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EDGE DOMINATING SETS AND VERTEX COVERS

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Abstract

Bipartite graphs with equal edge domination number and maximum matching cardinality are characterized. These two parameters are used to develop bounds on the vertex cover and total vertex cover numbers of graphs and a resulting chain of vertex covering, edge domination, and matching parameters is explored. In addition, the total vertex cover number is compared to the total domination number of trees and grid graphs.

Keywords: edge dominating set, matching, total dominating set, vertex cover.

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1. Introduction

Let $G = (V, E)$ be a graph with n vertices. Denote the open and closed neighborhoods of a vertex $x \in V$ by $N(x)$ and $N[x]$, respectively. That is, $N(x) = \{v : xv \in E\}$ and $N[x] = N(x) \cup \{x\}$. For a set of vertices $X \subseteq V$, let $N[X]$ denote the union of $N[x]$ over all $x \in X$. Let $[H]$ denote the subgraph induced by a set of vertices $H \subseteq V(G)$. For a set $C \subseteq V$, let $E(C)$ denote the set of edges in $[C]$. For an edge uv, we call u and v the *endvertices* of uv. For a set of edges $S \subseteq E(G)$, let $V(S) = \{u : uv \in S\}$. In other words, $V(S)$ consists of both endvertices of each edge in S. A vertex *covers* the edges incident to it and *dominates* itself and the vertices adjacent to it. An edge *covers* its endvertices and *dominates* itself and the edges adjacent to it (edges incident to its endvertices). Two edges are *independent* if they have no endvertices in common. For $v \in C$, x is a *private neighbor* of v if $N(x) \cap C = \{v\}$. For two sets X, Y, let $\nabla(X,Y) = (X - Y) \cup (Y - X)$.

A *dominating set* of G is a set $D \subseteq V$ with the property that for each $u \in V - D$, there exists $x \in D$ adjacent to u. The minimum cardinality amongst all dominating sets is the *domination number* $\gamma(G)$. A *total dominating set* of G is a set $X \subseteq V$ with the property that for each $u \in V$, there exists $x \in X$ adjacent to u. The minimum cardinality amongst all total dominating sets is the *total domination number* $\gamma_t(G)$ and is only defined for graphs without isolated vertices.

A *vertex cover* of G is a set $C \subseteq V$ such that for each edge $uv \in E$ at least one of u and v is in C. Let $\alpha(G)$ be the *vertex cover number* of G, the minimum number of vertices required to cover all edges of G. A *total vertex cover* of G is a vertex cover $C \subseteq V$ with the property that for each $u \in C$, there exists $x \in C$ adjacent to u. The minimum cardinality amongst all total vertex covers is the *total vertex cover number* $\alpha_t(G)$. Total vertex covers were studied in [3, 4, 6]. As observed in [3], a total vertex cover is simultaneously a vertex cover and a total dominating set.

A *matching* is any independent set of edges. A *maximal matching* is a matching X so that $V - V(X)$ is an independent set of vertices. A *perfect matching* in graph G is a matching X so that $V(X) = V(G)$. Let $\beta_1(G)$ denote the size of a *maximum matching* in G. The number of edges in a smallest maximal independent set of edges in G is denoted by $\beta_1'(G)$. Of course, $\beta_1'(G) \leq \beta_1(G) \leq \frac{n}{2}$. It is well-known that $\alpha(G) \geq \beta_1(G)$ and the two quantities are equal when G is bipartite [2]. A set of edges M is a *minimum matching* when M is a maximal matching and $|M| \leq |M'|$ for all maximal matchings M'.

An *edge dominating set* of a graph is a set of edges E' such that each edge in E is either in E' or shares an endvertex with some edge in E'. The *edge domination number* of a graph is the number of edges of a smallest edge dominating set. Corollary 1.2 of [1] states that for all graphs G, the edge domination number of G is equal to $\beta_1'(G)$. In other words, if E' is a minimum edge dominating set of G, there is an independent edge dominating set of G of cardinality $|E'|$.

In Section 2, we prove some basic results, present a chain of inequalities, and state a conjecture. Sections 3 and 4 focus on matchings and edge dominating sets and characterize the bipartite graphs with $\beta_1'(G) = \beta_1(G)$. Then, in Sections 5 and 6, we study bounds on the vertex cover and total vertex cover numbers including bounds that relate them to the matching, edge domination, and total domination numbers. Additionally, in Section 6, we consider total vertex covers in grid graphs.

