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FRACTIONAL $\mathcal{Q}$-EDGE-COLORING OF GRAPHS

Július Czap

Department of Applied Mathematics and Business Informatics,
Faculty of Economics, Technical University of Košice,
Němcovej 32, SK-040 01 Košice, Slovakia

e-mail: julius.czap@tuke.sk

and

Peter Mihók†

Department of Applied Mathematics and Business Informatics,
Faculty of Economics, Technical University of Košice,
Němcovej 32, SK-040 01 Košice, Slovakia
and
Mathematical Institute of the Slovak Academy of Sciences,
Grešákova 6, SK-040 01 Košice, Slovakia

Abstract

An additive hereditary property of graphs is a class of simple graphs which is closed under unions, subgraphs and isomorphism. Let $\mathcal{Q}$ be an additive hereditary property of graphs. A $\mathcal{Q}$-edge-coloring of a simple graph is an edge coloring in which the edges colored with the same color induce a subgraph of property $\mathcal{Q}$. In this paper we present some results on fractional $\mathcal{Q}$-edge-colorings. We determine the fractional $\mathcal{Q}$-edge chromatic number for matroidal properties of graphs.

Keywords: fractional coloring, graph property.

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1. Introduction

Our terminology and notation will be standard. The reader is referred to [1, 11] for undefined terms. All graphs considered in this paper are simple, i.e. they have no loops or multiple edges.

† Peter Mihók passed away on March 27, 2012.

We denote the class of all finite simple graphs by $\mathcal{I}$. A *graph property* $\mathcal{Q}$ is a non-empty isomorphism-closed subclass of $\mathcal{I}$. We also say that a graph G has property $\mathcal{Q}$ whenever $G \in \mathcal{Q}$. The fact that H is a subgraph of G is denoted by $H \subseteq G$ and the disjoint union of two graphs G and H is denoted by $G \cup H$. A property $\mathcal{Q}$ is called *additive* if $G \cup H \in \mathcal{Q}$ whenever $G \in \mathcal{Q}$ and $H \in \mathcal{Q}$. A property $\mathcal{Q}$ is called *hereditary* if $G \in \mathcal{Q}$ and $H \subseteq G$ implies $H \in \mathcal{Q}$. The set of all additive hereditary properties will be denoted by $\mathbb{L}$.

We list several well-known additive hereditary properties

$\mathcal{D}_k = \{G \in \mathcal{I} : \text{each subgraph of } G \text{ contains a vertex of degree at most } k\}$,
$\mathcal{I}_k = \{G \in \mathcal{I} : G \text{ does not contain } K_{k+2}\}$,
$\mathcal{J}_k = \{G \in \mathcal{I} : \chi'(G) \leq k\}$,
$\mathcal{O}_k = \{G \in \mathcal{I} : \text{each component of } G \text{ has at most } k+1 \text{ vertices}\}$,
$\mathcal{S}_k = \{G \in \mathcal{I} : \Delta(G) \leq k\}$,
$\mathcal{B} = \{G \in \mathcal{I} : G \text{ is a bipartite graph}\}$,

where K_{k+2} denotes the complete graph on $k+2$ vertices, $\chi'(G)$ is the *edge chromatic number* (chromatic index) and $\Delta(G)$ is the *maximum degree* of the graph G.

Generalized colorings of edges or/and vertices of graphs under restrictions given by graph properties have recently attracted much attention, see e.g. [2, 3, 4, 6, 7, 8, 10] and references therein.

By using the class of additive hereditary properties, there is the following generalization of edge coloring. Let $\mathcal{Q} \in \mathbb{L}$ and let t be a positive integer. A *t-fold $\mathcal{Q}$-edge-coloring* of a graph is an assignment of t distinct colors to each edge such that each color class induces a subgraph of property $\mathcal{Q}$. The smallest number k such that G admits a t-fold $\mathcal{Q}$-edge-coloring with k colors is the *$(t, \mathcal{Q})$-chromatic index* of G, denoted by $\chi'_{t,\mathcal{Q}}(G)$. Clearly, a 1-fold $\mathcal{O}_1$-edge-coloring is a usual proper edge coloring and hence $\chi'_{1,\mathcal{O}_1}(G) = \chi'(G)$.

