INTERVAL EDGE-COLORINGS OF CARTESIAN PRODUCTS OF GRAPHS I

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Abstract

A proper edge-coloring of a graph $G$ with colors $1, \ldots, t$ is an interval $t$-coloring if all colors are used and the colors of edges incident to each vertex of $G$ form an interval of integers. A graph $G$ is interval colorable if it has an interval $t$-coloring for some positive integer $t$. Let $\mathcal{I}$ be the set of all interval colorable graphs. For a graph $G \in \mathcal{I}$, the least and the greatest values of $t$ for which $G$ has an interval $t$-coloring are denoted by $w(G)$ and $W(G)$, respectively. In this paper we first show that if $G$ is an $r$-regular graph and $G \in \mathcal{I}$, then $W(G \square P_m) \geq W(G) + W(P_m) + (m-1)r$ ($m \in \mathbb{N}$) and $W(G \square C_{2n}) \geq W(G) + W(C_{2n}) + nr$ ($n \geq 2$). Next, we investigate interval edge-colorings of grids, cylinders and tori. In particular, we prove that if $G \square H$ is planar and both factors have at least 3 vertices, then $G \square H \in \mathcal{I}$ and $w(G \square H) \leq 6$. Finally, we confirm the first author’s conjecture on the $n$-dimensional cube $Q_n$ and show that $Q_n$ has an interval $t$-coloring if and only if $n \leq t \leq n(n+1)/2$.
Keywords: edge-coloring, interval coloring, grid, cylinder, torus, \( n \)-dimensional cube.

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1. Introduction

A proper edge-coloring of a graph \( G \) with colors \( 1, \ldots, t \) is an interval \( t \)-coloring if all colors are used and the colors of edges incident to each vertex of \( G \) form an interval of integers. A graph \( G \) is interval colorable if it has an interval \( t \)-coloring for some positive integer \( t \). Let \( \mathcal{G} \) be the set of all interval colorable graphs [1, 14].

For a graph \( G \in \mathcal{G} \), the least and the greatest values of \( t \) for which \( G \) has an interval \( t \)-coloring are denoted by \( w(G) \) and \( W(G) \), respectively. The concept of interval edge-coloring was introduced by Asratian and Kamalian [1]. In [1], they proved the following:

**Theorem 1.** Let \( G \) be a regular graph. Then

1. \( G \in \mathcal{G} \) if and only if \( \chi'(G) = \Delta(G) \).
2. If \( G \in \mathcal{G} \) and \( w(G) \leq t \leq W(G) \), then \( G \) has an interval \( t \)-coloring.

In [2], Asratian and Kamalian investigated interval edge-colorings of connected graphs. In particular, they obtained the following two results.

**Theorem 2.** If \( G \) is a connected graph and \( G \in \mathcal{G} \), then
\[
W(G) \leq \text{diam}(G) + (\Delta(G) - 1) + 1.
\]

**Theorem 3.** If \( G \) is a connected bipartite graph and \( G \in \mathcal{G} \), then
\[
W(G) \leq \text{diam}(G) (\Delta(G) - 1) + 1.
\]

Recently, Kamalian and the first author [16] showed that these upper bounds cannot be significantly improved.

In [13], Kamalian investigated interval colorings of complete bipartite graphs and trees. In particular, he proved the following:

**Theorem 4.** For any \( r, s \in \mathbb{N} \), the complete bipartite graph \( K_{r,s} \) is interval colorable, and

1. \( w(K_{r,s}) = r + s - \gcd(r, s) \),
2. \( W(K_{r,s}) = r + s - 1 \),
3. if \( w(K_{r,s}) \leq t \leq W(K_{r,s}) \), then \( K_{r,s} \) has an interval \( t \)-coloring.

In [21], the first author investigated interval colorings of complete graphs and \( n \)-dimensional cubes. In particular, he obtained the following two results.
Theorem 5. If \( n = p2^q \), where \( p \) is odd and \( q \) is nonnegative, then
\[
W(K_{2n}) \geq 4n - 2 - p - q.
\]

Theorem 6. If \( n \in \mathbb{N} \), then \( W(Q_n) \geq \frac{n(n+1)}{2} \).

The \( NP \)-completeness of the problem of the existence of an interval edge-coloring of an arbitrary bipartite graph was shown in [24]. A similar result for regular graphs was obtained in [1, 2]. In [19, 22, 23], interval edge-colorings of various products of graphs were investigated. Some interesting results on interval colorings were also obtained in [3, 4, 6, 7, 8, 9, 10, 14, 15, 16, 17, 18, 19, 20]. Surveys on this topic can be found in some books [3, 12, 19].

In this paper we focus only on interval edge-colorings of Cartesian products of graphs.

2. Notations, Definitions and Auxiliary Results

Throughout this paper all graphs are finite, undirected, and have no loops or multiple edges. Let \( V(G) \) and \( E(G) \) denote the sets of vertices and edges of graph \( G \), respectively. The degree of a vertex \( v \) in \( G \) is denoted by \( d_G(v) \), the maximum degree of \( G \) by \( \Delta(G) \), and the chromatic index of \( G \) by \( \chi'(G) \). If \( G \) is a connected graph, then the distance between two vertices \( u \) and \( v \) in \( G \), we denote by \( d(u, v) \), and the diameter of \( G \) by \( \text{diam}(G) \). We use the standard notations \( P_n, C_n, K_n \) and \( Q_n \) for the path, cycle, complete graph on \( n \) vertices and the \( n \)-dimensional cube, respectively. A partial edge-coloring of a graph \( G \) is a coloring of some edges of \( G \) such that no two adjacent edges receive the same color. If \( \alpha \) is a partial edge-coloring of \( G \) and \( v \in V(G) \), then \( S(v, \alpha) \) denotes the set of colors appearing on colored edges incident to \( v \). Clearly, if \( \alpha \) is a proper edge-coloring of a graph \( G \), then \( |S(v, \alpha)| = d_G(v) \) for every \( v \in V(G) \).

Let \([t]\) denote the set of the first \( t \) natural numbers. Let \([a] \ (\lceil a \rceil) \) denote the largest (least) integer less (greater) than or equal to \( a \). For two positive integers \( a \) and \( b \) with \( a \leq b \), the set \( \{a, \ldots, b\} \) is denoted by \([a, b]\). The terms and concepts that we do not define can be found in [25].

