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DECOMPOSITIONS OF PLANE GRAPHS UNDER PARITY CONSTRAINS GIVEN BY FACES

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Abstract

An edge coloring of a plane graph G is *facially proper* if no two face-adjacent edges of G receive the same color. A *facial* (*facially proper*) *parity edge coloring* of a plane graph G is an (facially proper) edge coloring with the property that, for each color c and each face f of G, either an *odd* number of edges incident with f is colored with c, or color c does not occur on the edges of f. In this paper we deal with the following question: For which integers k does there exist a facial (facially proper) parity edge coloring of a plane graph G with exactly k colors?

Keywords: plane graph, parity partition, edge coloring.

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1. Introduction and Notations

The following definitions are taken from [2]. A *parity partition* of a set is a partition such that the parity of all partition classes is the same. A partition is *odd* (*even*) if every class has an odd (even) number of elements. Note that every set of even cardinality admits both odd and even partitions, but a set of odd cardinality only has odd ones. Let $\mathcal{F}$ be a set system over a finite underlying set X. Assume that $\mathcal{F}$ is the disjoint union of four set systems,

$$\mathcal{F} = \mathcal{O} \cup \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2,$$

where all sets in $\mathcal{O}$ have odd cardinality and all sets in $\mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2$ have even cardinality. We say that a mapping $\varphi : X \to \mathbb{N}$ is a *parity coloring* of $\mathcal{F}$ if it induces

- an even partition on each $F \in \mathcal{E}_0$,
- an odd partition on each $F \in \mathcal{E}_1 \cup \mathcal{O}$,
- a parity partition on each $F \in \mathcal{E}_2$.

If a set system $\mathcal{F}$ is given but the partition $\mathcal{O} \cup \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2$ is not specified, then we assume that all $F \in \mathcal{F}$ of even cardinality belong to $\mathcal{E}_2$. Another approach interesting in its own right is if one assumes that all $F \in \mathcal{F}$ of even cardinality belong to $\mathcal{E}_1$. In this case we use the term *odd coloring*. On the other hand, if all $F \in \mathcal{F}$ have even cardinalities, then it is also natural to consider *even colorings* that means $\mathcal{F} = \mathcal{E}_0$.

A set system $\mathcal{F}$ is *parity colorable* if it admits at least one parity coloring; and otherwise it is called *uncolorable*. Assuming that $\mathcal{F}$ is parity colorable, we denote by $\chi_p(\mathcal{F})$ and $\overline{\chi}_p(\mathcal{F})$ the minimum and maximum possible numbers of colors in a parity coloring of $\mathcal{F}$.

The *feasible set* $\Phi_p = \Phi_p(\mathcal{F})$ of $\mathcal{F}$ consists of those integers k for which $\mathcal{F}$ admits a parity coloring with exactly k colors. Hence, $\chi_p = \chi_p(\mathcal{F})$ and $\overline{\chi}_p = \overline{\chi}_p(\mathcal{F})$ are the smallest and the largest elements of Φ_p, respectively.

We say that the feasible set of $\mathcal{F}$ is

- *continuous* if it is an interval of integers, i.e. $\Phi_p = \{i \mid \chi_p \leq i \leq \overline{\chi}_p\}$;
- *ℓ-continuous* if $\{i \mid \ell \leq i \leq \overline{\chi}_p\} \subseteq \Phi_p$;
- *semi-continuous* if, for every $k \in \Phi_p$ with $k \leq \overline{\chi}_p - 2$, also $k + 2 \in \Phi_p$ holds.

The notions above are meaningful for *odd colorings*, too. In this case the notation χ_o, $\overline{\chi}_o$, Φ_o has a natural interpretation accordingly, where subscript refers to 'odd' instead of 'parity'. Similarly, if all $F \in \mathcal{F} = \mathcal{E}_0$ have even cardinality, the corresponding *even colorings* with χ_e, $\overline{\chi}_e$, Φ_e are of interest, too.