Another generalization of edge coloring is fractional edge coloring. The *fractional chromatic index* of a graph G is defined in the following way: $\chi'_f(G) = \lim_{t\to\infty} \frac{\chi'_{t,\mathcal{O}_1}(G)}{t}$. If we replace the property $\mathcal{O}_1$ by any other additive hereditary graph property $\mathcal{Q}$ in the definition of the fractional chromatic index, then we obtain the *fractional $\mathcal{Q}$-chromatic index* of a graph G and we denote it $\chi'_{f,\mathcal{Q}}(G)$.

A *hypergraph* $\mathcal{H}$ is a pair (S, X), where S is a finite set and X is a family of subsets of S. The elements of X are called *hyperedges*. A *t-fold covering* of a hypergraph $\mathcal{H}$ is a collection (multiset) of hyperedges which includes every element of S at least t times. The smallest cardinality of such a multiset is called the *t-fold covering number* of $\mathcal{H}$ and is denoted $k_t(\mathcal{H})$. The *fractional covering number* of $\mathcal{H}$ is defined as $k_f(\mathcal{H}) = \lim_{t\to\infty} \frac{k_t(\mathcal{H})}{t}$.

For given simple graph $G = (V, E)$ and additive hereditary property $\mathcal{Q}$, let $\mathcal{H}_G = (E_G, \mathcal{Q}_G)$ denote the hypergraph whose vertex set is the edge set of G and the hyperedges are those subsets of $E(G) = E_G$ which induce a graph of property $\mathcal{Q}$. Since $\mathcal{Q}$ is hereditary, we can formulate the $(t, \mathcal{Q})$-chromatic index of the graph G as the t-fold covering number of the hypergraph $\mathcal{H}_G$. There is a natural one-to-one correspondence between the color classes of G and the hyperedges of $\mathcal{H}_G$. Therefore the following assertion holds.

Claim 1. *The fractional $\mathcal{Q}$-chromatic index of the graph G is equal to the fractional covering number of the hypergraph $\mathcal{H}_G = (E_G, \mathcal{Q}_G)$.*

A *matroid* $\mathcal{M} = (S, I)$ is a hypergraph which satisfies the following three conditions:

1. $\emptyset \in I$,
2. if $X \in I$ and $Y \subseteq X$, then $Y \in I$,
3. if $X, Y \in I$ and $|X| > |Y|$, then there is an $x \in X \setminus Y$ such that $Y \cup \{x\} \in I$.

In [12] the fractional covering number of matroids is determined. Let X be a subset of the ground set S of a matroid $\mathcal{M}$. The rank of X, denoted $\rho(X)$, is defined as the maximum cardinality of an independent subset of X (a subset of X which belongs to I).

Theorem 2 [12]**.** *If $\mathcal{M} = (S, I)$ is a matroid, then*

$$k_f(\mathcal{M}) = \max_{X \subseteq S; X \neq \emptyset} \frac{|X|}{\rho(X)}.$$

In this paper, by combining Claim 1 and Theorem 2, we give a general formula for the fractional $\mathcal{Q}$-chromatic index. Afterwards, by this formula and with other results from the literature, we determine the exact values of $\chi'_{f,\mathcal{Q}}(G)$ for so-called $\mathcal{Q}$-matroidal graphs.

2. Results

Let $G = (V, E)$ be a graph and let $\mathcal{Q}$ be an additive hereditary property. If the hypergraph $(E_G, \mathcal{Q}_G)$ is a matroid, then G is called *$\mathcal{Q}$-matroidal*. Let $\mathcal{Q}^{\mathcal{M}}$ denote the set of all $\mathcal{Q}$-matroidal graphs. A property $\mathcal{Q}$ is called *matroidal* if every graph G is $\mathcal{Q}$-matroidal. Schmidt [13] proved the existence of uncountably many matroidal properties.

