in $K^\star_{2m} - F$, then we define a 2-factor C as $C = (1, \bar{2}, 3, \bar{4}, \ldots, \overline{2m})(\bar{1}, 2, \bar{3}, 4, \ldots, 2m)$ in $K_{2m,2m} - \{F, F'\}$. So corresponding to each $H_{m+i} \in \mathcal{H}$ we can define a C^i as above. Hence $\{C^i \mid 1 \leq i \leq m-1\}$ gives a C_{2m}-factorization of $K_{2m,2m} - \{F, F'\}$. Since by Remark 6, each $H_{m+i} \in \mathcal{H}$ contain the edges $\{i(i+1), (m+i)(m+i+1)\}$, C^i also contain the edges $\{i(\overline{i+1}), \bar{i}(i+1), (m+i)(\overline{m+i+1}), (\overline{m+i})(m+i+1)\}$. ■

Lemma 19. *The graph $(K^\star_{2m} - I) \oplus K_{2m,2m} - \{F, F'\} \oplus (\overline{K}^\star_{2m} - \bar{I})$, where $I = \{i(m+i) \in E(K^\star_{2m}) \mid 1 \leq i \leq m\}$, $\bar{I} = \{\bar{i}(\overline{m+i}) \in E(\overline{K}^\star_{2m}) \mid 1 \leq i \leq m\}$, $F = \{i\bar{i} \in E(K_{2m,2m}) \mid 1 \leq i \leq 2m\}$, $F' = \{i(\overline{m+i}), \bar{i}(m+i) \in E(K_{2m,2m}) \mid 1 \leq i \leq m\}$ admits a symmetric Hamilton cycle decomposition for all $m \geq 1$.*

Proof. We know by Remark 6, $K^\star_{2m} - I$ has a Hamilton cycle decomposition $\{H_{m+1}, H_{m+2}, \ldots, H_{2m-1}\}$ such that each H_{m+i} contain the edges $\{i(i+1), (m+i)(m+i+1)\}$. Similarly, $\overline{K}^\star_{2m} - \bar{I}$ has a Hamilton cycle decomposition $\{\overline{H}_{m+1}, \overline{H}_{m+2}, \ldots, \overline{H}_{2m-1}\}$ such that each $\overline{H}_{m+j}$ contain the edges $\{\bar{j}(\overline{j+1}), (\overline{m+j})(\overline{m+j+1})\}$. Let $\{C^1, C^2, \ldots, C^{m-1}\}$ be a C_{2m}-factorization of $K_{2m,2m} - \{F, F'\}$ as obtained in Lemma 18. Note that each factor C^j contain the edges $\{j(\overline{j+1}), \bar{j}(j+1), (m+j)(\overline{m+j+1}), (\overline{m+j})(m+j+1)\}$.

For each integer j, $1 \leq j \leq m-1$, we construct symmetric Hamilton cycles D^j_1, D^j_2 as follows:

$$\begin{aligned} D^j_1 &= (H_j \setminus \{j(j+1)\}) \cup (\overline{H}_j \setminus \{\bar{j}(\overline{j+1})\}) \oplus \{j(\overline{j+1}), \bar{j}(j+1)\}, \\ D^j_2 &= (C^j \setminus \{j(\overline{j+1}), \bar{j}(j+1)\}) \oplus \{j(j+1), \bar{j}(\overline{j+1})\}. \end{aligned}$$

Then $\{D^j_1, D^j_2 \mid 1 \leq j \leq m-1\}$ gives a symmetric Hamilton cycle decomposition of $(K^\star_{2m} - I) \oplus K_{2m,2m} - \{F, F'\} \oplus (\overline{K}^\star_{2m} - \bar{I})$. ■

Theorem 20. *For all $n \equiv 0 \pmod 8$, there exists a symmetric Hamilton cycle decomposition of $3K_n - F$.*

Proof. Let $n = 8m$ and $V(3K_{8m}) = \{1, 2, \ldots, 4m, \bar{1}, \bar{2}, \ldots, \overline{4m}\}$. Now we write $3K_{8m} - F$, where $F = \{i\bar{i} \in E(K_{4m,4m}) \mid 1 \leq i \leq 4m\}$ as follows:

$$\begin{aligned} 3K_{8m} - F &= ((K^\star_{4m} \oplus I) \oplus K_{4m,4m} \oplus (\overline{K}^\star_{4m} \oplus \bar{I})) \oplus (K^\star_{4m} \oplus F' \oplus \overline{K}^\star_{4m}) \\ &\quad \oplus ((K^\star_{4m} - I) \oplus K_{4m,4m} - \{F, F'\} \oplus (\overline{K}^\star_{4m} - \bar{I})) \oplus K_{4m,4m}. \end{aligned}$$

Where $I = \{i(2m+i) \in E(K^{\star}_{4m}) \mid 1 \leq i \leq 2m\}$, $\overline{I} = \{\overline{i}(\overline{2m+i}) \in E(\overline{K}^{\star}_{4m}) \mid 1 \leq i \leq 2m\}$, $F' = \{i(\overline{2m+i}), \overline{i}(2m+i) \in E(K_{4m,4m}) \mid 1 \leq i \leq 2m\}$. The remaining proof follows from Lemmas 16, 17, 19 and Remark 5. ∎

Theorem 21. *For all $\lambda \equiv 1 \pmod 2 \geq 3$ and $n \equiv 0 \pmod 2 \geq 4$, there exists a symmetric Hamilton cycle decomposition of $\lambda K_n - F$.*

Proof. Follows from Theorems 14, 15 and 20. ∎

5. Conclusion

From the results of Sections 3 and 4 together with the known results of Section 2, we have the following:

Theorem 22. *For $n \geq 3$, there exists a symmetric Hamilton cycle decomposition of λK_n if and only if*

(i) *λ is even and n is odd,* (*or*)

(ii) *λ is odd and n is odd,* (*or*)

(iii) *λ is even and n is even.*

Theorem 23. *For $n \geq 3$, there exists a symmetric Hamilton cycle decomposition of $\lambda K_n - F$ with respect to the 1-factor F if and only if λ is odd and n is even except the non-existence cases $n \equiv 0$ or $6 \pmod 8$ when $\lambda = 1$.*

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