

CHARACTERIZATIONS OF THE FAMILY OF ALL  
GENERALIZED LINE GRAPHS—FINITE AND  
INFINITE—AND CLASSIFICATION OF THE FAMILY OF  
ALL GRAPHS WHOSE LEAST EIGENVALUES  $\geq -2$

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**Abstract**

The infimum of the least eigenvalues of all finite induced subgraphs of an infinite graph is defined to be its least eigenvalue. In [P.J. Cameron, J.M. Goethals, J.J. Seidel and E.E. Shult, *Line graphs, root systems, and elliptic geometry*, J. Algebra **43** (1976) 305–327], the class of all finite graphs whose least eigenvalues  $\geq -2$  has been classified: (1) *If a (finite) graph is connected and its least eigenvalue is at least  $-2$ , then either it is a generalized line graph or it is represented by the root system  $E_8$ .* In [A. Torgašev, *A note on infinite generalized line graphs*, in: Proceedings of the Fourth Yugoslav Seminar on Graph Theory, Novi Sad, 1983 (Univ. Novi Sad, 1984) 291–297], it has been found that (2) *any countably infinite connected graph with least eigenvalue  $\geq -2$  is a generalized line graph.* In this article, the family of all generalized line graphs—countable and uncountable—is described algebraically and characterized structurally and an extension of (1) which subsumes (2) is derived.

**Keywords:** generalized line graph, enhanced line graph, representation of a graph, extended line graph, least eigenvalue of a graph.

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For unexplained graph theoretic terms and notation, the reader is referred to [15]. For information on Hilbert spaces, we rely on [6]. Each graph considered in this article is simple; its order need not be finite. Let  $G$  be a graph; let  $a, b$  be two vertices of  $G$ ; we write  $\langle\langle a, b \rangle\rangle = 1$  to mean that  $a, b$  are adjacent whereas  $\langle\langle a, b \rangle\rangle = 0$  implies that they are not adjacent. The set  $N(a) \cup \{a\}$ , known as the closed neighbourhood of  $a$ , is denoted by  $N[a]$  or  $N_G[a]$ . The order of  $G$  is denoted by  $|G|$ . Any graph obtained from  $G$  by deleting some vertices—known

as an *induced subgraph* of  $G$ —is called in this article an *innergraph* of  $G$ . To denote that a graph  $H$  is an innergraph of  $G$ , we write  $H \preceq G$ . If  $X \subseteq V(G)$ , then the innergraph of  $G$  with vertex set  $X$  is denoted by  $G[X]$ ; any innergraph with a finite vertex set  $\{x_1, x_2, \dots, x_n\}$  is denoted by  $G[x_1, x_2, \dots, x_n]$  also. Let  $H$  be a finite subgraph of  $G$ ; if there is a vertex  $p$  in  $V(G) \setminus V(H)$  such that  $|N(p) \cap V(H)|$  is odd, then  $H$  is called *odd* in  $G$ ; otherwise it is *even* in  $G$ . Let  $e$  be an edge of  $G$ ; when the degree of one of its endpoints is 1, it is called a *pendant edge*; in that case, the other endpoint is said to *support*  $e$ . As in [11], the infimum of the least eigenvalues of all finite innergraphs of  $G$ —note that when  $G$  is finite, this infimum coincides with the least eigenvalue of  $G$ —is defined to be the *least eigenvalue* of  $G$  and denoted by  $\lambda(G)$ . The *line graph*  $L(G)$  is defined as follows: its vertex set is  $E(G)$ ; two vertices of  $L(G)$  are adjacent if and only if they have a common vertex in  $G$ . A family  $\mathcal{F}$  of subgraphs of  $G$  is called a *decomposition* of  $G$ , if every edge of  $G$  appears in exactly one member of  $\mathcal{F}$ . The following results have been obtained in [8] and [12], respectively:

**Theorem 1.** *A graph  $G$  is a line graph if and only if it decomposes into complete subgraphs such that each vertex of  $G$  appears in at most two of these subgraphs.*

**Theorem 2.** *A graph  $G$  is a line graph if and only if no innergraph is  $K_{1,3}$  and whenever an innergraph is  $K_{1,1,2}$ , one of its triangles is even in  $G$ .*

The above results are usually confined to finite graphs. However, since their proofs involve neither counting arguments nor induction, they work for infinite graphs also. (See [15, Pages 280–82]; for proving the second result for infinite graphs, we need the following: If  $H$  is a complete subgraph of a graph  $G$ , then there exists a maximal complete subgraph of  $G$  containing  $H$ ; this fact can be proved by using Zorn's lemma.)

Let  $\mathfrak{L}$  denote the family of all graphs with least eigenvalues  $\geq -2$ . A notable property of finite line graphs is that they belong to  $\mathfrak{L}$ . (For information in this regard and for a proof in particular, see [2]. If  $G$  is an infinite line graph, then for any finite innergraph  $H$  of  $G$ ,  $\lambda(H) \geq -2$  because  $H$  also is a line graph whence  $\lambda(G) = \inf\{\lambda(H) : H \preceq G \text{ and } |H| < \infty\} \geq -2$ . Therefore infinite line graphs also belong to  $\mathfrak{L}$ .) This fact has prompted many authors to study intensively the set of all finite graphs in  $\mathfrak{L}$ , denoted by  $\mathfrak{L}_f$  in this article. Hoffman has found an important subfamily of  $\mathfrak{L}_f$ , whose members are called generalized line graphs. (See [7].) For our purpose, we extend Hoffman's definition of these graphs, as done in [11].