Given a sample graph H, in any graph G we may consider the subgraphs isomorphic to H, and view them (their vertex sets or edge sets) as the sets in a set system. Also more generally, instead of a fixed graph H, one can consider a fixed type of subgraphs; for example, the cycles of G, induced cycles of G, subpaths, or subtrees, etc. Some classes of graphs offer families of subgraphs in a very natural way, notably the facial cycles in plane graphs.

Let $G = (V, E, F)$ be a connected plane graph with vertex set V, edge set E, and face set F. For a face $f \in F$ let $E(f)$ denote the set of all edges incident with f. We say that two edges are *face-adjacent* if they are consecutive edges in the facial walk of some face of G. For a face $f \in F$ let $E_2(f)$ denote the set of all face-adjacent pairs of edges incident with f. Let us also introduce the further notation

$$\mathcal{F}_1 = \{E(f) \mid f \in F\}$$

for the set system consisting of all facial cycles of G, and

$$\mathcal{F}_2 = \{E(f) \mid f \in F\} \cup \{e_1e_2 \in E_2(f) \mid f \in F\}$$

for the set system consisting of all facial cycles and all pairs of face-adjacent edges of G.

In this paper we deal with the following questions:

Question 1. *For which plane graphs $G = (V, E, F)$ is the feasible set $\Phi_o(\mathcal{F}_1)$ continuous or at least semi-continuous?*

Question 2. *For which plane graphs $G = (V, E, F)$ is the feasible set $\Phi_o(\mathcal{F}_2)$ continuous or at least semi-continuous?*

In this paper we prove that $\Phi_o(\mathcal{F}_1)$ and $\Phi_o(\mathcal{F}_2)$ are semi-continuous for any 2-edge-connected plane graph G. Moreover, if G is 3-edge-connected, then $\Phi_o(\mathcal{F}_1)$ is 5-continuous and $\Phi_o(\mathcal{F}_2)$ is 12-continuous.

The partitions of the edge set in Question 1 (Question 2) correspond to facial (facially proper) parity edge colorings. In this terminology we say that an edge coloring of a plane graph G is *facially proper* if no two face-adjacent edges of G receive the same color. A *facial* (*facially proper*) *parity edge coloring* of a plane graph G is an (facially proper) edge coloring with the property that, for each color c and each face f of G, either an *odd* number of edges incident with f is colored with c, or color c does not occur on the edges of f.

2. Results

In this paper we consider only 2-edge-connected plane graphs since any non-2-edge-connected plane graph contains a face whose boundary is not a cycle.

2.1. Facial parity edge coloring

In this section we deal with Question 1. We show that the feasible set $\Phi_o(\mathcal{F}_1)$ of every 2-edge-connected plane graph is semi-continuous. Moreover, if G is 3-edge-connected, then $\Phi_o(\mathcal{F}_1)$ is 5-continuous.

Let φ be a facial parity edge coloring of a 2-edge-connected plane graph G. This coloring induces a coloring of the dual graph G^* in a natural way. Observe that in G^*, the edges in each color class induce a factor of G^* with the degrees of all the vertices either odd or zero, i.e. it is an *odd* subgraph.

We say that an edge coloring of a plane graph is *odd* if each color class induces an odd subgraph.

Observation 3. *Let G be a 2-edge-connected plane graph. Then G has a facial parity edge coloring with k colors if and only if the dual graph G^* has an odd edge coloring with k colors.*

2.1.1. 2-edge-connected plane graphs

Proposition 4. *There exist 2-edge-connected plane graphs for which the feasible set $\Phi_o(\mathcal{F}_1)$ is not continuous.*

Proof. For any cycle C_ℓ of length ℓ, a facial parity edge coloring with exactly k colors exists if and only if $1 \leq k \leq \ell$ and $k \equiv \ell \pmod 2$. ■

Proposition 5. *The feasible set $\Phi_o(\mathcal{F}_1)$ is semi-continuous for any 2-edge-connected plane graph.*

Proof. Let G be a 2-edge-connected plane graph and let G^* be its dual. Let φ be a facial parity edge coloring of G which uses k colors. Clearly, we can assume that $k \leq |E(G)| - 2$. This coloring induces a coloring of G^* in a natural way. From Observation 3 it follows that each color class induces an odd subgraph of G^*. Consider one such subgraph, say H, on at least three edges.