A subset of the edge set of a graph is called *$\mathcal{Q}$-independent* if it induces a graph of property $\mathcal{Q}$. For a graph H let $\mathcal{Q}(H)$ denote the maximum cardinality of a $\mathcal{Q}$-independent subset of $E(H)$.

Lemma 3. *Let $a_i, b_i > 0$ for $i = 1, \dots, n$. Then* $\dfrac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \leq \max_i \left\{ \dfrac{a_i}{b_i} \right\}$.

Proof. By induction on n. ■

Theorem 4. *Let $\mathcal{Q} \in \mathbb{L}$ and let $G \in \mathcal{Q}^{\mathcal{M}}$. Then*

$$\chi'_{f,\mathcal{Q}}(G) = \max \frac{|E(H)|}{\mathcal{Q}(H)}\,, \tag{1}$$

where the maximum is taken over all connected nontrivial subgraphs H of G.

Proof. Since G is $\mathcal{Q}$-matroidal, the hypergraph $\mathcal{H}_G = (E_G, \mathcal{Q}_G)$ is a matroid. Claim 1 with Theorem 2 imply that

$$\chi'_{f,\mathcal{Q}}(G) = \max_{X \subseteq E_G; X \neq \emptyset} \frac{|X|}{\rho(X)} = \max \frac{|E(H)|}{\mathcal{Q}(H)}\,,$$

where the maximum is taken over all nontrivial subgraphs H of G.

Now we show that we may restrict our attention to connected H. Suppose that the maximum on the right-hand side of (1) is achieved for a graph H with more than one component. Let $H = H_1 \cup \cdots \cup H_n$, where H_i are the components of H. If one of these components, say H_j, is an empty graph (set of isolated vertices), then $\dfrac{|E(H)|}{\mathcal{Q}(H)} = \dfrac{|E(H - H_j)|}{\mathcal{Q}(H - H_j)}$. Thus we can assume that each component has at least one edge. Then $\dfrac{|E(H)|}{\mathcal{Q}(H)} = \dfrac{|E(H_1)| + \cdots + |E(H_n)|}{\mathcal{Q}(H_1) + \cdots + \mathcal{Q}(H_n)} \leq \max_i \left\{ \dfrac{|E(H_i)|}{\mathcal{Q}(H_i)} \right\}$. ■

We can now determine the fractional $\mathcal{Q}$-chromatic index for $\mathcal{Q}$-matroidal graphs. The following question arises: Which graphs are $\mathcal{Q}$-matroidal for given properties $\mathcal{Q}$?

Each hereditary property $\mathcal{Q}$ can be determined by the set of *minimal forbidden subgraphs* $F(\mathcal{Q}) = \{G \in \mathcal{I}; G \notin \mathcal{Q} \text{ but } G \setminus \{e\} \in \mathcal{Q} \text{ for each } e \in E(G)\}$. For example: $F(\mathcal{O}_k) = \{H; H \text{ is a tree on } k+2 \text{ vertices }\}$; $F(\mathcal{I}_k) = \{K_{k+2}\}$. Simões-Pereira [14] proved that if $F(\mathcal{Q})$ is finite, then $\mathcal{Q}$ is not matroidal.

In [9] there is the following characterization of $\mathcal{Q}$-matroidal graphs.

Theorem 5 [9]**.** *A graph $G = (V, E)$ is $\mathcal{Q}$-matroidal if and only if for each $\mathcal{Q}$-independent set $I \subseteq E$ and for each edge $e \in E \setminus I$ the graph $G[I \cup \{e\}]$ induced by $I \cup \{e\}$ contains at most one minimal forbidden subgraph of $\mathcal{Q}$.*

By Theorem 5 each graph G which contains either at most one minimal forbidden subgraph of $\mathcal{Q}$ or only edge-disjoint minimal forbidden subgraphs of $\mathcal{Q}$ is $\mathcal{Q}$-matroidal.