Let \( G \) and \( H \) be graphs. The Cartesian product \( G \square H \) is defined as follows:
\[
V(G \square H) = V(G) \times V(H), \quad E(G \square H) = \{(u_1, v_1)(u_2, v_2) : u_1 = u_2 \land v_1v_2 \in E(H) \text{ or } v_1 = v_2 \land u_1u_2 \in E(G)\}.
\]

Let \( V(G) = \{u_1, \ldots, u_n\} \) and \( V(H) = \{w_1, \ldots, w_m\} \). We use the following notation for vertex and edge sets of the Cartesian product \( G \square H \):
\[
V(G \square H) = \bigcup_{i=1}^m V^i, \quad E(G \square H) = \bigcup_{i=1}^m E^i \cup \bigcup_{j=1}^n E_j,
\]
where \( E^i = \{v^{(i)}_j v^{(i)}_k : u_j u_k \in E(G)\} \) and \( E_j = \{v^{(i)}_j v^{(k)}_j : w_j w_k \in E(H)\} \).
We define subgraphs $G_i$ of $G$ as follows: $G_i = (V^i, E^i)$. Clearly, $G_i$ is isomorphic to $G$ for $1 \leq i \leq m$.

Clearly, if $G$ and $H$ are connected graphs, then $G \square H$ is connected, too. Moreover, $\Delta(G \square H) = \Delta(G) + \Delta(H)$ and $\text{diam}(G \square H) = \text{diam}(G) + \text{diam}(H)$.

The $k$-dimensional grid $G(n_1, \ldots, n_k)$, $n_i \in \mathbb{N}$, is the Cartesian product of paths $P_{n_1} \square P_{n_2} \square \cdots \square P_{n_k}$. The cylinder $C(n_1, n_2)$ is the Cartesian product $P_{n_1} \square C_{n_2}$, and the torus $T(n_1, n_2)$ is the Cartesian product of cycles $C_{n_1} \square C_{n_2}$.

We also need the following two lemmas.

**Lemma 7.** If $\alpha$ is an edge-coloring of a connected graph $G$ with colors $1, \ldots, t$ such that the edges incident to each vertex $v \in V(G)$ are colored by distinct and consecutive colors, and $\min_{e \in E(G)} \{\alpha(e)\} = 1$, $\max_{e \in E(G)} \{\alpha(e)\} = t$, then $\alpha$ is an interval $t$-coloring of $G$.

**Proof.** For the proof of the lemma, it suffices to show that all colors are used in the coloring $\alpha$ of $G$.

Let $u$ and $w$ be vertices such that $1 \in S(u, \alpha)$ and $t \in S(w, \alpha)$. Also, let $P = v_1, \ldots, v_k$, where $u = v_1$ and $v_k = w$ be a $u,w$-path in $G$. If $k = 1$, then $t \in S(u, \alpha)$ and all colors appear on edges incident to $u$. Assume that $k \geq 2$. The sets $S(v_i, \alpha)$ for $v_i \in V(P)$ are intervals, and for $2 \leq i \leq k$, intervals $S(v_{i-1}, \alpha)$ and $S(v_i, \alpha)$ share a color. Thus, the sets $S(v_1, \alpha), \ldots, S(v_k, \alpha)$ cover $[1, t]$. □

The next lemma was proved by Behzad and Mahmoodian in [5].

**Lemma 8.** If both $G$ and $H$ have at least 3 vertices, then the Cartesian product $G \square H$ is planar if and only if $G \square H = G(m, n)$ or $G \square H = C(m, n)$.

### 3. The Cartesian Product of Regular Graphs

Interval edge-colorings of Cartesian products of graphs were first investigated by Giaro and Kubale in [7], where they proved the following:

**Theorem 9.** If $G \in \mathcal{R}$, then $G \square P_m \in \mathcal{R}$ ($m \in \mathbb{N}$) and $G \square C_{2n} \in \mathcal{R}$ ($n \geq 2$).

It is well-known that $P_m \square C_{2n} \in \mathcal{R}$ and $W(P_m) = m - 1$, $W(C_{2n}) = n + 1$ for $m \in \mathbb{N}$ and $n \geq 2$. Later, Giaro and Kubale [9, 19] proved a more general result.

**Theorem 10.** If $G, H \in \mathcal{R}$, then $G \square H \in \mathcal{R}$. Moreover, $w(G \square H) \leq w(G) + w(H)$ and $W(G \square H) \geq W(G) + W(H)$.

Let us note that if $G \in \mathcal{R}$ and $H = P_m$ or $H = C_{2n}$, then, by Theorem 10, we obtain $w(G \square H) \leq w(G) + 2$ and $W(G \square P_m) \geq W(G) + m - 1$, $W(G \square C_{2n}) \geq W(G) + n + 1$. Now we improve the lower bound in Theorem 10 for $W(G \square P_m)$ and $W(G \square C_{2n})$ when $G$ is a regular graph and $G \in \mathcal{R}$. More precisely, we show that the following two theorems hold.
Theorem 11. If $G$ is an $r$-regular graph and $G \in \mathfrak{M}$, then $G \square P_m \in \mathfrak{M}$ ($m \in \mathbb{N}$) and $W(G \square P_m) \geq W(G) + W(P_m) + (m - 1)r$.

Proof. For the proof, we construct an edge-coloring of the graph $G \square P_m$ that satisfies the specified conditions.

Since $G \in \mathfrak{M}$, there exists an interval $W(G)$-coloring $\alpha$ of $G$. Now we define an edge-coloring $\beta$ of the subgraphs $G^1, \ldots, G^m$. For $1 \leq i \leq m$ and for every edge $v^{(i)}_j v^{(i)}_k \in E(G^i)$, let

$$\beta(v^{(i)}_j v^{(i)}_k) = \alpha(v_j v_k) + (i - 1)(r + 1).$$

It is easy to see that the color of each edge of the subgraph $G^i$ is obtained by shifting the color of the associated edge of $G$ by $(i - 1)(r + 1)$. Thus the set $S(v^{(i)}_j, \beta)$ is an interval for each vertex $v^{(i)}_j \in V(G^i)$, where $1 \leq i \leq m$, $1 \leq j \leq n$. Now we define an edge-coloring $\gamma$ of the graph $G \square P_m$. For every edge $e \in E(G \square P_m)$, let

$$\gamma(e) = \begin{cases} 
\beta(e), & \text{if } e \in E(G^i), \\
\max S(v^{(i)}_j, \beta) + 1, & \text{if } e = v^{(i)}_j v^{(i+1)} \in E_j,
\end{cases}$$

where $1 \leq i \leq m, 1 \leq j \leq n$.

Let us prove that $\gamma$ is an interval $(W(G) + W(P_m) + (m - 1)r)$-coloring of the graph $G \square P_m$ for $m \in \mathbb{N}$.

First we prove that the set $S(v^{(i)}_j, \gamma)$ is an interval for each vertex $v^{(i)}_j \in V(G \square P_m)$, where $1 \leq i \leq m, 1 \leq j \leq n$.