Let P be a longest path in H; denote its ends by x and y. If the length of P equals one, then H is a matching. In this case we recolor two edges of H with two new colors.

Suppose that P is incident with at least 3 vertices. If both vertices x and y have degree one in H, then we recolor the edges of this path alternately with two new colors. Otherwise, since P is a longest path in H, at least one end of P has at least three neighbors on P, therefore H contains a cycle of even length. We recolor all the edges of this cycle alternately with two new colors.

Now assume that each color class contains at most two (independent) edges. We select two classes, say H_1 and H_2, both of cardinality 2. They exist since $k \leq |E(G)| - 2$. We recolor two edges, one from H_1 and one from H_2, with two new colors.

In each case we obtain an odd edge coloring of the dual graph with $k+2$ colors. From Observation 3 it follows that G has a facial parity edge coloring with $k+2$ colors. ■

2.1.2. 3-edge-connected plane graphs

In 1991, Pyber [8] proved the following:

Theorem 6 [8]. *The edges of any simple graph can be colored with at most 4 colors so that each color class induces a graph with all vertices having odd degree. Moreover, if a graph has an even number of vertices, then 3 colors are sufficient.*

Mátrai [6] constructed an infinite sequence of finite simple graphs which require 4 colors in any such coloring.

Observe that if G is a 3-edge-connected plane graph, then its dual G^* is a simple plane graph. Hence, for this class of graphs, Pyber's result can be stated as follows:

Theorem 7. *Let G be a 3-edge-connected plane graph. Then the edges of G can be colored with at most 4 colors so that, for any color c and any face f of G, either no edge or an odd number of edges on the boundary of f is colored with color c.*

Corollary 8. *If G is a 3-edge-connected plane graph, then $3 \in \Phi_o(\mathcal{F}_1)$ or $4 \in \Phi_o(\mathcal{F}_1)$.*

Proof. The assertion immediately follows from Theorem 7 and Proposition 5. ■

Proposition 9. *There exist 3-edge-connected plane graphs for which $\Phi_o(\mathcal{F}_1)$ is not continuous.*

Proof. It is sufficient to consider a 3-edge-connected plane graph G which has an odd dual G^*. Clearly, $1 \in \Phi_o(\mathcal{F}_1)$, while $2 \notin \Phi_o(\mathcal{F}_1)$ because an odd coloring of G^* with at most two colors means monochromatic stars at each vertex, and then the number of colors is 1, due to connectivity. ■

Theorem 10. *If G is a 3-edge-connected plane graph, then $\Phi_o(\mathcal{F}_1)$ is 5-continuous.*

Proof. From Proposition 5 it follows that $\Phi_o(\mathcal{F}_1)$ is semi-continuous. So it is sufficient to show that $\Phi_o(\mathcal{F}_1)$ contains two consecutive integers from $\{2,3,4,5\}$.

Let G^* be the dual of G. The graph G^* is a simple plane graph since G is 3-edge-connected. Let e_1 and e_2 be two non-adjacent edges of G^*, and let $H = G^* \setminus \{e_1, e_2\}$ be the graph obtained from G^* by deleting the edges e_1 and e_2. Theorem 6 implies that H has an odd coloring with k colors for some $k \in \{1,2,3,4\}$. We can extend this coloring to odd colorings of G^* with $k+1$ and $k+2$ colors by assigning the edges e_1 and e_2 to one new common color or two new colors, respectively. From Observation 3 it follows that $k+1, k+2 \in \Phi_o(\mathcal{F}_1)$ for some $k \in \{1,2,3,4\}$. ■

2.2. Facially proper parity edge coloring

In this Section we deal with Question 2: For which plane graphs the feasible set $\Phi_o(\mathcal{F}_2)$ is continuous or at least semi-continuous, where the set system $\mathcal{F}_2$ consists of all facial cycles and all pairs of face-adjacent edges of G?