**Definition 3.** Any graph in which every vertex is adjacent to all other vertices except one—i.e., any graph obtained from a complete graph by removing a perfect matching—is called a *cocktail party graph*. For any nonnegative integer  $n$ , the cocktail party graph with  $2n$  vertices is denoted by  $CP(n)$ . Let  $G$  be a graph and

$\{H_\alpha : \alpha \in V(G)\}$  be a collection of cocktail party graphs such that for each  $\alpha \in V(G)$ ,  $V(H_\alpha) \cap E(G) = \emptyset$  and for all distinct  $\alpha, \beta \in V(G)$ ,  $V(H_\alpha) \cap V(H_\beta) = \emptyset$ . Then the *generalized line graph*  $L[G; H_\alpha, \alpha \in V(G)]$  is obtained from the union of  $L(G)$  and the graphs  $H_\alpha, \alpha \in V(G)$  by forming additional edges: a vertex  $e$  in  $L(G)$  is adjacent to all vertices in  $H_\alpha$  whenever  $\alpha$  is an endpoint of  $e$  in  $G$ . When  $V(G)$  is finite, say  $\{v_1, v_2, \dots, v_n\}$  and for each  $i \leq n$ , the cocktail party graph associated with  $v_i$  is a finite graph, say  $CP(\sigma_i)$ , then the generalized line graph is denoted by  $L(G; \sigma_1, \sigma_2, \dots, \sigma_n)$  also. Let  $\mathcal{G}$  denote the family of all generalized line graphs.

- Remark 4.** (1) The generalized line graph described above coincides with  $L(G)$  when all cocktail party graphs are null; it is a cocktail party graph when  $G = K_1$ . Thus line graphs and cocktail party graphs belong to  $\mathcal{G}$ .
- (2) Let  $\Gamma$  be an innergraph of a generalized line graph  $L[G; H_\alpha, \alpha \in V(G)]$ . For each  $\alpha \in V(G)$ , let  $A_\alpha = \{v \in V(\Gamma) \cap V(H_\alpha) : \text{for some } x \in V(\Gamma) \cap V(H_\alpha) \setminus \{v\}, \langle x, v \rangle = 0\}$  and  $B_\alpha = (V(\Gamma) \cap V(H_\alpha)) \setminus A_\alpha$ . (Note that for each  $\alpha \in V(G)$ ,  $\Gamma[A_\alpha]$  is a cocktail party graph.) Let  $G'$  be the graph defined according to the following three conditions:  $V(G') = V(G) \cup [\cup_{\alpha \in V(G)} B_\alpha]$ ;  $V(\Gamma) \cap E(G) \subset E(G')$ ; if  $\alpha \in V(G)$ , then for each  $x \in B_\alpha, \alpha x \in E(G')$ . Now, for each  $\alpha \in V(G')$ , let  $H'_\alpha$  be the cocktail party graph defined as follows: if  $\alpha \in V(G)$ , then  $H'_\alpha = \Gamma[A_\alpha]$ ; if  $\alpha \in (V(G') \setminus V(G))$ , then  $H'_\alpha = CP(0)$ . It can be verified that  $\Gamma$  can be taken as the generalized line graph  $L[G'; H'_\alpha, \alpha \in V(G')]$ . Thus it follows that any innergraph of a generalized line graph also is a generalized line graph.
- (3) Note that  $L(K_2; 1, 1) = K_{1,4}$  whence  $K_{1,3}$  also belongs to  $\mathcal{G}$  and  $L(K_3; 1, 0, 0) = K_{1,1,3}$ .
- (4) Since any finite generalized line graph belongs to  $\mathcal{L}_f$ —see [7] or [2] for a proof—it follows that  $\mathcal{G} \subset \mathcal{L}$ .

Let  $\mathcal{G}_f$  be the family of all finite generalized line graphs; a characterization of this family, analogous to that of the family of all finite line graphs given by Theorem 1 has been obtained in [4]. The following definition conceptualizes the characterization given by [4].

**Definition 5.** A graph  $\Omega$  is called an *extended line graph*, if there exists a decomposition  $\mathfrak{F} = \{F_j : j \in J\}$  of  $\Omega$  such that the following hold.

- (1) For any  $j \in J$ , every vertex of  $F_j$  is adjacent to all other vertices of  $F_j$  except at most one vertex, i.e.,  $F_j$  can be obtained from a complete graph by removing a matching.
- (2) For all distinct  $j, k \in J, F_j$  and  $F_k$  have at most one common vertex.
- (3) Every vertex lies in at most two members of  $\mathfrak{F}$ .

- (4) If a vertex  $v$  lies in two distinct members  $F_j, F_k$  of  $\mathfrak{F}$ , then  $v$  is adjacent to all vertices in  $[V(F_j) \cup V(F_k)] \setminus \{v\}$ .

Let  $\mathcal{X}$  be the family of all extended line graphs; note that by Theorem 1, line graphs belong to this family.

**Remark 6.** Let  $\Omega = L[G; H_\alpha, \alpha \in V(G)]$  be a generalized line graph; for each  $\alpha \in V(G)$ , let  $F_\alpha = \Omega[V(H_\alpha) \cup E_\alpha]$  where  $E_\alpha$  is the set of all edges in  $G$  which are incident with  $\alpha$ . Taking  $J = V(G)$ , it is easy to check that (1), (2), (3) and (4) of Definition 5 hold, i.e.,  $\Omega$  is an extended line graph. Therefore,  $\mathfrak{G} \subseteq \mathcal{X}$ .