For each vertex $v^{(i)}_j \in V(G \square P_m)$, the set $S(v^{(i)}_j, \gamma)$ can be represented as a union of three sets, $S(v^{(i)}_j, \gamma) = A^{(i)}_j \cup B^{(i)}_j \cup C^{(i)}_j$, where $A^{(i)}_j$ corresponds to the edges of $i$-th layer, $B^{(i)}_j$ corresponds to the edges from the vertices of lower layer and $C^{(i)}_j$ corresponds to the edges from the vertices of higher layer. More specifically, for $1 \leq i \leq m, 1 \leq j \leq n$, define sets $A^{(i)}_j$, $B^{(i)}_j$ and $C^{(i)}_j$ as follows:

$$A^{(i)}_j = \{ \gamma(v^{(i)}_j u) : v^{(i)}_j u \in E^i \},$$

$$B^{(i)}_j = \begin{cases} 
\emptyset, & \text{if } i = 1, \\
\{ \gamma(v^{(i)}_j u) : v^{(i)}_j u \in E_j, u \in V^{i-1} \}, & \text{if } 2 \leq i \leq m,
\end{cases}$$

$$C^{(i)}_j = \begin{cases} 
\{ \gamma(v^{(i)}_j u) : v^{(i)}_j u \in E_j, u \in V^{i+1} \}, & \text{if } 1 \leq i \leq m - 1, \\
\emptyset, & \text{if } i = m.
\end{cases}$$

By the definition of $\gamma$, we have that for $1 \leq i \leq m, 1 \leq j \leq n,$
\[ A_j^{(i)} = \min S(v_j, \alpha) + (i-1)(r+1), \max S(v_j, \alpha) + (i-1)(r+1) \]

for \( 2 \leq i \leq m, 1 \leq j \leq n, \)

\[ B_j^{(i)} = \{ \max S(v_j, \alpha) + (i-2)(r+1) \}, \]

and for \( 1 \leq i \leq m-1, 1 \leq j \leq n, \)

\[ C_j^{(i)} = \{ \max S(v_j, \alpha) + (i-1)(r+1) \}. \]

By this and taking into account that \( \max S(v_j, \alpha) - \min S(v_j, \alpha) = r - 1 \) for \( 1 \leq j \leq n, \) we have that \( A_j^{(i)} \cup B_j^{(i)} \cup C_j^{(i)} \) is an interval for each vertex \( v_j^{(i)} \in V(G^i), \) where \( 1 \leq i \leq m, 1 \leq j \leq n. \)

Next we show that in the coloring \( \gamma \) all colors are used. Clearly, there exists an edge \( v_{j_0} v_{k_0}^{(1)} \in E(G^1) \) such that \( \gamma(v_{j_0} v_{k_0}^{(1)}) = 1, \) since in the coloring \( \alpha \) there exists an edge \( v_{j_0} v_{k_0} \) with \( \alpha(v_{j_0} v_{k_0}) = 1 \) and \( \gamma(v_{j_0} v_{k_0}^{(1)}) = \beta(v_{j_0} v_{k_0}) = \alpha(v_{j_0} v_{k_0}). \) Similarly, there exists an edge \( v_{j_1} v_{k_1}^{(m)} \in E(G^m) \) such that \( \gamma(v_{j_1} v_{k_1}^{(m)}) = W(G) + (m-1)(r+1) = W(G) + W(P_m) + (m-1)r, \) since in the coloring \( \alpha \) there exists an edge \( v_{j_1} v_{k_1} \) with \( \alpha(v_{j_1} v_{k_1}) = W(G) \) and \( \gamma(v_{j_1} v_{k_1}^{(m)}) = \beta(v_{j_1} v_{k_1}) = \alpha(v_{j_1} v_{k_1}) + (m-1)(r+1). \)

Now, by Lemma 7, we have that \( \gamma \) is an interval \((W(G) + W(P_m) + (m-1)r)\)-coloring of the graph \( G \Box P_m \) for \( m \in \mathbb{N}. \)

**Corollary 12.** If \( G \) is an \( r \)-regular graph and \( G \in \mathfrak{N}, \) then \( G \Box Q_n \in \mathfrak{N} \ (n \in \mathbb{N}) \)

and

\[ W(G \Box Q_n) \geq W(G) + \frac{n(n+2r+1)}{2}. \]

**Proof.** By Theorem 11 and using associativity of the Cartesian product, we get

\[ W(G \Box Q_n) = W(\cdots((G \Box K_2)\Box K_2)\Box \cdots \Box K_2) \geq W(G) + \frac{n(n+2r+1)}{2}. \]

**Theorem 13.** If \( G \) is an \( r \)-regular graph and \( G \in \mathfrak{N}, \) then \( G \Box C_{2n} \in \mathfrak{N} \ (n \geq 2) \)

and \( W(G \Box C_{2n}) \geq W(G) + W(C_{2n}) + nr. \)

**Proof.** For the proof, we construct an edge-coloring of the graph \( G \Box C_{2n} \) that satisfies the specified conditions.

Since \( G \in \mathfrak{N}, \) there exists an interval \( W(G) \)-coloring \( \alpha \) of \( G. \) Now we define an edge-coloring \( \beta \) of the subgraphs \( G^1, \ldots, G^{2n}. \)

For \( 1 \leq i \leq 2n \) and for every edge \( v_j^{(i)} v_k^{(i)} \in E(G^i), \) let
\[
\beta \left( v_j^{(i)} v_k^{(i)} \right) = \begin{cases} 
\alpha (v_j v_k), & \text{if } i = 1, \\
\alpha (v_j v_k) + (i - 1)(r + 1) + 1, & \text{if } 2 \leq i \leq n + 1, \\
\alpha (v_j v_k) + (2n + 1 - i)(r + 1), & \text{if } n + 2 \leq i \leq 2n.
\end{cases}
\]

It is easy to see that the color of each edge of the subgraph \( G' \) is obtained by shifting the color of the associated edge of \( G \) by \((i - 1)(r + 1) + 1\) for \(2 \leq i \leq n + 1\), and by \((2n - i + 1)(r + 1)\) for \(n + 2 \leq i \leq 2n\). Thus the set \( S \left( v_j^{(i)}, \beta \right) \) is an interval for each vertex \( v_j^{(i)} \in V(G') \), where \(1 \leq i \leq 2n\), \(1 \leq j \leq p\). Now we define an edge-coloring \( \gamma \) of the graph \( G \boxtimes C_{2n} \).

For every \( e \in E(G \boxtimes C_{2n}) \), let

\[
\gamma(e) = \begin{cases} 
\beta(e), & \text{if } e \in E(G'), \\
\max S \left( v_j^{(1)}, \beta \right) + 1, & \text{if } e = v_j^{(1)} v_k^{(2n)} \in E_j, \\
\max S \left( v_j^{(1)}, \beta \right) + 2, & \text{if } e = v_j^{(1)} v_k^{(2)} \in E_j, \\
\max S \left( v_j^{(i)}, \beta \right) + 1, & \text{if } e = v_j^{(i)} v_j^{(i+1)} \in E_j, 2 \leq i \leq n, \\
\max S \left( v_j^{(i)}, \beta \right) + 1, & \text{if } e = v_j^{(i-1)} v_j^{(i)} \in E_j, n + 2 \leq i \leq 2n,
\end{cases}
\]

where \(1 \leq i \leq 2n\), \(1 \leq j \leq p\).