Recall that a facially proper parity edge coloring of a plane graph G is a facially proper edge coloring with the following property: for each color c and each face f of G either no edge or an odd number of edges incident with f is colored with color c.

Let φ be a facially proper parity edge coloring of a 2-edge-connected plane graph G. Then in the dual graph G^*, the edges in each color class induce an odd graph. Moreover, since φ is a facially proper edge coloring in G, it induces a facially proper edge coloring in G^* as well.

Observation 11. *Let G be a 2-edge-connected plane graph. Then G has a facially proper parity edge coloring with k colors if and only if the dual graph G^* has a facially proper odd edge coloring with k colors.*

Proposition 12. *The feasible set $\Phi_o(\mathcal{F}_2)$ is semi-continuous for any 2-edge-connected plane graph.*

Proof. We can use the same argument as in the proof of Proposition 5. ■

2.2.1. 3-edge-connected plane graphs

In 1965, Vizing [10] proved that planar simple graphs with maximum degree at least eight have chromatic index (edge chromatic number) equal to their maximum degree. He conjectured the same if the maximum degree is seven or six. For $\Delta = 7$, this conjecture was proved independently by Grünewald [4], Sanders and Zhao [9], and Zhang [11].

Note that, by Vizing's classic theorem, every graph with maximum degree Δ has chromatic index equal to Δ or $\Delta + 1$. These results can be reformulated in the following way:

Theorem 13. *Let G be a 3-edge-connected plane graph with maximum face size $\Delta^*(G)$. Then the edges of G can be colored with $\Delta^*(G) + 1$ colors in such a way that any two edges on the boundary of any face of G are colored distinctly. Moreover, if $\Delta^*(G) \geq 7$ also holds, then an edge coloring of this kind exists with $\Delta^*(G)$ colors.*

Corollary 14. *Let G be a 3-edge-connected plane graph and let $\Delta^*(G)$ be the maximum face size of G. Then the feasible set $\Phi_o(\mathcal{F}_2)$ is $(\Delta^*(G)+1)$-continuous, and also $\Delta^*(G)$-continuous if $\Delta^*(G) \geq 7$.*

Proof. It follows from Theorem 13 that the edges of G can be colored with $\Delta^*(G) + 1$ colors (and also with $\Delta^*(G)$ colors if $\Delta^*(G) \geq 7$) in such a way that the edges bounding every face of G are colored distinctly. Using this coloring of G we can find a facially proper parity edge coloring which uses k colors, for any k in the range $\Delta^*(G) < k \leq |E(G)|$, since if we recolor any edge with a new color, we again get a facially proper parity edge coloring of G. ■

Kotzig [5] proved that every simple planar graph with minimum degree at least 3 has an edge such that the sum of degrees of its two ends is small.

Theorem 15 [5]**.** *Let G be a simple planar graph with minimum degree at least 3. Then it contains an edge uv such that $\deg(u) + \deg(v) \leq 13$.*

Corollary 16. *Let G be a 3-edge-connected simple plane graph. Then it contains two adjacent faces f and h such that $\deg(f) + \deg(h) \leq 13$.*

Proof. The dual of G is a simple plane graph with minimum vertex degree at least 3, hence it has an edge uv such that $\deg(u) + \deg(v) \leq 13$. The ends u, v of this edge correspond to two adjacent faces which have the required property. ■

Nash-Williams [7] proved that each planar graph has an edge decomposition into at most three forests. Gonçalves [3] proved a similar theorem, replacing trees with outerplanar graphs.

Theorem 17 [3]**.** *Let $G = (V, E)$ be a simple planar graph. Then its edge set has a bipartition $E = A \cup B$ such that the graphs induced by these subsets, $G[A]$ and $G[B]$, are outerplanar.*

Recall that a (planar) graph is *outerplanar* if it can be embedded in the plane in such a way that all the vertices are on the boundary of the outer face. Note that for a given plane embedding of a planar graph G, the two outerplanar graphs given in Theorem 17 need not be outerplanarly embedded.