Let  $W$  be a subset of a Hilbert space such that the norm of each vector in  $W$  is  $\sqrt{2}$ . (Every Hilbert space  $\mathbb{H}$  considered in this article is real; the inner product of any two vectors  $\alpha, \beta \in \mathbb{H}$  is denoted by  $\langle \alpha, \beta \rangle$ .) If  $\psi$  is a map from the vertex set of a graph  $G$  to  $W$  such that for all distinct  $x, y \in V(G)$ ,  $\langle \psi(x), \psi(y) \rangle = \langle x, y \rangle$ , then  $\psi$  is called a *representation* of  $G$  in  $W$ . The family of all representable graphs is denoted by  $\mathcal{R}$ . If a graph has a representation in some set  $\{\pm\mu \pm \nu : \mu, \nu \in \mathcal{O} \text{ and } \mu \neq \nu\}$  where  $\mathcal{O}$  is an orthonormal set in a Hilbert space, then it is called *amicable*. The family of all amicable graphs is denoted by  $\mathcal{A}$ .

Let  $\{e_1, e_2, \dots, e_8\}$  be an orthonormal basis of  $\mathbb{R}^8$ . The set  $E_8$ —known as an exceptional root system in the literature—is defined to be

$$\{\pm e_i \pm e_j : 1 \leq i < j \leq 8\} \cup \left\{ \frac{1}{2} \sum_{i=1}^8 \varepsilon_i e_i : \text{for all } i \leq 8, \varepsilon_i = \pm 1 \text{ and } \prod_{i=1}^8 \varepsilon_i = 1 \right\}.$$

A representation of a graph in  $E_8$  is shown in Figure 1. Note that this graph appears in Figure 2. Later, we will prove that no graph in this figure is amicable. Therefore,  $\mathcal{A} \subsetneq \mathcal{R}$ .

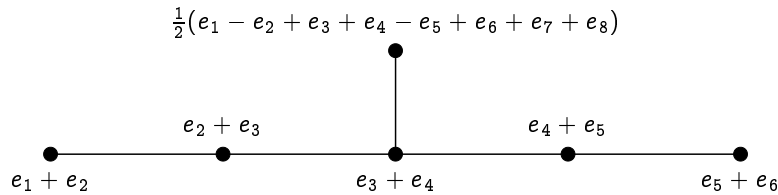


Figure 1. A representation of a graph in  $E_8$ .

**Lemma 7.** *Every extended line graph is amicable.*

**Proof.** Let  $\Omega$  be an extended line graph; then there exists a decomposition  $\{F_j : j \in J\}$  of  $\Omega$  such that the conditions (1), (2), (3) and (4) of Definition 5 hold. For each  $j \in J$ , let  $A_j$  be a subset of  $V(F_j)$  such that for each  $v \in V(F_j)$ , there is exactly one vertex  $x \in A_j$  with  $\langle v, x \rangle = 0$  ( $x$  and  $v$  may be same). We can assume that  $\mathcal{O} := J \cup V(\Omega)$  is an orthonormal set in a Hilbert space and  $\Omega$  does not have isolated vertices. Let  $\xi : V(\Omega) \rightarrow \{\pm\mu \pm \nu : \mu, \nu \in \mathcal{O} \text{ and } \mu \neq \nu\}$  be the

map defined as follows: Let  $u \in V(\Omega)$ . If  $u$  is a vertex of two members of  $\mathfrak{F}$ , say  $F_\alpha, F_\beta$ , then let  $\xi(u) = \alpha + \beta$ . Suppose that  $u$  belongs to exactly one member, say  $F_\alpha$ . If  $u \in A_\alpha$ , let  $\xi(u) = \alpha + u$ ; otherwise let  $\xi(u) = \alpha - v$  where  $v$  is the vertex in  $A_\alpha$  such that  $\langle\langle u, v \rangle\rangle = 0$ . It is easy to verify that  $\xi$  is a representation of  $\Omega$ . ■

It has been proved that  $\{G \in \mathfrak{L}_f \setminus \mathfrak{G}_f : G \text{ is connected}\}$  is finite—see [2, Theorem 4.10]. This fact and the natural relationship between line graphs and generalized line graphs have motivated various authors to study  $\mathfrak{G}_f$  comprehensively. Let  $\mathcal{M}_f$  be the class of all minimal forbidden graphs for  $\mathfrak{G}_f$ . (Let  $\mathfrak{G}$  be a hereditary family of graphs; if a graph  $G$  does not belong to  $\mathfrak{G}$ , whereas every proper innergraph of  $G$  belongs to  $\mathfrak{G}$ , then  $G$  is called a *minimal forbidden graph* for  $\mathfrak{G}$ .) Note that  $\mathcal{M}_f$  determines  $\mathfrak{G}_f$ : A finite graph  $G$  is a generalized line graph if and only if no innergraph of  $G$  belongs to  $\mathcal{M}_f$ . Various algebraic properties of  $\mathfrak{G}_f$  have been found in [4, 5] and five different methods of computing  $\mathcal{M}_f$  have been found in [4, 9, 13, 5, 14]. Countably infinite graphs in  $\mathfrak{L}$  also have been studied: in [10], it has been shown that any countably infinite connected graph with least eigenvalue  $\geq -2$  is a generalized line graph and in [11], all countably infinite connected graphs with least eigenvalues  $> -2$  have been determined. The current article generalizes many of the results on  $\mathfrak{L}_f$ . We describe  $\mathfrak{G}$  by using vectorial representability of its members, characterize it structurally and find the set of all minimal forbidden graphs for  $\mathfrak{G}$ , denoted in this article by  $\mathcal{M}$ . In this process we obtain a classification of  $\mathfrak{L}$ , and prove that  $\mathcal{M} = \mathcal{M}_f$  and that  $\mathfrak{G}$  is determined by  $\mathcal{M}$ . The main tool for proving our results is a notion introduced in [14]:

**Definition 8.** A graph  $G$  is called an *enhanced line graph* when the following conditions hold for every innergraph  $H$  of  $G$ .