Let us prove that \( \gamma \) is an interval \((W(G) + W(C_{2n}) + nr)\)-coloring of the graph \( G \boxtimes C_{2n} \) for \( n \geq 2 \).

First we prove that the set \( S \left( v_j^{(i)}, \gamma \right) \) is an interval for each vertex \( v_j^{(i)} \in V(G \boxtimes C_{2n}) \), where \(1 \leq i \leq 2n\), \(1 \leq j \leq p\).

**Case 1.** \( i = 1, 1 \leq j \leq p \). By the definition of \( \gamma \) and taking into account that \( \max S(v_j, \alpha) - \min S(v_j, \alpha) = r - 1 \) for \(1 \leq j \leq p\), we have

\[
S \left( v_j^{(1)}, \gamma \right) = \{ \min S(v_j, \alpha), \ldots, \max S(v_j, \alpha) \} \cup \{ \max S(v_j, \alpha) + 2 \}
\]

\[
= \{ \min S(v_j, \alpha), \max S(v_j, \alpha) + 1 \} = [\min S(v_j, \alpha), \max S(v_j, \alpha) + 2].
\]

**Case 2.** \( 2 \leq i \leq n, 1 \leq j \leq p \). By the definition of \( \gamma \) and taking into account that \( \max S(v_j, \alpha) - \min S(v_j, \alpha) = r - 1 \) for \(1 \leq j \leq p\), we have

\[
S \left( v_j^{(i)}, \gamma \right) = \{ \min S(v_j, \alpha) + (i - 1)(r + 1) + 1, \ldots, \max S(v_j, \alpha) + (i - 1)(r + 1) + 2 \}
\]

\[
= [\min S(v_j, \alpha) + (i - 1)(r + 1), \max S(v_j, \alpha) + (i - 1)(r + 1) + 2].
\]

**Case 3.** \( i = n + 1, 1 \leq j \leq p \). By the definition of \( \gamma \) and taking into account that \( \max S(v_j, \alpha) - \min S(v_j, \alpha) = r - 1 \) for \(1 \leq j \leq p\), we have
Next we show that in the coloring and Kubale in [7], where they proved the following:

Interval edge-colorings of grids, cylinders and tori were first considered by Giaro and Kubale in [7], where they proved the following:

From Theorems 5 and 13, we have:

Next we show that in the coloring \( \gamma \) all colors are used. Clearly, there exists an edge \( v_j^{(1)}v_k^{(1)} \in E(G) \) such that \( \gamma(v_j^{(1)}v_k^{(1)}) = 1 \), since in the coloring \( \alpha \) there exists an edge \( v_jv_k \) with \( \alpha(v_jv_k) = 1 \) and \( \gamma(v_j^{(1)}v_k^{(1)}) = \beta(v_j^{(1)}v_k^{(1)}) = \alpha(v_jv_k) \). Similarly, there exists an edge \( v_j^{(n+1)}v_k^{(n+1)} \in E(G^{n+1}) \) such that \( \gamma(v_j^{(n+1)}v_k^{(n+1)}) = W(G) + n(r + 1) + 1 = W(G) + W(C_{2n}) + nr \), since in the coloring \( \alpha \) there exists an edge \( v_jv_k \) with \( \alpha(v_jv_k) = W(G) \) and \( \gamma(v_j^{(n+1)}v_k^{(n+1)}) = \beta(v_j^{(n+1)}v_k^{(n+1)}) = \alpha(v_jv_k) + n(r + 1) + 1 \).

Now, by Lemma 7, we have that \( \gamma \) is an interval \( (W(G) + W(C_{2n}) + nr) \)-coloring of the graph \( G \square C_{2n} \) for \( n \geq 2 \).

From Theorems 5 and 13, we have:

**Corollary 14.** If \( n = p2^q \), where \( p \) is odd and \( q \) is nonnegative, then

\[
W(K_{2n} \square C_{2n}) \geq 2n^2 + 4n - 1 - p - q.
\]

Note that the lower bound in Corollary 14 is close to the upper bound for \( W(K_{2n} \square C_{2n}) \), since \( \Delta(K_{2n} \square C_{2n}) = 2n + 1 \) and \( \text{diam}(K_{2n} \square C_{2n}) = n + 1 \), by Theorem 2, we have \( W(K_{2n} \square C_{2n}) \leq 2n^2 + 4n + 1 \).

4. **Grids, Cylinders and Tori**

Interval edge-colorings of grids, cylinders and tori were first considered by Giaro and Kubale in [7], where they proved the following:
Theorem 15. If \( G = G(n_1, \ldots, n_k) \) or \( G = C(m, 2n) \), \( m \in \mathbb{N}, n \geq 2 \), or \( G = T(2m, 2n) \), \( m, n \geq 2 \), then \( G \in \mathfrak{N} \) and \( w(G) = \Delta(G) \).

For the greatest possible number of colors in interval colorings of grid graphs, the first author and Karapetyan [20] proved the following theorems:

Theorem 16. For any \( m \in \mathbb{N}, n \geq 2 \), we have \( W(C(m, 2n)) \geq 3m + n - 2 \).

Theorem 17. For any \( m, n \geq 2 \), we have \( W(T(2m, 2n)) \geq \max\{3m+n, 3n+m\} \).

First we consider grids. It is easy to see that \( W(G(2, n)) = 2n - 1 \) for any \( n \in \mathbb{N} \).

Now we provide a lower bound for \( W(G(m, n)) \) when \( m, n \geq 2 \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{interval_14 Coloring.png}
\caption{Interval 14-coloring of the graph \( G(4, 6) \).}
\end{figure}

Theorem 18. For any \( m, n \geq 2 \), we have \( W(G(m, n)) \geq 2(m + n - 3) \).

\textbf{Proof.} For the proof, we are going to construct an edge-coloring of the graph \( G(m, n) \) that satisfies the specified conditions.

Define an edge-coloring \( \alpha \) of \( G(m, n) \) as follows:

1. for \( i = 1, \ldots, m - 1, j = 1, \ldots, n - 1 \), let
\[
\alpha (v_j^{(i)}, v_j^{(i+1)}) = 2(i + j) - 3;
\]
2. for \( i = 1, \ldots, m - 1 \), let
\[
\alpha (v_i^{(i)}, v_i^{(i+1)}) = 2(n + i) - 5;
\]
3. for \( j = 1, \ldots, n - 1 \), let
\[ \alpha(v_j^{(1)}v_{j+1}^{(1)}) = 2j; \]

(4) for \( i = 2, \ldots, m, j = 1, \ldots, n - 1 \), let

\[ \alpha(v_j^{(i)}v_{j+1}^{(i)}) = 2(i + j) - 4. \]

It is easy to see that \( \alpha \) is an interval \((2(m + n - 3))-coloring\) of \( G(m, n) \) when \( m, n \geq 2 \).