2. Fundamentals

The first bound on the total vertex cover number shown in the literature is stated next.

Theorem 1 [6]**.** *For all connected graphs G, $\alpha_t(G) \le 2\alpha(G)$ with equality if and only if $G = K_{1,m}$ with $m \ge 1$.*

Note that this implies $\alpha_t(G) \le 2\alpha(G) - 1$ if G is not a star. A *star* is a graph isomorphic to $K_{1,m}$ with $m \ge 1$.

Proposition 2. *Let G be a graph. Then, $\frac{\alpha(G)}{2} \le \beta_1'(G) \le \alpha(G)$.*

Proof. Since $\beta_1'(G) \le \beta_1(G)$ and $\beta_1(G) \le \alpha(G)$, the right inequality follows. For the left inequality, consider a minimum edge dominating set D. Let $C = V(D)$. Then, C is a vertex cover, otherwise if it were note there would be an edge not dominated by D. Therefore $|C| \le 2|D| = 2\beta_1'(G)$. ■

It is possible to combine P_4's together in a "ladder" fashion to obtain infinitely many graphs where the left inequality in Proposition 2 is sharp. For example, take a P_4 with vertices v_1, v_2, v_3, v_4, make a copy of it and add an edge between the two vertices labeled v_3. Then $\frac{\alpha(G)}{2} = \beta_1'(G) = 2$ in this graph. Note that in these examples, there exist minimum vertex covers that are also total vertex covers. Furthermore $\alpha_t(G) = 2\beta_1'(G)$ if G is a star, but there exists graphs that are not stars, such as P_4, with $\alpha_t(G) = 2\beta_1'(G)$.

The following are from [3].

Theorem 3 [3]**.** *Let G be a graph with n vertices,*

(i) *if $n \ge 2$ then $\alpha_t(G) \le 2\beta_1'(G)$,*

(ii) *if $n \ge 2$ then $\alpha_t(G) \le \alpha(G) + \gamma(G)$,*

(iii) *if $n \ge 3$ then $\alpha_t(G) \le \frac{n+\alpha(G)}{2}$.*

Proposition 4. *For any graph G, C is a total vertex cover if and only if $E(C)$ is an edge dominating set.*

Proof. Suppose C is a set such that $E(C)$ is an edge dominating set. Then from the definitions, C must be a total vertex cover, since if $uv \in E(C)$, both u and v must be in C.

Let C be a total vertex cover of G. Since C is a vertex cover, $V(G) - C$ is an independent set. Therefore, $E(C)$ is an edge dominating set. ■

It follows immediately from Proposition 4 that $\alpha_t(G)$ is the minimum number of vertices in the set of endvertices of any edge dominating set. Let $\beta_3(G)$ denote the minimum cardinality of $V(D)$ over all minimum edge dominating sets D. (We use the notation $\beta_3(G)$, as $\beta_2(G)$ is used for other purposes in [7]). Note that $\alpha_t(G) \leq \beta_3(G) \leq 2\beta_1'(G)$ since some vertices may be the endvertices of more than one edge in an edge dominating set.

The above results give rise to a chain of inequalities, reminiscent of the famous domination chain of inequalities [5]. An "edge domination" chain is described in [7].

$$\beta_1'(G) \leq \beta_1(G) \leq \alpha(G) \leq \alpha_t(G) \leq \beta_3(G) \leq 2\beta_1'(G). \tag{1}$$

While the following may seem obvious, its formal verification has thus far eluded us.

Conjecture 1. *For every graph G, $\alpha_t(G) = \beta_3(G)$.*

For any graph G and a total vertex cover C, partition the vertex set of C into $C_1, C_2, \ldots, C_k$ where each C_i has a spanning subgraph S_i that is a star with at least two vertices. Since C is a total vertex cover, such a partitioning exists. In other words, each total vertex cover can be reduced, by edge removals, to a forest of stars whose edges form an edge dominating set of G. With the validity of Conjecture 1, there is an α_t-set with a spanning forest containing exactly $\beta_1'(G)$ edges.