- (E1) If  $H = C_4$ , then it is even in  $G$ .
- (E2) If  $H = K_{1,3}$ , then two vertices of  $H$  have same neighbourhood in  $G$ .
- (E3) If  $H = K_{1,1,2}$ , then either its nonadjacent vertices have same neighbourhood in  $G$  or one of its triangles is even in  $G$ .
- (E4) If  $H = K_{1,1,1,3}$ , then for some  $x, y \in V(H)$ ,  $N_G(x) = N_G(y)$  and  $H - \{x, y\}$  is an even triangle in  $G$ .
- (E5) If  $H$  is connected and  $V(H)$  has three distinct vertices such that their neighbourhoods in  $H$  are same, then  $H$  is either  $K_{1,3}$  or  $K_{1,4}$  or  $K_{1,1,3}$ .

Let  $\mathcal{E}$  denote the family of all enhanced line graphs. By using Theorem 2, it can be easily verified that every line graph belongs to this family; we can even prove a generalization of this fact:  $\mathcal{A}$  is a subfamily of  $\mathcal{E}$ ; the latter fact itself is

subsumed by Proposition 9, to derive which, we need the following: Let  $\psi$  be a representation of a graph  $G$  in a set  $\{\pm\mu \pm \nu : \mu, \nu \in \mathcal{O} \text{ and } \mu \neq \nu\}$  where  $\mathcal{O}$  is an orthonormal set in a Hilbert space. If  $a, b$  are two distinct vertices such that  $\psi(a)$  and  $\psi(b)$  are linear combinations of the same vectors in  $\mathcal{O}$ —i.e.,  $\psi(a)$  and  $\psi(b)$  are of the forms  $\pm\alpha \pm \beta$  and  $\pm\alpha \mp \beta$ —then  $a$  and  $b$  are called *associates with respect to  $\psi$* . Note that a vertex cannot be an associate of two different vertices. If two vertices  $a, b$  are associates, then  $N(a) = N(b)$  because for any  $x \in V(G) \setminus \{a, b\}$ ,  $\langle \psi(a), \psi(x) \rangle = \langle \psi(b), \psi(x) \rangle$  and  $\langle \psi(a), \psi(b) \rangle = 0$ .

**Proposition 9.** *Let  $G$  be a graph such that every finite innergraph of  $G$  is amicable. Then  $G$  is an enhanced line graph.*

**Proof.** Let  $\mathcal{O}$  be a countably infinite orthonormal set in a Hilbert space. Let  $\mathcal{O}^* = \{-\nu, \nu : \nu \in \mathcal{O}\}$  and  $W = \{\pm\mu \pm \nu : \mu, \nu \in \mathcal{O} \text{ and } \mu \neq \nu\}$ ; note that by the hypothesis, every finite innergraph of  $G$  has a representation in  $W$ . First, suppose that  $a, b, c, d$  are vertices of  $G$  such that  $H := G[a, b, c, d] = C_4$  and  $\langle\langle a, c \rangle\rangle = 0$ . Let  $p \in V(G) \setminus \{a, b, c, d\}$  and  $\psi$  be a representation of  $G[a, b, c, d, p]$  in  $W$ . It is easy to verify that  $\|\psi(a) - \psi(b) + \psi(c) - \psi(d)\|^2 = 0$  whence  $\psi(a) + \psi(c) = \psi(b) + \psi(d)$ ; therefore  $|N(p) \cap V(H)| = \sum_{x \in V(H)} \langle\langle p, x \rangle\rangle = 2\langle\langle \psi(p), \psi(a) + \psi(c) \rangle\rangle$ ; so, it follows that  $H$  is even, i.e., the conclusion of (E1) holds.

Now, suppose that (E2) does not hold. Then, there are vertices  $p, a, b, c$  such that  $H := G[p, a, b, c] = K_{1,3}$ ,  $\deg_H p = 3$  and  $a, b, c$  have different neighbourhoods in  $G$ . Therefore, there are three vertices  $x, y, z$  such that in  $K := G[p, a, b, c, x, y, z]$ ,  $a, b, c$  have different neighbourhoods. Let  $\psi$  be a representation of  $K$  in  $W$ . It is easy to verify that two of  $\{a, b, c\}$  are associates with respect to  $\psi$ ; these two have same neighbourhood in  $K$ —a contradiction.