Lemma 6 [9]**.** *The property $\mathcal{Q}^{\mathcal{M}}$ belongs to $\mathbb{L}$ for every $\mathcal{Q} \in \mathbb{L}$.*

By Lemma 6 we can characterize the structure of $\mathcal{Q}$-matroidal graphs by describing the set of minimal forbidden subgraphs $F(\mathcal{Q}^{\mathcal{M}})$.

For any two given graphs G_1 and G_2 with a common induced subgraph H we construct the graph $G = (G_1; H; G_2)$ by amalgamation of G_1 and G_2 with respect to H so that $V(G) = V(G_1) \cup V(G_2)$, $E(G) = E(G_1) \cup E(G_2)$ and $H = (V(G_1) \cap V(G_2), E(G_1) \cap E(G_2))$.

In the following P_n and C_n will denote the path and the cycle on n vertices, respectively. D_n will denote the complement of K_n.

Theorem 7 [9]**.** *Let G be a graph and let $k \geq 1$. Then*

- *$G \in F(\mathcal{O}_k^{\mathcal{M}})$ if and only if $G \in T \setminus \{K_{1,k+2}; C_{k+2}\}$, where T is the set of all trees on $k+3$ vertices and all unicyclic graphs on $k+2$ vertices,*
- *$G \in F(\mathcal{S}_k^{\mathcal{M}})$ if and only if $G = (K_{1,k+1}; K_2 \cup D_p; K_{1,k+1})$ for some $0 \leq p \leq k$ and $k \geq 2$, where K_2 joins the central vertices of the stars,*
- *$G \in F(\mathcal{I}_k^{\mathcal{M}})$ if and only if $G = (K_{k+2}; K_r; K_{k+2})$ for some $2 \leq r \leq k+1$,*
- *$G \in F(\mathcal{B}^{\mathcal{M}})$ if and only if $G = (C_{2p+1}; P_q; C_r)$ for some $p \geq 1$, $q \geq 2$ and $r \geq 3$.*

The seminal result on fractional edge colorings is due to Edmonds [5]. For a graph G we define $\Gamma(G) = \max \dfrac{2|E(H)|}{|V(H)|-1}$, where the maximization is over every induced subgraph H of G with $|V(H)| \geq 3$ and $|V(H)|$ odd.

Theorem 8 [5]**.** *Let G be a graph. Then*

$$\chi'_{f,\mathcal{J}_1}(G) = \chi'_{f,\mathcal{S}_1}(G) = \chi'_{f,\mathcal{O}_1}(G) = \chi'_f(G) = \max\{\Delta(G), \Gamma(G)\}.$$

Lemma 9. *Every graph is $\mathcal{D}_1$-matroidal.*

Proof. Clearly, $F(\mathcal{D}_1)$ is a set of cycles. Moreover, if we add an edge to a tree (forest) we obtain exactly (at most) one cycle. So the claim follows from Theorem 5. ■

Although all graphs are $\mathcal{D}_1$-matroidal, for $k \geq 2$ the characterization of $\mathcal{D}_k$-matroidal graphs seems to be difficult.

Theorem 10. *Let G be a graph. Then*

$$\chi'_{f,\mathcal{D}_1}(G) = \max \frac{|E(H)|}{|V(H)|-1},$$

where the maximum is taken over all connected nontrivial subgraphs H of G.

Proof. From Lemma 9 it follows that G is $\mathcal{D}_1$-matroidal. Any spanning tree of a connected graph H on n vertices has $n-1$ edges, therefore $\mathcal{D}_1(H) = |V(H)| - 1$. Theorem 4 implies $\chi'_{f,\mathcal{D}_1}(G) = \max_{H \subseteq G} \frac{|E(H)|}{\mathcal{D}_1(H)} = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|-1}$. ■

Corollary 11. *Let G be a graph and let $\mathcal{Q} \in \mathbb{L}$ such that $\mathcal{D}_1 \subseteq \mathcal{Q}$. Then*

$$\chi'_{f,\mathcal{Q}}(G) \leq \max \frac{|E(H)|}{|V(H)|-1},$$

where the maximization is over all connected nontrivial subgraphs H of G.