Proposition 5. $\alpha_t(G) = 2\beta_1'(G)$ *if and only if there exists an α_t-set C of G such that $[C]$ contains a perfect matching.*

Proof. Assume $\alpha_t(G) < 2\beta_1'(G)$ and let C be an α_t-set of G, where $[C]$ contains a perfect matching, M_C. From the preceding comments, M_C is an edge dominating set, since $V(M_C) = C$. Thus, $\alpha_t(G) = 2|M_C| > 2\beta_1'(G)$, contradicting equation (1).

Next, suppose $\alpha_t(G) = 2\beta_1'(G)$. Let $V(M)$ be the endvertices of a minimum matching M. Then, $V(M)$ is an α_t-set of G and $V(M)$ is a total vertex cover satisfying $|V(M)| = 2\beta_1'(G)$. The edges of M comprise a perfect matching of the subgraph induced by $V(M)$. ∎

Proposition 6. $\alpha(G) = 2\beta_1'(G)$ *if and only if there exists an α-set C of G such that $[C]$ contains a perfect matching.*

Proof. Suppose G has an α-set C that possesses a perfect matching of $[C]$. Thus, $\alpha(G) = 2|M_C| \geq 2\beta_1'(G)$. From equation (1), equality must hold.

For the other direction, suppose $\alpha(G) = 2\beta_1'(G)$. Then, since $\alpha(G) = \alpha_t(G) = 2\beta_1'(G)$, equality holds and the result follows from Proposition 5. ∎

The bound of Proposition 6 is sharp for K_4 minus an edge and for P_4. Graphs with $\alpha(G) = 2\beta_1'(G)$ must have $\beta_1(G) = 2\beta_1'(G)$. Such graphs have at least two edge-disjoint maximal matchings: one is of size β_1' and one is of size $2\beta_1'$.

3. Matchings

Section 4 characterizes bipartite graphs G with equal maximum and minimum matching numbers, $\beta_1(G)$ and $\beta_1'(G)$, respectively. It is helpful to have bounds on the change in these values upon the removal of any edge and its endpoints. It is well known that if e is an edge of a maximum matching, then removing e and its endvertices, $V(e)$, results in a graph $G - V(e)$ with a maximum matching number that is one less than that of the original graph. When e is not a member of a maximum matching, $G - V(e)$ may or may not have a reduced maximum matching number. In this section, we show that when $\beta_1'(G) = \beta_1(G)$, the minimum and maximum numbers are both reduced by one upon the removal of any edge e and its endvertices.

Observations.
(a) Every independent set of k edges, $1 \leq k < \beta_1'(G)$, is a subset of a maximal matching M where $\beta_1'(G) \leq |M| \leq \beta_1(G)$.
(b) $\beta_1'(G) = \beta_1(G) = 1$ if and only if $G = K_{1,n-1}$, for $n \geq 2$.

Proposition 7. *$\beta_1'(G) = \beta_1(G)$ if and only if every matching is subset of a maximum matching.*

Proof. When $\beta_1'(G) = \beta_1(G)$, by Observation (a) every matching is a subset of a maximal matching with $\beta_1'(G) = \beta_1(G)$ edges. When $\beta_1'(G) < \beta_1(G)$, a minimum matching cannot be a subset of a larger matching. ■

Besides Observation (b), the only other characterization we know to exist for graphs G with $\beta_1'(G) = \beta_1(G)$ is due to Arumugam and Velammal and is given next.

Theorem 8 [1]**.** *For any graph G, $\beta_1'(G) = \beta_1(G) = n/2$ if and only if n is even and $G = K_n$ or $G = K_{n/2,n/2}$.*

For arbitrary graphs G, Lemma 9 places bounds on the change in $\beta_1(G)$ when an edge and its endvertices are removed from G. Lemma 10 provides similar bounds for minimum matchings.

Lemma 9. *For any graph G and edge e*
(i) $\beta_1(G) - 2 \leq \beta_1(G - V(e)) \leq \beta_1(G) - 1$, *and*

(ii) $\beta_1(G - V(e)) = \beta_1(G) - 1$ *if and only if* e *is a member of a maximum matching.*

Proof. Let $e = vw$ be any edge of G, M any maximum matching of G, and M' any maximum matching of $G - V(e)$. Notice that $M' \cup e$ is a maximal matching in G.