Now suppose that (E3) does not hold. Then there are vertices  $a, b, p, q$  such that  $G[a, b, p, q] = K_{1,1,2}$ ,  $\langle\langle p, q \rangle\rangle = 0$ ,  $N(p) \neq N(q)$  and both  $G[a, b, p]$  and  $G[a, b, q]$  are odd. Therefore, there are vertices  $x, y, z$  such that in  $K := G[a, b, p, q, x, y, z]$ ,  $p, q$  have different neighbourhoods and both  $G[p, a, b]$  and  $G[q, a, b]$  are odd. Let  $\psi$  be a representation of  $K$  in  $W$ . Then for some  $\alpha, \beta, \gamma \in \mathcal{O}^*$ ,  $\{\psi(a), \psi(b)\} = \{\alpha + \beta, \alpha + \gamma\}$ . If  $\langle\langle \alpha, \psi(p) \rangle\rangle \neq 0 \neq \langle\langle \alpha, \psi(q) \rangle\rangle$ , then  $\psi(p), \psi(q)$  are associates whence  $N_K(p) = N_K(q)$ —a contradiction. Therefore, we can assume that  $\langle\langle \alpha, \psi(p) \rangle\rangle = 0$ . Then  $\psi(p) = \beta + \gamma$  whence  $G[a, b, p]$  is even in  $K$ —again, a contradiction.

Next, suppose that (E4) does not hold; i.e., there are vertices  $p, q, a, b, c$  such that  $H := G[p, q, a, b, c] = K_{1,1,3}$ ,  $\deg_H p = \deg_H q = 4$  and the conclusion of (E4) does not hold. Then, there are vertices  $x, y, z \in [V(G) \setminus V(H)]$  such that the following statements hold:

- either  $\langle\langle x, b \rangle\rangle \neq \langle\langle x, c \rangle\rangle$  or  $|N(x) \cap \{p, q, a\}|$  is odd;
- either  $\langle\langle y, c \rangle\rangle \neq \langle\langle y, a \rangle\rangle$  or  $|N(y) \cap \{p, q, b\}|$  is odd;
- either  $\langle\langle z, a \rangle\rangle \neq \langle\langle z, b \rangle\rangle$  or  $|N(z) \cap \{p, q, c\}|$  is odd.

Thus the conclusion of (E4) does not hold when  $G$  is replaced by  $K := G[p, q, a, b, c, x, y, z]$ . Let  $\psi$  be a representation of  $K$  in  $W$ . Then for some  $\alpha, \beta, \gamma, \delta \in \mathcal{O}^*$ ,  $\{\psi(p), \psi(a), \psi(b), \psi(c)\} = \{\alpha + \beta, \beta + \gamma, \alpha + \delta, \alpha - \delta\}$ ; from this we find that  $\psi(q) = \alpha + \gamma$  whence the vertices which are associates have same neighbourhood in  $K$  and the rest of  $V(H)$  form an even triangle in  $K$ —a contradiction.

Now, suppose that  $a, b, c$  are distinct vertices of a connected innergraph  $H$  such that  $N_H(a) = N_H(b) = N_H(c)$ . Let  $p \in N_H(a)$ ; then  $G[p, a, b, c] = K_{1,3}$ . Let  $K$  be a maximal connected innergraph of  $H$  such that  $p, a, b, c \in V(K)$  and  $|K| \leq 6$ . Let  $\psi$  be a representation of  $K$  in  $W$ . Then for some  $\alpha, \beta, \gamma, \delta \in \mathcal{O}^*$ ,  $\{\psi(p), \psi(a), \psi(b), \psi(c)\} = \{\alpha + \beta, \beta + \gamma, \alpha + \delta, \alpha - \delta\}$ . If  $|K| = 4$ , then  $H = K$  whence the conclusion of (E5) holds. So, let  $q$  be a vertex in  $V(K) \setminus \{p, a, b, c\}$  such that  $G[p, q, a, b, c]$  is connected; note that  $\psi(q)$  is either  $\alpha + \gamma$  or  $\beta - \gamma$ ; in each case, the connectivity of  $K$  and the property of  $a, b, c$  ensure that  $V(K)$  does not have any other vertex of  $G$ , whence  $K = K_{1,4}$  or  $K_{1,1,3}$ . Now, by the choice of  $K$ , it follows that  $H = K$ . Thus the conclusion of (E5) holds.

Summarizing, we find that all the conditions of Definition 8 hold; therefore,  $G$  is an enhanced line graph. ■

Using Theorem 2 effectively, the set of all minimal forbidden graphs for the family of all line graphs has been found in [1]. In this article, the next result—its finite version has been derived in [14]—is similarly used for determining the set of all minimal forbidden graphs for the family of all generalized line graphs.

**Theorem 10.** *Any enhanced line graph  $G$ —possibly infinite—is a generalized line graph.*

**Proof.** Since every component of  $G$  is an enhanced line graph and the conclusion holds for  $G$  when it does for every component, we can assume that  $G$  is connected; let its vertex set be  $V$ . If three distinct vertices of  $G$  have same neighbourhood, then by connectivity of  $G$  and by (E5),  $G \in \{K_{1,3}, K_{1,4}, K_{1,1,3}\}$  whence by Remark 4,  $G \in \mathcal{G}$ . Therefore, we assume that for each  $v \in V$ ,  $|\{x \in V : N(x) = N(v)\}| \leq 2$ . Let  $U$  be a subset of  $V$  such that for each  $x \in V$ , there is exactly one vertex  $u \in U$  with  $N(u) = N(x)$ . It is easy to verify that  $F := G[U]$  is connected. We can assume that  $F \not\cong K_3$  for otherwise being an innergraph of  $CP(3)$ ,  $G \in \mathcal{G}$  by Remark 4. Now from (E2), (E3) and Theorem 2 it follows that  $F$  is a line graph, i.e., there is a graph  $I$  with  $L(I) = F$ .