Lemma 12. *Let $k \geq 1$. The graph G is $\mathcal{I}_k$-matroidal if and only if any two complete graphs on $k+2$ vertices are edge-disjoint in G.*

Proof. Assume that G contains two complete graphs on $k+2$ vertices which have $r \geq 2$ vertices in common. These r vertices induce K_r, hence G contains $(K_{k+2}; K_r; K_{k+2})$ as a subgraph. So $G \notin \mathcal{I}_k^{\mathcal{M}}$ since $(K_{k+2}; K_r; K_{k+2}) \in F(\mathcal{I}_k^{\mathcal{M}})$ (see Theorem 7).

If $G \notin \mathcal{I}_k^{\mathcal{M}}$, then G contains a forbidden subgraph $(K_{k+2}; K_r; K_{k+2})$ for some $2 \leq r \leq k+1$, thus it contains two complete graphs on $k+2$ vertices which share an edge. ■

Let H_{k+2} denote the number of complete graphs on $k+2$ vertices in the graph H.

Theorem 13. *Let G be an $\mathcal{I}_k$-matroidal graph, $k \geq 1$. Then*

$$\chi'_{f,\mathcal{I}_k}(G) = \max \frac{|E(H)|}{|E(H)| - H_{k+2}},$$

where the maximum is taken over all connected nontrivial subgraphs H of G.

Proof. From Theorem 4 it follows that $\chi'_{f,\mathcal{I}_k}(G) = \max_{H \subseteq G} \frac{|E(H)|}{\mathcal{I}_k(H)}$. So it is sufficient to show that $\mathcal{I}_k(H) = |E(H)| - H_{k+2}$.

Lemma 12 implies that any two complete graphs on $k+2$ vertices are edge-disjoint in every subgraph H of G. Hence, if we remove less than H_{k+2} edges from H, then the obtained graph still contains at least one K_{k+2}. Therefore $\mathcal{I}_k(H) \leq |E(H)| - H_{k+2}$.

On the other hand, if we remove one edge from each K_{k+2}, then the remaining edges form an $\mathcal{I}_k$-independent set, hence $\mathcal{I}_k(H) \geq |E(H)| - H_{k+2}$. ■

Lemma 14. *Let $k \geq 2$. The graph G is $\mathcal{S}_k$-matroidal if and only if no two vertices of degree at least $k+1$ are incident in G.*

Proof. Let uv be an edge of G such that its endvertices have degree at least $k+1$. Let G_1 be a subgraph of G which contains only the edges incident with u or v. Clearly, G_1 contains a subgraph G_2 in which the vertices u and v are joined by an edge and they have degree $k+1$. Let p denote the number of common neighbors of u and v in G_2. Observe that $G_2 = (K_{1,k+1}; K_2 \cup D_p; K_{1,k+1})$, consequently $G_2 \in F(\mathcal{S}_k^{\mathcal{M}})$. So G cannot be $\mathcal{S}_k$-matroidal.

If $G \notin \mathcal{S}_k^{\mathcal{M}}$, then it contains a minimal forbidden subgraph $(K_{1,k+1}; K_2 \cup D_p; K_{1,k+1})$ for some $0 \leq p \leq k$. The central vertices of these stars are joined by an edge and they have degree $k+1$. ■

Theorem 15. *Let G be an $\mathcal{S}_k$-matroidal graph, $k \geq 2$. Then*

$$\chi'_{f,\mathcal{S}_k}(G) = \max \frac{|E(H)|}{|E(H)| - \sum_{\substack{v \in V(H) \\ \deg_H(v) \geq k+1}} (\deg_H(v) - k)},$$

where the maximum is taken over all connected nontrivial subgraphs H of G.