(i) The endvertices of e are endvertices of at most two edges of M. Therefore, at least $\beta_1(G) - 2$ edges of M are independent edges in $G - V(e)$ and establishes the left inequality of (i).

For the right inequality, since $M' \cup e$ is a maximal matching of G, $\beta_1(G) \geq |M' \cup e| = \beta_1(G - V(e)) + 1$, and establishes the right inequality. Thus, (i) holds.

(ii) Suppose $e \in M$. Then, $M - e$ is a maximal matching of $G - V(e)$ and, thus, $\beta_1(G - V(e)) \geq |M - e| = \beta_1(G) - 1$. From (i), $\beta_1(G) - 1 \geq \beta_1(G - V(e))$. Therefore, equality holds. Finally, assume e is not a member M or any other maximum matching of G. Since $M' \cup e$ is a maximal matching in G that contains e, $\beta_1(G) > |M' \cup e|$. That is, $\beta_1(G - V(e)) < \beta_1(G) - 1$, and shows that (ii) holds. ■

Lemma 10. *For any graph* G *and edge* e

(i) $\beta_1'(G) - 1 \leq \beta_1'(G - V(e)) \leq \beta_1'(G)$,

(ii) $\beta_1'(G - V(e)) = \beta_1'(G) - 1$ *if and only if* e *is a member of a minimum matching of* G.

Proof. Let $e = vw$ be any edge of G, M any minimum matching of G, and M' any minimum matching of $G - V(e)$. We may assume $v \in V(M)$. Notice that $M' \cup e$ is a maximal matching in G.

(i) Since $M' \cup e$ is a maximal matching in G, $|M' \cup e| \geq \beta_1'(G)$. That is, $\beta_1'(G - V(e)) + 1 \geq \beta_1'(G)$, and establishes the left inequality of (i).

For the right inequality, when $w \notin V(M)$, then $vv' \in M$ and $M - vv'$ is an independent set of $\beta_1'(G) - 1$ edges in $G - V(e)$. If $N(v') \subseteq V(M)$, then $M - vv'$ is a maximal matching of $G - V(e)$, hence, $\beta_1'(G - V(e)) \leq |M - vv'| = \beta_1'(G) - 1$. If v' has a neighbor $x \notin V(M)$, then $(M - vv') \cup v'x$ is a maximal matching of $G - V(e)$, with $\beta_1'(G)$ edges. Therefore, $\beta_1'(G - V(e)) \leq \beta_1'(G)$.

When $w \in V(M)$, we consider two cases.

Case 1. When $w = v'$, $M - e$ is a maximal matching of $G - V(w)$. Therefore, $\beta_1'(G - V(e)) \leq |M - e| = \beta_1'(G) - 1$.

Case 2. When $w \neq v'$, vv' and ww' are both members of M. Then, $M' = M - \{vv', ww'\}$ is an independent set of $\beta_1'(G) - 2$ edges in $G - V(e)$. All undominated edges have either v' or w' as one endvertex and the other in $V - V(M)$. Therefore, M', and at most one edge with an endvertex v', and one edge with an endvertex

w' is a maximal matching in $G - V(e)$. Again, $\beta_1'(G - V(e)) \leq \beta_1'(G)$. Thus, the right inequality holds and establishes the validity of (i).

(ii) Assume $e \in M$. Since $M - e$ is a maximal matching of $G - V(e)$, $\beta_1'(G - V(e)) \leq |M - e| = \beta_1'(G) - 1$. From (i), $\beta_1'(G) - 1 \leq 1'(G - V(e))$. Therefore, equality holds. Now, assume e is an edge that is not a member of M or any other minimum matching of G. Since $M' \cup e$ is a maximal matching of G that contains e, $\beta_1'(G) < |M' \cup e| = \beta_1'(G - V(e)) + 1$. Thus, $\beta_1'(G) - 1 < \beta_1'(G - V(e))$ and completes the proof of (ii). ∎

The next result follows directly from Lemmas 9 and 10 and is given without additional justification.

Theorem 11. *$\beta_1'(G) = \beta_1(G)$ if and only if, for every edge e of G, $\beta_1'(G - V(e)) = \beta_1(G - V(e)) = \beta_1(G) - 1$.*

Theorem 11 implies $\beta_1'(G) = \beta_1(G)$ is a "hereditary" property.