Let  $q$  be any vertex in  $V \setminus U$ . Then there is a (unique) vertex  $p \in U$  with  $N(p) = N(q)$ . If  $a, b$  are two distinct nonadjacent vertices in  $N_F(p)$ , then by the choice of  $U$ , there is a vertex  $x$  with  $\langle\langle x, a \rangle\rangle \neq \langle\langle x, b \rangle\rangle$  whence  $G[p, a, q, b]$  is odd—a contradiction to (E1). Therefore,  $N_F[p]$  is a clique. Now we claim that  $p$  is a pendant edge in the graph  $I$ . Otherwise, there are edges  $a, b$  in  $E(I) \setminus \{p\}$  such that they are incident with different ends of  $p$ . Since  $G[p, a, b] = K_3$ , in  $I$ ,  $p, a, b$  form a triangle. Therefore,  $G[p, a, b]$  is even in  $F$ . Since  $N_F[p]$  is a clique, it

follows that  $\deg_F p = 2$ . Since  $F \neq K_3$ , there is a vertex  $r \in U \setminus \{p, a, b\}$  such that  $G[p, a, b, r]$  is connected. Obviously  $\langle\langle a, r \rangle\rangle = \langle\langle b, r \rangle\rangle = 1$  and  $\langle\langle r, p \rangle\rangle = 0$ . Since  $N(r) \neq N(p)$ , there must be a vertex  $x \in U \setminus \{a, b\}$  with  $\langle\langle r, x \rangle\rangle = 1$ . Since  $G[p, a, b]$  is even,  $\langle\langle a, x \rangle\rangle = \langle\langle b, x \rangle\rangle$ . Now we find that (E4) is violated. So, our claim holds.

For each  $\alpha \in V(I)$ , let  $X_\alpha = \{u \in U : \text{for some } v \in (V \setminus U), N(u) = N(v) \text{ and } \alpha \text{ supports } u \text{ in } I\}$  and  $Y_\alpha = \{v \in (V \setminus U) : \text{for some } u \in X_\alpha, N(v) = N(u)\}$ . Let  $K$  be the spanning subgraph of  $I$  whose edge set is  $U \setminus \cup_{\alpha \in V(I)} X_\alpha$ . Now we observe the following:  $L(K) = G[U \setminus \cup_{\alpha \in V(K)} X_\alpha]$ ;  $\{E(K), X_\alpha, Y_\alpha : \alpha \in V(K)\}$  is a partition of  $V$ ; for each  $\alpha \in V(K)$ ,  $H_\alpha := G[X_\alpha \cup Y_\alpha]$  is a cocktail party graph; if  $\alpha \in V(K)$ ,  $u \in E(K)$  and  $v \in V(H_\alpha)$ , then  $\langle\langle u, v \rangle\rangle = 1$  if and only if  $u$  is incident with  $\alpha$  in  $K$ ; for all distinct  $\alpha, \beta \in V(K)$ , there is no edge from  $X_\alpha \cup Y_\alpha$  to  $X_\beta \cup Y_\beta$ . From the preceding five facts, it follows that  $G$  is  $L[K; H_\alpha, \alpha \in V(K)]$ . ■

The next result yields a set of different descriptions of  $\mathcal{G}$ , including a characterization using which  $\mathcal{M}$  is computed by Theorem 13.

**Theorem 11.** *For any graph  $G$ —possibly infinite—the following are equivalent.*

- (1)  $G$  is a generalized line graph.
- (2)  $G$  is an extended line graph.
- (3)  $G$  is an amicable graph.
- (4)  $G$  is an enhanced line graph.

**Proof.** Remark 6 is (1)  $\Rightarrow$  (2); Lemma 7 is (2)  $\Rightarrow$  (3); by Proposition 9, (3)  $\Rightarrow$  (4); Theorem 10 is (4)  $\Rightarrow$  (1). ■

Theorem 12 generalizes the result that every countably infinite connected graph with least eigenvalue  $\geq -2$  is a generalized line graph [10] and yields an objective of this article, viz., the classification of  $\mathcal{L}$ .

**Theorem 12.** *For a connected graph  $G$ —possibly infinite—the following are equivalent.*

- (1) The least eigenvalue of  $G$  is at least  $-2$ .
- (2)  $G$  is representable.
- (3) Either  $G$  is a generalized line graph or it has a representation in  $E_8$ .

**Proof.** It is a well known fact that the result holds when  $G$  is finite. (For a proof, see [2]; for additional details, see [3].) So, let us assume that  $G$  is infinite. First, suppose that (1) holds. Let  $X$  be a finite subset of  $V(G)$ . We can choose a finite subset  $Y$  of  $V(G)$  such that  $X \subset Y$ ,  $G[Y]$  is connected and it cannot be represented by  $E_8$ . (Note that  $E_8$  is finite.) Since  $\lambda(G[Y]) \geq -2$ , by the



fact mentioned in the beginning of the proof,  $G[Y]$  is a generalized line graph; therefore,  $G[X]$  also is a generalized line graph whence by Theorem 11, we find that  $G[X] \in \mathcal{A}$ ; now by Proposition 9,  $G \in \mathcal{E}$  whence by Theorem 10,  $G \in \mathcal{G}$ ; therefore, (3) holds.