Proof. Let H be a subgraph of G. If for every vertex v of H of degree at least $k+1$ we remove $\deg_H(v) - k$ edges incident with it, then we obtain a graph whose maximum degree is at most k. Therefore

$$\mathcal{S}_k(H) \geq |E(H)| - \sum_{\substack{v \in V(H) \\ \deg_H(v) \geq k+1}} (\deg_H(v) - k).$$

The opposite inequality follows from the fact that no two vertices of degree at least $k+1$ are incident in G, thus neither in $H \subseteq G$ (see Lemma 14). Therefore the claim follows from Theorem 4. ■

Lemma 16. *The graph G is $\mathcal{B}$-matroidal if and only if no odd cycle of G shares an edge with any other cycle of G.*

Proof. $G \notin \mathcal{B}^{\mathcal{M}}$ if and only if G contains a minimal forbidden subgraph $(C_{2p+1}; P_q; C_r)$ for some $p \geq 1$, $q \geq 2$ and $r \geq 3$. Equivalently, G contains an odd cycle which shares an edge with an other cycle. ■

Corollary 17. *If $G \in \mathcal{B}^{\mathcal{M}}$, then the odd cycles of G are edge-disjoint.*

Let $oc(G)$ denote the number of odd cycles in the graph G.

Theorem 18. *Let G be a $\mathcal{B}$-matroidal graph. Then*

$$\chi'_{f,\mathcal{B}}(G) = \max \frac{|E(H)|}{|E(H)| - oc(H)},$$

where the maximum is taken over all connected nontrivial subgraphs H of G.

Proof. Let H be a subgraph of G. If we remove one edge from every odd cycle of H, then the remaining edges induce a bipartite graph, hence $\mathcal{B}(H) \geq |E(H)| - oc(H)$.

The odd cycles in H are edge-disjoint (see Corollary 17), thus we must remove at least $oc(H)$ edges from $E(H)$ to obtain a $\mathcal{B}$-independent set. Therefore $\mathcal{B}(H) \leq |E(H)| - oc(H)$.

Consequently, $\mathcal{B}(H) = |E(H)| - oc(H)$ and hence the assertion follows from Theorem 4. ∎

Lemma 19. *Let $k \geq 1$. The graph G is $\mathcal{O}_k$-matroidal if and only if G either belongs to $\mathcal{O}_k$ or it is isomorphic to $K_{1,p}$, $p \geq k+1$, to C_{k+2} or to a tree on $k+2$ vertices.*

Proof. G is $\mathcal{O}_k$-matroidal if and only if it does not contain any subgraph from $F(\mathcal{O}_k^{\mathcal{M}})$. So the claim follows from Theorem 7. ∎

Clearly, if $G \in \mathcal{O}_k$, then its fractional $\mathcal{O}_k$-edge chromatic number equals one. If $G \in \mathcal{O}_k^{\mathcal{M}} \setminus \mathcal{O}_k$, then it has $k+2$ vertices or it is a star on at least $k+3$ vertices.

Theorem 20. *Let $G \in \mathcal{O}_k^{\mathcal{M}} \setminus \mathcal{O}_k$ and let $|V(G)| = k+2$, $k \geq 2$. Then*

$$\chi'_{f,\mathcal{O}_k}(G) = \frac{|E(G)|}{|E(G)| - \lambda(G)},$$

where $\lambda(G)$ is the edge-connectivity of G.