Figure 1 shows the interval 14-coloring \( \alpha \) of the graph \( G(4, 6) \) described in the proof of Theorem 18.

Note that the lower bound in Theorem 18 is not far from the upper bound for \( W(G(m, n)) \), since \( G(m, n) \) is bipartite, \( 2 \leq \Delta(G(m, n)) \leq 4 \) and \( \text{diam}(G(m, n)) = m + n - 2 \), by Theorem 3, we have \( W(G(m, n)) \leq 3(m + n - 2) + 1 \).

From Theorems 10 and 18, we have:

**Corollary 19.** If \( n_1 \geq \cdots \geq n_{2k} \geq 2 \) \((k \in \mathbb{N})\), then

\[ W(G(n_1, \ldots, n_{2k})) \geq 2 \sum_{i=1}^{2k} n_i - 6k, \]

and if \( n_1 \geq \cdots \geq n_{2k+1} \geq 2 \) \((k \in \mathbb{N})\), then

\[ W(G(n_1, \ldots, n_{2k+1})) \geq 2 \sum_{i=1}^{2k} n_i + n_{2k+1} - 6k - 1. \]

Next we consider cylinders. In [18], Khchoyan proved the following:

**Theorem 20.** For any \( n \geq 3 \), we have

1. \( C(2, n) \in \mathcal{G} \),
2. \( w(C(2, n)) = 3 \),
3. \( W(C(2, n)) = n + 2 \),
4. if \( w(C(2, n)) \leq t \leq W(C(2, n)) \), then \( C(2, n) \) has an interval \( t \)-coloring.

Now we prove some general results on cylinders.

**Theorem 21.** For any \( m \geq 3, n \in \mathbb{N} \), we have \( C(m, 2n + 1) \in \mathcal{G} \) and

\[ w(C(m, 2n + 1)) = \begin{cases} 4, & \text{if } m \text{ is even,} \\ 6, & \text{if } m \text{ is odd.} \end{cases} \]

**Proof.** First we show that if \( m \) is even, then \( C(m, 2n + 1) \) has an interval 4-coloring. For \( 1 \leq i \leq \frac{m}{2} \), define a subgraph \( C^i \) of the graph \( C(m, 2n + 1) \) as follows:

\[ C^i = \left( V^{2i-1} \cup V^{2i}, E^{2i-1} \cup E^{2i} \cup \{v_j^{(2i-1)}v_j^{(2i)} : 1 \leq j \leq 2n + 1 \} \right). \]
Clearly, \( C^i \) is isomorphic to \( C(2, 2n + 1) \) for \( 1 \leq i \leq \frac{m}{2} \). By Theorem 20, \( C(2, 2n + 1) \in \mathfrak{N} \) and there exists an interval 3-coloring \( \alpha \) of \( C(2, 2n + 1) \). Now we define an edge-coloring \( \beta \) of \( C(m, 2n + 1) \). First we color the edges of \( C^i \) according to \( \alpha \) for \( 1 \leq i \leq \frac{m}{2} \). Then we color the edges \( v_j^{(2i)}v_j^{(2n+1)} \in E_j \) with color 4 for \( 1 \leq i \leq \frac{m}{2} - 1, 1 \leq j \leq 2n + 1 \). It is easy to see that \( \beta \) is an interval 4-coloring of \( C(m, 2n + 1) \). This shows that \( C(m, 2n + 1) \in \mathfrak{N} \) and \( w(C(m, 2n + 1)) \leq 4 \). On the other hand, \( w(C(m, 2n + 1)) \geq \Delta(C(m, 2n + 1)) = 4 \); thus \( w(C(m, 2n + 1)) = 4 \) for even \( m \).

Now assume that \( m \) is odd. First we show that \( C(3, 2n + 1) \) has an interval 6-coloring. Define an edge-coloring \( \gamma \) of \( C(3, 2n + 1) \) as follows:

1. \( \gamma(v_1^{(1)}v_1^{(2)}) = 6 \) and for \( j = 2, \ldots, 2 \left\lfloor \frac{n+1}{2} \right\rfloor \), let \( \gamma(v_j^{(1)}v_j^{(2)}) = 4 \);
2. \( \gamma(v_2^{(1)}v_2^{(2)}) = 2 \) and for \( j = 2 \left\lfloor \frac{n+1}{2} \right\rfloor + 2, \ldots, 2n + 1 \), let \( \gamma(v_j^{(1)}v_j^{(2)}) = 3 \);
3. \( \gamma(v_1^{(2)}v_1^{(3)}) = 3 \) and for \( j = 2, \ldots, 2 \left\lfloor \frac{n+1}{2} \right\rfloor \), let \( \gamma(v_j^{(2)}v_j^{(3)}) = 2 \);
4. for \( j = 2 \left\lfloor \frac{n+1}{2} \right\rfloor + 1, \ldots, 2n + 1 \), let \( \gamma(v_j^{(2)}v_j^{(3)}) = 1 \);
5. \( j = 1, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor \), let
   \[ \gamma(v_{2j-1}^{(1)}v_{2j}^{(1)}) = \gamma(v_{2j-1}^{(2)}v_{2j}^{(2)}) = 5 \] and \( \gamma(v_{2j}^{(1)}v_{2j+1}^{(1)}) = \gamma(v_{2j}^{(2)}v_{2j+1}^{(2)}) = 3 \);
6. for \( j = \left\lfloor \frac{n+1}{2} \right\rfloor + 1, \ldots, n \), let
   \[ \gamma(v_{2j-1}^{(1)}v_{2j}^{(1)}) = \gamma(v_{2j-1}^{(2)}v_{2j}^{(2)}) = 4 \] and \( \gamma(v_1^{(1)}v_{2n+1}^{(1)}) = \gamma(v_1^{(2)}v_{2n+1}^{(2)}) = 4 \);
7. for \( j = \left\lfloor \frac{n+1}{2} \right\rfloor + 1, \ldots, n \), let \( \gamma(v_{2j}^{(1)}v_{2j+1}^{(1)}) = \gamma(v_{2j}^{(2)}v_{2j+1}^{(2)}) = 2 \);
8. for \( j = 1, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor \), let \( \gamma(v_{2j-1}^{(3)}v_{2j}^{(3)}) = 1 \) and \( \gamma(v_{2j}^{(3)}v_{2j+1}^{(3)}) = 3 \);
9. for \( j = \left\lfloor \frac{n+1}{2} \right\rfloor + 1, \ldots, n \), let \( \gamma(v_{2j-1}^{(3)}v_{2j}^{(3)}) = 2 \) and \( \gamma(v_1^{(3)}v_{2n+1}^{(3)}) = 2 \);
10. for \( j = \left\lfloor \frac{n+1}{2} \right\rfloor + 1, \ldots, n \), let \( \gamma(v_{2j}^{(3)}v_{2j+1}^{(3)}) = 3 \).
It is not difficult to see that $\gamma$ is an interval 6-coloring of $C(3, 2n + 1)$ for which $S(v_j^{(3)}, \gamma) = [1, 3]$ when $1 \leq j \leq 2n + 1$.