Now, suppose that (3) holds. Since  $G$  cannot have a representation in  $E_8$ ,  $G \in \mathcal{G}$ ; therefore, by Theorem 11,  $G \in \mathcal{A}$ ; since  $\mathcal{A} \subset \mathcal{R}$ , (2) holds. Next, suppose that (2) holds. Let  $X$  be any finite subset of  $V(G)$ . Then  $G[X] \in \mathcal{R}$  whence by the fact mentioned in the beginning of this proof,  $\lambda(G[X]) \geq -2$ . Therefore,  $\lambda(G) \geq -2$ , i.e., (1) holds. ■

It is easy to verify that  $\mathcal{M}_f \subseteq \mathcal{M}$ ; let  $G \in \mathcal{M}$ ; then for each  $v \in V(G)$ ,  $G - v \in \mathcal{G}$ ; therefore,  $G$  is finite for otherwise by Proposition 9 and Theorem 11,  $G$  would belong to  $\mathcal{G}$ . Therefore,  $G \in \mathcal{M}_f$ . Thus it follows that  $\mathcal{M}_f$  and  $\mathcal{M}$  are same. In [14],  $\mathcal{M}_f$  has been computed by using the finite version of Theorem 10 but not directly. (See [4, 9, 5] for different methods.) This has been done in [13] and [3] by using results for finite graphs which are similar to Theorem 10. For the sake of completeness, we give a method of computing  $\mathcal{M}$  which is similar to the method of [13] but shorter.

**Theorem 13.** *A graph  $G$ —possibly infinite—is a generalized line graph if and only if no innergraph of  $G$  belongs to  $\mathfrak{M} := \{G_i : 1 \leq i \leq 31\}$ . (See Figure 2.)*

**Proof.** Routinely but easily, the following can be verified.

- For each  $i \in \{10, 11\}$ , (E1) does not hold in  $G_i$ .
- For each  $i \in \{18, 23, 24, 25, 26, 27, 28, 29, 30, 31\}$ , (E2) does not hold in  $G_i$ .
- For each  $i \in \{12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 25, 26, 28, 29, 30\}$ , (E3) does not hold in  $G_i$ .
- For each  $i \in \{4, 7, 8, 9\}$ , (E4) does not hold in  $G_i$ .
- For each  $i \in \{1, 2, 3, 4, 5, 6, 7\}$ , (E5) does not hold in  $G_i$ .

From this information, it follows that for each  $i \leq 31$ ,  $G_i \notin \mathcal{E}$ ; therefore, if  $G \in \mathcal{G}$ , then by Theorem 11, no innergraph of  $G$  belongs to  $\mathfrak{M}$ . Conversely, proving the following is enough: if a graph is not an enhanced line graph, then one of its innergraphs belongs to  $\mathfrak{M}$ . Let  $G \notin \mathcal{E}$ . Then for an innergraph  $H$  of  $G$ , one of the conditions of Definition 8 fails.

*Case E1.*  $H$  is isomorphic to  $C_4$  and odd in  $G$ . Then, there is a vertex  $p \in V(G) \setminus V(H)$  such that  $|N(p) \cap V(H)|$  is odd whence  $G[\{p\} \cup V(H)]$  is either  $G_{10}$  or  $G_{11}$ .

*Case E2.*  $H = K_{1,3}$  and all vertices of  $H$  have different neighbourhoods. Let  $V(H) = \{p, a, b, c\}$  and assume that  $\deg_H p = 3$ . Since  $N(a) \neq N(b)$ , there is a vertex  $x \in V(G)$  with  $\langle\langle x, a \rangle\rangle \neq \langle\langle x, b \rangle\rangle$ . Let  $S = G[p, a, b, c, x]$ . Leaving the

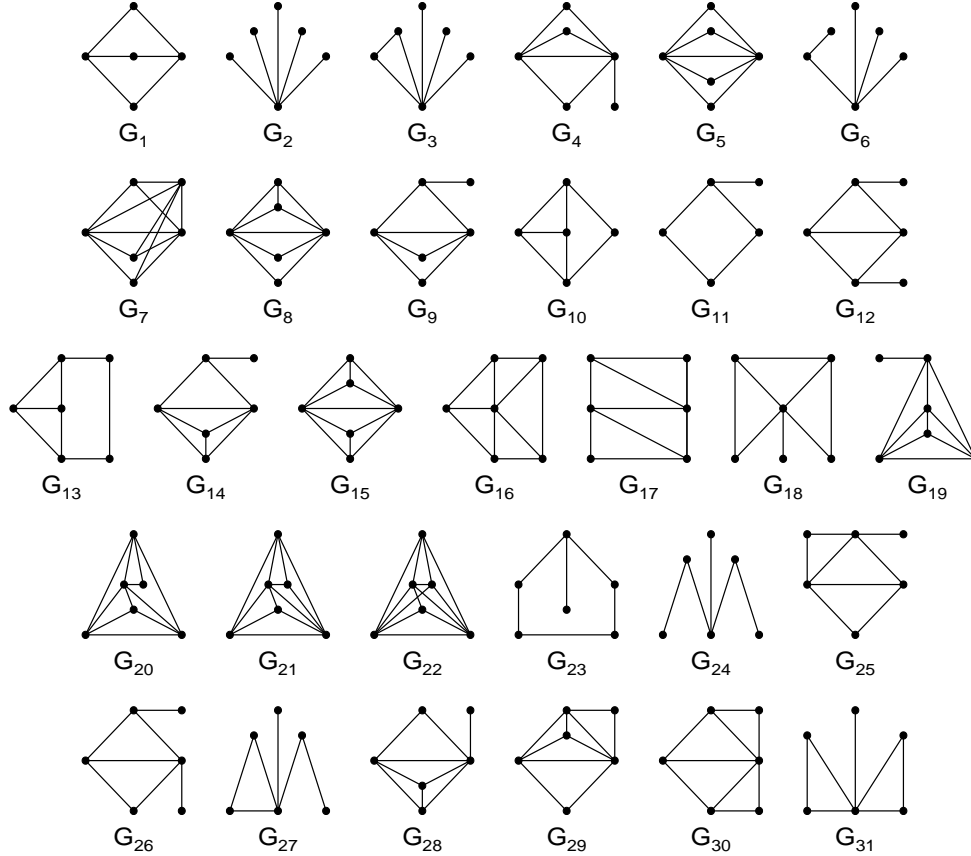


Figure 2. The set of all minimal forbidden graphs for  $\mathcal{E}$ .