Proof. Let H be a connected subgraph of G. If $E(H)$ is not $\mathcal{O}_k$-independent, then either $|E(H)| = k+2$ or $|E(H)| = k+1$. In the first case $H = C_{k+2}$, hence $\mathcal{O}_k(H) = |E(H)| - 2$. In the second case H is a tree, therefore $\mathcal{O}_k(H) = |E(H)| - 1$. Thus the claim follows from Theorem 4. ∎

Theorem 21. *Let $G \in \mathcal{O}_k^{\mathcal{M}} \setminus \mathcal{O}_k$ and let $|V(G)| = k+i$, $k \geq 2$, $i \geq 3$. Then*

$$\chi'_{f,\mathcal{O}_k}(G) = \frac{|E(G)|}{|E(G)| - i + 1} = \frac{k+i-1}{k}.$$

Proof. It follows from Theorem 4 and from the fact that G is a star. ∎

3. Examples

Example 22. *Let $K_{2,3}$ denote the complete bipartite graph on $2+3$ vertices. We will show that $\chi'_{f,\mathcal{S}_2}(K_{2,3}) = \frac{3}{2}$.*

Solution 1.

From Lemma 14 it follows that $K_{2,3} \in \mathcal{S}_2^{\mathcal{M}}$. From Theorem 15 we have $\chi'_{f,\mathcal{S}_2}(K_{2,3}) = \max \frac{|E(H)|}{|E(H)| - \sum_{\substack{v \in V(H) \\ \deg_H(v)=3}} 1}$, where the maximum is taken over all connected nontrivial subgraphs H of G.

If H is a connected subgraph of G, then either $H \in \mathcal{S}_2$ or it is a graph from Figure 1. So $\chi'_{f,\mathcal{S}_2}(K_{2,3}) = \max\{1, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}\} = \frac{3}{2}$.

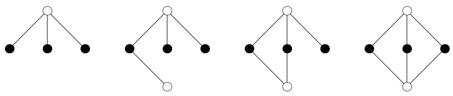

Figure 1. Connected subgraphs of $K_{2,3}$ which are not in $\mathcal{S}_2$.

Solution 2.

Fractional $\mathcal{Q}$-edge-colorings may be viewed in several ways. We present an equivalent definition. Let r, s be positive integers with $r \geq s$. An (r, s)-fractional $\mathcal{Q}$-edge-coloring of G is an assignment of s-element subsets of $\{1, \ldots, r\}$ to the edges of G such that each color class induces a graph of property $\mathcal{Q}$. Then the fractional $\mathcal{Q}$-edge chromatic number of G is defined as

$$\chi'_{f,\mathcal{Q}}(G) = \inf\left\{\frac{r}{s} : G \text{ has an } (r,s)\text{-fractional } \mathcal{Q}\text{-edge-coloring}\right\}.$$

Note that in this definition we can replace the infimum by the minimum.

For each (r, s)-fractional $\mathcal{S}_2$-edge-coloring of $K_{2,3}$ and for each color $i \in \{1, \ldots, r\}$ the following holds: at most four edges are colored with sets containing the color i. On the other hand, every edge is assigned with an s-element color set. This implies that $4r \geq 6s$, hence $\chi'_{f,\mathcal{S}_2}(K_{2,3}) \geq \frac{3}{2}$.

To prove the inequality $\chi'_{f,\mathcal{S}_2}(K_{2,3}) \leq \frac{3}{2}$ we construct a $(3, 2)$-fractional $\mathcal{S}_2$-edge-coloring of $K_{2,3}$, see Figure 2.

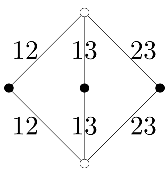

Figure 2. A $(3, 2)$-fractional $\mathcal{S}_2$-edge-coloring of the graph $K_{2,3}$.

The following results immediately follows from Theorems 13, 15 and 18.

Example 23. *If* $k \geq 1$, *then* $\chi'_{f,\mathcal{I}_k}(K_{k+2}) = \frac{\binom{k+2}{2}}{\binom{k+2}{2}-1}$, $\chi'_{f,\mathcal{S}_k}(K_{1,k+1}) = \frac{k+1}{k}$ *and* $\chi'_{f,\mathcal{B}}(C_{2k+1}) = \frac{2k+1}{2k}$.

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