Next we define an edge-coloring $\phi$ of $C(m, 2n + 1)$ as follows: first we color the edges of the subgraph $C(3, 2n + 1)$ of $C(m, 2n + 1)$ according to $\gamma$. Secondly, we color the edges of the remaining subgraph $C(m - 3, 2n + 1)$ of $C(m, 2n + 1)$ according to $\beta$, and finally, we color the edges $v_j^{(3)}v_j^{(4)} \in E_j$ with color 4 for $1 \leq j \leq 2n + 1$. It is easy to see that $\phi$ is an interval 6-coloring of $C(m, 2n + 1)$.

This shows that $C(m, 2n + 1) \in \mathcal{N}$ and $w(C(m, 2n + 1)) \leq 6$.

Now we prove that $w(C(m, 2n + 1)) \geq 6$ for odd $m$. Let $\psi$ be an interval $w(C(m, 2n + 1))$-coloring of $C(m, 2n + 1)$ and $w(C(m, 2n + 1)) \leq 5$. Consider the set $S(v_j^{(i)}, \psi)$ for $1 \leq i \leq m, 1 \leq j \leq 2n + 1$. It is easy to see that if $d(v_j^{(i)}) = 3$, then $1 \leq \min S(v_j^{(i)}, \psi) \leq 3$, and if $d(v_j^{(i)}) = 4$, then $1 \leq \min S(v_j^{(i)}, \psi) \leq 2$.

Hence, $3 \in S(v_j^{(i)}, \psi)$ for $1 \leq i \leq m, 1 \leq j \leq 2n + 1$, but this implies that the edges with color 3 form a perfect matching in $C(m, 2n + 1)$, which contradicts the fact that $C(m, 2n + 1)$ does not have one. Thus $w(C(m, 2n + 1)) = 6$ for odd $m$.

![Figure 2. Interval 6-coloring of the graph C(3, 7).](image)

Figure 2 shows the interval 6-coloring $\gamma$ of the graph $C(3, 7)$ described in the proof of Theorem 21.

Before we derive lower bounds for $W(C(2m, 2n))$ and $W(C(2m, 2n + 1))$, let us note that Lemma 8, Theorems 15 and 21 imply the following:

**Corollary 22.** If $G \square H$ is planar and both factors have at least 3 vertices, then $G \square H \in \mathcal{N}$ and $w(G \square H) \leq 6$.

**Theorem 23.** If $m \in \mathbb{N}, n \geq 2$, then $W(C(2m, 2n)) \geq 4m + 2n - 2$, and if $m, n \in \mathbb{N}$, then $W(C(2m, 2n + 1)) \geq 4m + 2n - 1$.

**Proof.** For the proof of the theorem, it suffices to construct edge-colorings that satisfies the specified conditions. First we construct an interval $(4m + 2n - 2)$-coloring of $C(2m, 2n)$ when $m \in \mathbb{N}, n \geq 2$. 
Define an edge-coloring $\alpha$ of $C(2m, 2n)$ as follows:

(1) for $i = 1,\ldots, m$, $j = 1,\ldots, n$, let
\[
\alpha \left( v_j^{(2i-1)} v_j^{(2i-1)} \right) = \alpha \left( v_j^{(2i)} v_j^{(2i)} \right) = 4i + 2j - 4;
\]

(2) for $i = 1,\ldots, m$, $j = n + 1,\ldots, 2n - 1$, let
\[
\alpha \left( v_j^{(2i-1)} v_j^{(2i-1)} \right) = \alpha \left( v_j^{(2i)} v_j^{(2i)} \right) = 4i - 2j + 4n - 1;
\]

(3) for $i = 1,\ldots, m$, let
\[
\alpha \left( v_1^{(2i-1)} v_{2n}^{(2i-1)} \right) = \alpha \left( v_1^{(2i)} v_{2n}^{(2i)} \right) = 4i - 1;
\]

(4) for $i = 1,\ldots, m$, $j = 1,\ldots, n$, let
\[
\alpha \left( v_j^{(2i-1)} v_j^{(2i)} \right) = 4i + 2j - 5;
\]

(5) for $i = 1,\ldots, m$, $j = n + 1,\ldots, 2n$, let
\[
\alpha \left( v_j^{(2i-1)} v_j^{(2i)} \right) = 4i - 2j + 4n;
\]

(6) for $i = 1,\ldots, m - 1$, $j = 2,\ldots, n + 1$, let
\[
\alpha \left( v_j^{(2i)} v_j^{(2i+1)} \right) = 4i + 2j - 3;
\]

(7) for $i = 1,\ldots, m - 1$, $j = n + 2,\ldots, 2n$, let
\[
\alpha \left( v_j^{(2i)} v_j^{(2i+1)} \right) = 4i - 2j + 4n + 2;
\]

(8) for $i = 1,\ldots, m - 1$, let
\[
\alpha \left( v_1^{(2i)} v_1^{(2i+1)} \right) = 4i.
\]

Next we construct an interval $(4m + 2n - 1)$-coloring of $C(2m, 2n + 1)$ when $m, n \in \mathbb{N}$. Define an edge-coloring $\beta$ of $C(2m, 2n + 1)$ as follows:

(1) for $i = 1,\ldots, m$, $j = 1,\ldots, n + 1$, let
\[
\beta \left( u_j^{(2i-1)} u_j^{(2i-1)} \right) = \beta \left( u_j^{(2i)} u_j^{(2i)} \right) = 4i + 2j - 4;
\]

(2) for $i = 1,\ldots, m$, $j = n + 2,\ldots, 2n$, let
\[ \beta(u_j^{(2i-1)} u_{j+1}^{(2i-1)}) = \beta(u_j^{(2i)} u_{j+1}^{(2i)}) = 4i - 2j + 4n + 1; \]  
(3) for \( i = 1, \ldots, m, \) let
\[ \beta(u_1^{(2i-1)} u_{2n+1}^{(2i-1)}) = \beta(u_1^{(2i)} u_{2n+1}^{(2i)}) = 4i - 1; \]

\[ \beta(u_1^{(2i-1)} u_j^{(2i)}) = 4i + 2j - 5; \]

(4) for \( i = 1, \ldots, m, j = 1, \ldots, n + 2, \) let
\[ \beta(u_1^{(2i-1)} u_j^{(2i)}) = 4i - 2j + 4n + 2; \]

\[ \beta(u_j^{(2i)} u_j^{(2i+1)}) = 4i + 2j - 3; \]