possibility that  $x \in (N(c) \setminus N(p))$ , for then  $S = G_{11}$ , we find that  $S$  is one of the graphs  $H_1, H_2, H_3$  in Figure 3. Note that  $N_S(c) \neq N_S(a)$  or  $N_S(b)$ . Therefore,

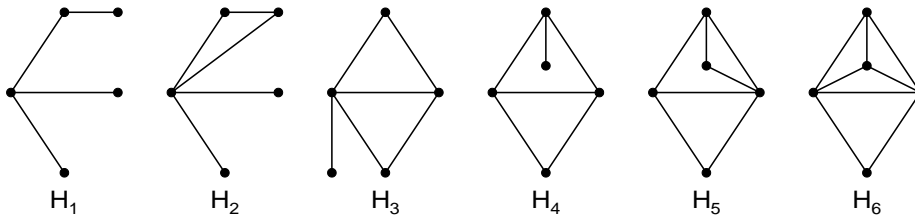


Figure 3. Some auxiliary graphs in  $\mathcal{E}$ .

by the hypothesis there is a vertex  $y$  such that  $N_T(b) \neq N_T(c) \neq N_T(a)$  where  $T = G[p, a, b, c, x, y]$ . It can be verified that either  $T \in \{G_{18}\} \cup \{G_i : 23 \leq i \leq 31\}$  or  $G_{11} \preceq T$ .

*Case E3.* There are vertices  $a, b, p, q$  such that  $G[a, b, p, q] = K_{1,1,2}$ ,  $\langle\langle p, q \rangle\rangle = 0$ ,  $N(p) \neq N(q)$  and both  $G[a, b, p]$  and  $G[a, b, q]$  are odd. Then there is a vertex  $x \in V(G)$  with  $\langle\langle x, p \rangle\rangle \neq \langle\langle x, q \rangle\rangle$ . Note that  $I := G[a, b, p, q, x]$  is one of the graphs  $H_4, H_5, H_6$  in Figure 3 and either  $G[a, b, p]$  or  $G[a, b, q]$  is odd in  $I$ . By the hypothesis, there is a vertex  $y$  such that both  $G[a, b, p]$  and  $G[a, b, q]$  are odd in  $K := G[a, b, p, q, x, y]$ . It can be verified that either  $G_{10} \preceq K$  or  $G_{11} \preceq K$  or  $K \in \{G_i : 12 \leq i \leq 22\} \cup \{G_{25}, G_{26}, G_{28}, G_{29}, G_{30}\}$ .

*Case E4.* There are vertices  $p, q, a, b, c$  such that  $H := G[p, q, a, b, c] = K_{1,1,3}$ ,  $\deg_H p = \deg_H q = 4$  and the conclusion of (E4) does not hold. If  $a, b, c$  have different neighbourhoods, then Case E2 holds. So, let us assume that  $N(a) = N(b)$ . Then by the hypothesis,  $G[p, q, c]$  is odd whence for some  $x \in V(G)$ ,  $|N(x) \cap \{p, q, c\}|$  is odd. It is easy to verify that for some  $i \in \{4, 7, 8, 9, 10, 11\}$ ,  $G_i \preceq G[p, q, a, b, c, x]$ .

*Case E5.* There are three distinct vertices  $a, b, c$  in a connected innergraph  $H$  of  $G$  such that  $N_H(a) = N_H(b) = N_H(c)$  and  $H \notin \{K_{1,3}, K_{1,4}, K_{1,1,3}\}$ . By connectivity of  $H$ , there is a vertex  $p \in N_H(a)$ ; then  $G[p, a, b, c] = K_{1,3}$ . Since  $H \neq K_{1,3}$ , there is a vertex  $x \in V(H) \setminus \{p, a, b, c\}$  such that  $S := G[p, a, b, c, x]$  is connected. If  $x \notin N(p)$ , then  $S = G_1$ ; so let  $\langle\langle x, p \rangle\rangle = 1$ ; then  $S$  is  $K_{1,4}$  or  $K_{1,1,3}$ . Then there has to be a vertex  $y \in V(G) \setminus \{p, a, b, c, x\}$  such that  $T = G[p, a, b, c, x, y]$  is connected whence we find that either  $G_1 \preceq T$  or  $T \in \{G_i : 2 \leq i \leq 7\}$ . ■

Let  $G \in \mathfrak{M}$ ; then by Theorem 13,  $G \notin \mathfrak{G}$  and for each  $v \in V(G)$ , it is easy to verify that  $G - v \in \mathcal{A}$  whence by Theorem 11,  $G - v \in \mathfrak{G}$ ; therefore  $G \in \mathcal{M}$ . Conversely, let  $G \in \mathcal{M}$ ; then by Theorem 13, an innergraph  $H$  of  $G$  belongs to  $\mathfrak{M}$ ; since  $H \notin \mathfrak{G}$ , we find that  $G = H$ , i.e.,  $G \in \mathfrak{M}$ . Thus, it follows that  $\mathcal{M} = \mathfrak{M}$ .

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