Lemma 18 [1]**.** *Any plane embedding of a simple outerplanar graph has a facially proper odd edge coloring which uses at most 6 colors.*

Lemma 19. *Let G be an arbitrary plane embedding of a simple outerplanar graph with m edges. If G has a facially proper odd edge coloring with k colors, $k \leq m-2$, then it also has such a coloring with $k+2$ colors.*

Proof. We can use similar arguments as in the proof of Proposition 5. ∎

Theorem 20. *Let G be a 3-edge-connected plane graph. Then the feasible set $\Phi_o(\mathcal{F}_2)$ is 12-continuous.*

Proof. Since $\Phi_o(\mathcal{F}_2)$ is semi-continuous, it is sufficient to show that $12, 13 \in \Phi_o(\mathcal{F}_2)$. To prove this, let e be an edge of G such that the incident faces f, h have size together at most 13 (see Corollary 16). Let G^* be the dual of G and let A, B be a bipartition of its edge set such that the graphs induced by these subsets, $G^*[A]$ and $G^*[B]$, are outerplanar. We can assume that the edge e^* of G^* corresponding to e in G belongs to A. The graphs $G^*[A] \setminus \{e^*\}$ and $G^*[B]$ are outerplanar, hence they have facially proper odd edge colorings using at most 6 colors (see Lemma 18). We can assume that these colorings use together ℓ colors for some $\ell \in \{11, 12\}$ (see Lemma 19). The coloring of $G^* \setminus \{e^*\}$ corresponds to a coloring of $G \setminus \{e\}$ in a natural way.

Case 1. Assume that $\ell = 12$. Note that $\deg((f \cup h) \setminus \{e\}) \leq 11$. If we color the edge e with a color which occurs in $G \setminus \{e\}$ but does not appear on the faces f and h, we obtain a facially proper parity edge coloring of G with 12 colors. If we color e with a (new) color not used in the 12-coloring of $G \setminus \{e\}$, we obtain a facially proper parity edge coloring of G with 13 colors.

Case 2. Let $\ell = 11$. If we color the edge e with a (new) color, say c, not used in the 11-coloring of $G \setminus \{e\}$, we obtain a facially proper parity edge coloring of G which uses 12 colors. So there is a facially proper parity edge 14-coloring of G which uses the color c exactly once (see Proposition 12 and its proof). Hence, if we recolor e with an existing color which does not appear on the faces f and h, we obtain a facially proper parity edge coloring of G which uses 13 colors. ∎

3. OPEN PROBLEMS

Let G be a 3-edge-connected plane graph. We know that $3 \in \Phi_o(\mathcal{F}_1)$ or $4 \in \Phi_o(\mathcal{F}_1)$ (see Corollary 8). On the other hand, we proved that the feasible set $\Phi_o(\mathcal{F}_1)$ is 5-continuous.

Problem 21. Is it true that $\Phi_o(\mathcal{F}_1)$ is 4-continuous for every 3-edge-connected plane graph?

Observe that if $1 \in \Phi_o(\mathcal{F}_1)$, then $2 \notin \Phi_o(\mathcal{F}_1)$. So we can ask the following.

Problem 22. Let G be a 3-edge-connected plane graph such that $\chi_o(\mathcal{F}_1) \neq 1$. Is it true that $\Phi_o(\mathcal{F}_1)$ is continuous?

Pyber [8] proved that $\chi_o(\mathcal{F}_1) \leq 4$ for any plane graph; moreover, Mátrai [6] proved that this bound is attained for infinitely many graphs. Czap *et al.* [1] showed that $\chi_o(\mathcal{F}_2) \leq 20$ for 2-edge-connected plane graphs and $\chi_o(\mathcal{F}_2) \leq 12$ for 3-edge-connected plane graphs.

Problem 23. Determine the best upper bound on $\chi_o(\mathcal{F}_2)$ for the class of 2-edge-connected (3-edge-connected) plane graphs.

Problem 24. Let G be a 3-edge-connected plane graph. Is it true that $\Phi_o(\mathcal{F}_2)$ is continuous?

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