(5) for \( i = 1, \ldots, m, j = n + 3, \ldots, 2n + 1, \) let
\[ \beta(u_j^{(2i)} u_j^{(2i+1)}) = 4i - 2j + 4n + 4; \]

\[ \beta(u_1^{(2i)} u_1^{(2i+1)}) = 4i. \]

It is straightforward to check that \( \alpha \) is an interval \((4m + 2n - 2)\)-coloring of \( C(2m, 2n) \) when \( m \in \mathbb{N}, n \geq 2, \) and \( \beta \) is an interval \((4m + 2n - 1)\)-coloring of \( C(2m, 2n + 1) \) when \( m, n \in \mathbb{N}. \)

Note that the lower bound in Theorem 23 is not so far from the upper bound for \( W(C(m, n)). \) Indeed, since \( C(2m, 2n) \) is bipartite, \( 3 \leq \Delta(C(2m, 2n)) \leq 4 \) and \( \text{diam}(C(2m, 2n)) = 2m + n - 1, \) by Theorem 3, we have \( W(C(2m, 2n)) \leq 3(2m + n - 1) + 1. \) Similarly, since \( 3 \leq \Delta(C(2m, 2n + 1)) \leq 4 \) and \( \text{diam}(C(2m, 2n + 1)) = 2m + n - 1, \) by Theorem 2, we have \( W(C(2m, 2n + 1)) \leq 3(2m + n) + 1. \) Next we would like to compare obtained lower bounds for \( W(C(m, n)). \) If \( m \) is even and \( m < n, \) then the lower bound in Theorem 23 is better than in Theorem 16, if \( m \) is even and \( m > n, \) then the lower bound in Theorem 16 is better than in Theorem 23, and if \( m \) is even and \( m = n, \) then we obtain the same lower bound in both theorems.

In the following we consider tori. In [22], the first author proved that the torus \( T(m, n) \in \mathcal{K} \) if and only if \( mn \) is even. Since \( T(m, n) \) is 4-regular, by Theorem 1, we obtain that \( w(T(m, n)) = 4 \) when \( mn \) is even. Now we derive a new lower bound for \( W(T(m, n)) \) when \( mn \) is even.
Theorem 24. For any \( m, n \geq 2 \), we have \( W(T(2m, 2n)) \geq \max\{3m+n+2, 3n+m+2\} \), and for any \( m \geq 2, n \in \mathbb{N} \), we have

\[
W(T(2m, 2n+1)) \geq \begin{cases} 
2m + 2n + 2, & \text{if } m \text{ is odd}, \\
2m + 2n + 3, & \text{if } m \text{ is even}.
\end{cases}
\]

Proof. First note that the lower bound for \( W(T(2m, 2n)) \) \((m, n \geq 2)\) follows from Theorem 13. For the proof of a second part of the theorem, it suffices to construct an edge-coloring of \( T(2m, 2n+1) \) that satisfies the specified conditions.

Figure 3. Interval 13-coloring of the graph \( T(4, 7) \).

Define an edge-coloring \( \alpha \) of \( T(2m, 2n+1) \) as follows:

1. for \( j = 1, \ldots, n+1 \), let

\[
\alpha(v_j^{(1)}v_{j+1}^{(1)}) = \alpha(v_j^{(2m)}v_{j+1}^{(2m)}) = 2j;
\]

2. for \( j = n+2, \ldots, 2n \), let

\[
\alpha(v_j^{(1)}v_{j+1}^{(1)}) = \alpha(v_j^{(2m)}v_{j+1}^{(2m)}) = 2(2n+1-j) + 3
\]

and

\[
\alpha(v_1^{(1)}v_{2n+1}^{(1)}) = \alpha(v_1^{(2m)}v_{2n+1}^{(2m)}) = 3;
\]

3. for \( j = 1, \ldots, n+2 \), let

\[
\alpha(v_j^{(1)}v_j^{(2m)}) = 2j - 1;
\]

4. for \( j = n+3, \ldots, 2n+1 \), let
\[
\alpha \left( v_j^{(1)} v_j^{(2m)} \right) = 2(2n + 3 - j);
\]

(5) for \( i = 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor, j = 1, \ldots, n + 1, \) let

\[
\alpha \left( v_j^{(2i)} v_j^{(2i)} \right) = \alpha \left( v_j^{(2i+1)} v_j^{(2i+1)} \right) = \alpha \left( v_j^{(2m-2i)} v_j^{(2m-2i)} \right) = 4i + 2j;
\]

(6) for \( i = 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor, j = n + 2, \ldots, 2n, \) let

\[
\alpha \left( v_j^{(2i)} v_j^{(2i)} \right) = \alpha \left( v_j^{(2i+1)} v_j^{(2i+1)} \right) = \alpha \left( v_j^{(2m-2i)} v_j^{(2m-2i)} \right) = 4i + 2(2n + 1 - j) + 3
\]

and

\[
\alpha \left( v_1^{(2i)} v_1^{(2i)} \right) = \alpha \left( v_1^{(2i+1)} v_1^{(2i+1)} \right) = \alpha \left( v_1^{(2m-2i)} v_1^{(2m-2i)} \right) = 4i + 3;
\]

(7) for \( i = 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor, j = 2, \ldots, n + 1, \) let

\[
\alpha \left( v_j^{(2i-1)} v_j^{(2i)} \right) = \alpha \left( v_j^{(2m-2i+1)} v_j^{(2m-2i+2)} \right) = 4i + 2j - 3;
\]

(8) for \( i = 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor, j = n + 2, \ldots, 2n + 1, \) let

\[
\alpha \left( v_j^{(2i-1)} v_j^{(2i)} \right) = \alpha \left( v_j^{(2m-2i+1)} v_j^{(2m-2i+2)} \right) = 4(n + 1 + i) - 2j;
\]

(9) for \( i = 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor, \) let

\[
\alpha \left( v_1^{(2i-1)} v_1^{(2i)} \right) = \alpha \left( v_1^{(2m-2i+1)} v_1^{(2m-2i+2)} \right) = 4i;
\]

(10) for \( i = 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor, j = 1, \ldots, n + 2, \) let

\[
\alpha \left( v_j^{(2i)} v_j^{(2i+1)} \right) = \alpha \left( v_j^{(2m-2i)} v_j^{(2m-2i+1)} \right) = 4i + 2j - 1;
\]

(11) for \( i = 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor, j = n + 3, \ldots, 2n + 1, \) let

\[
\alpha \left( v_j^{(2i)} v_j^{(2i+1)} \right) = \alpha \left( v_j^{(2m-2i)} v_j^{(2m-2i+1)} \right) = 4i + 2(2n + 3 - j).\]
Figure 4. Interval 14-coloring of the graph $T(6, 7)$.

Let us show that the edges incident to any vertex of $T(2m, 2n + 1)$ are colored by four consecutive colors. For example, let $2 \leq i \leq \lfloor \frac{m}{2} \rfloor$ and $2 \leq j \leq n + 1$. By the points (5), (7) and (10) of the definition of $\alpha$, for $2 \leq i \leq \lfloor \frac{m}{2} \rfloor$, $2 \leq j \leq n + 1$, we have

$$S(v_j^{(2i)}, \alpha) = S(v_j^{(2m-2i)}, \alpha) = \{4i + 2j - 2, 4i + 2j\}$$

$$\cup \{4i + 2j - 3\} \cup \{4i + 2j - 1\} = [4i + 2j - 3, 4i + 2j].$$

Similarly, it can be verified that the edges incident to other vertices of $T(2m, 2n + 1)$ are also colored by four consecutive colors. It is easy to see that $\alpha(v_1^{(1)} v_1^{(2m)}) = 1$. Now if $m$ is odd, then $\alpha(v_{n+2}^{(m)} v_{n+2}^{(m+1)}) = 2m + 2n + 2$ and, by Lemma 7, $\alpha$ is an interval $(2m + 2n + 2)$-coloring of $T(2m, 2n + 1)$ when $m$ is odd. If $m$ is even, then $\alpha(v_{n+2}^{(m)} v_{n+2}^{(m+1)}) = 2m + 2n + 3$ and, by Lemma 7, $\alpha$ is an interval $(2m + 2n + 3)$-coloring of $T(2m, 2n + 1)$ when $m$ is even.

Figure 3 and 4 show the interval colorings of the graphs $T(4, 7)$ and $T(6, 7)$ described in the proof of Theorem 24.

From Theorems 1, 15 and 24, we have:

**Corollary 25.** If $G = T(2m, 2n)$ ($m, n \geq 2$) and $4 \leq t \leq \max\{3m + n + 2, 3n + m + 2\}$, then $G$ has an interval $t$-coloring. Also, if $H = T(2m, 2n + 1)$ ($m \geq 2$) and $4 \leq t \leq \max\{3m + 2n + 1, 3n + m + 2\}$, then $H$ has an interval $t$-coloring.
2, n ∈ N), m is odd and 4 ≤ t ≤ 2m + 2n + 2, then H has an interval t-coloring, and if H = T(2m, 2n + 1) (m ≥ 2, n ∈ N), m is even and 4 ≤ t ≤ 2m + 2n + 3, then H has an interval t-coloring.

Let us note that the lower bound in Theorem 24 is not so far from the upper bound for W(T(m, n)). Indeed, since T(2m, 2n) is bipartite, ∆(T(2m, 2n)) = 4 and diam(C(2m, 2n)) = m + n, by Theorem 3, we have W(T(2m, 2n)) ≤ 3(m + n) + 1. Similarly, since ∆(T(2m, 2n + 1)) = 4 and diam(T(2m, 2n + 1)) = m + n, by Theorem 2, we have W(T(2m, 2n + 1)) ≤ 3(m + n + 1) + 1.

5. n-DIMENSIONAL CUBES

It is well-known that the n-dimensional cube Q_n is the Cartesian product of n copies of K_2. In [21], the first author investigated interval colorings of n-dimensional cubes and proved that w(Q_n) = n and W(Q_n) ≥ n(n + 1) for any n ∈ N. In the same paper he also conjectured that W(Q_n) = n(n + 1) for any n ∈ N. Here, we prove this conjecture.

Let e, e′ ∈ E(Q_n) and e = u_1u_2, e′ = v_1v_2. The distance between two edges e and e′ in Q_n, we define as follows:

\[ d(e, e′) = \min_{1 \leq i < 2, 1 \leq j < 2} \{d(u_i, v_j)\}. \]

Let α be an interval t-coloring of Q_n. Define an edge span sp_α(e, e′) of edges e and e′ (e, e′ ∈ E(Q_n)) in coloring α as follows:

\[ sp_\alpha(e, e′) = |\alpha(e) - \alpha(e′)|. \]

For any k, 0 ≤ k ≤ n − 1, define an edge span at distance k sp_α,k in coloring α as follows:

\[ sp_\alpha,k = \max \{sp_\alpha(e, e′) : e, e′ ∈ E(Q_n) \text{ and } d(e, e′) = k\}. \]

Clearly, sp_α,0 = n − 1.

**Theorem 26.** If n ∈ N, then W(Q_n) ≤ \( \frac{n(n+1)}{2} \).

**Proof.** Let α be an interval W(Q_n)-coloring of Q_n. First we show that if 1 ≤ k ≤ n − 1, then sp_α,k ≤ sp_α,k−1 + n − k.

Let e, e′ ∈ E(Q_n) be any two edges of Q_n with d(e, e′) = k. Without loss of generality, we may assume that \( \alpha(e) ≥ \alpha(e′) \). Since d(e, e′) = k, there exist u and v vertices such that u ∈ e and v ∈ e′ and d(u, v) = k. There are v_1, v_2, . . . , v_k (v_i ≠ v_j when i ≠ j) vertices such that d(u, v_i) = k − 1 and vv_i ∈ E(Q_n) for i = 1, . . . , k. Since Q_n is n-regular, we have

\[ \text{(*) } \min_{1 \leq i \leq k} \{\alpha(v_i)\} ≤ \alpha(e′) + n - k. \]

Let \( \alpha(e′′) = \min_{1 \leq i \leq k}\{\alpha(v_i)\} \). By (⋆), we obtain
\[ \alpha(e') \geq \alpha(e'') - (n - k) \text{ and } d(e, e'') = k - 1. \]

Thus,
\[
sp_\alpha (e, e') = |\alpha(e) - \alpha(e')| \leq |\alpha(e) - \alpha(e'') + n - k| \leq |\alpha(e) - \alpha(e'')| + n - k \\
\leq sp_{\alpha,k-1} + n - k.
\]

Since \( e \) and \( e' \) were arbitrary edges with \( d(e, e') = k \), we obtain \( sp_{\alpha,k} \leq sp_{\alpha,k-1} + n - k \). Now by induction on \( k \) with \( sp_{\alpha,0} = n - 1 \), we obtain \( sp_{\alpha,n-1} \leq \frac{n(n+1)}{2} - 1 \). From this and taking into account that \( d(e, e') \leq n - 1 \) for all \( e, e' \in E(Q_n) \), we get \( W(Q_n) \leq \frac{n(n+1)}{2} \).

By Theorems 6 and 26, we obtain \( W(Q_n) = \frac{n(n+1)}{2} \) for any \( n \in \mathbb{N} \). Moreover, by Theorem 1, we have that \( Q_n \) has an interval \( t \)-coloring if and only if \( n \leq t \leq \frac{n(n+1)}{2} \).

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References


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