Corollary 12. *Let G be a graph for which $\beta_1'(G) = \beta_1(G)$. For any set X of independent edges, $\beta_1'(G - V(X)) = \beta_1(G - V(X)) = \beta_1(G) - |X|$.*

Following Chartrand *et al.* [2], let M_1 and M_2 be maximal matchings in a graph G. Let H be the subgraph of G for which $E(H) = \nabla(M_1, M_2)$. First, notice that

$$M_1 = \nabla(M_2, E(H)) \text{ and } M_2 = \nabla(M_1, E(H)).$$

Since M_1 and M_2 are independent sets of edges, no vertex in H can have degree greater than two. Therefore, the non-trivial components of H are cycles and/or paths where edges in a cycle or path alternate between M_1 and M_2. Thus, alternating cycles have even length and alternating paths have length at least two. One endvertex of each even length alternating path is in $V(M_1) - V(M_2)$ and the other is in $V(M_2) - V(M_1)$. Endvertices of odd length alternating paths are both in $V(M_1) - V(M_2)$ or both in $V(M_2) - V(M_1)$.

Let H^* be any subset of the set of components of H. Then, $M_3 = \nabla(M_1, E(H^*))$ is a maximal matching of G. Further, $|M_3| = |M_1| - O_1 + O_2$ edges, where O_1 is the number of odd length alternating paths with both endvertices in $V(M_1) - V(M_2)$ and O_2 is the number of odd length alternating paths with both end vertices in $V(M_2) - V(M_1)$.

Lemma 13. *Let M' be an arbitrary minimum matching of a graph G. Then, for any maximal matching M, $\nabla(M, M')$ contains $|M| - \beta_1'(G)$ odd alternating length paths, where the endvertices of each path lie in $V(M) - V(M')$. No other odd length alternating paths exist.*

Proof. Let H be the subgraph of G for which $E(H) = \nabla(M, M')$. Therefore, $M = \nabla(M', E(H))$. The degree one vertices of any odd length alternating path must both lie in $V(M') - V(M)$, or both lie in $V(M) - V(M')$. If both degree one vertices of alternating path P in H lie in $V(M') - V(M)$, then $\nabla(M', E(P))$ is a maximal matching with $|M'| - 1 = \beta_1'(G) - 1$ edges, a contradiction. Therefore, both endvertices of every odd length alternating path lie in $V(M) - V(M')$.

Let k be the number of odd length paths in H. Then, $M = \nabla(M', E(H))$ is a maximal matching and $|M| = |M'| + k = \beta_1'(G) + k$. That is, the number of odd length alternating paths in H is $k = |M| - \beta_1'(G)$ and each has both endvertices in $V(M) - V(M')$. ■

Theorem 14. *For any graph G, if $\beta_1(G) = \beta_1'(G) + k$, then there exists a minimum matching M' and a maximum matching M such that $\nabla(M, M')$ consists of k odd length paths with endvertices in $V(M) - V(M')$. There are no other alternating cycles or paths.*

Proof. Let M be an arbitrary maximum matching and M^* an arbitrary minimum matching of G. By the Lemma 13, $H = \nabla(M, M^*)$ contains a set of $k = \beta_1(G) - \beta_1'(G)$ odd length alternating paths with all endvertices in $V(M) - V(M^*)$, no other odd length alternating paths exist, and possibly a collection $H_e^* \subseteq H$ of even length alternating paths and cycles. Let $M' = \nabla(M^*, E(H_e^*))$. Since the number of edges of M' and M^* are equal in number in H_e^*, M' is a matching with $|M'| = |M^*|$. That is M' is a minimum matching and $H = \nabla(M', M)$ contains no even length cycles or paths, but does contain the original k odd length paths in H^*. That is, $H = H^* - H_e^*$. Therefore, M and M' establish the theorem. ■

Corollary 15. *For any graph G, $\beta_1(G) = 2\beta_1'(G)$ if and only if for any maximum matching M and minimum matching M', $(M \cap M') = \emptyset$, and $(M \cup M') = \nabla(M, M')$ is a collection of $\beta_1'(G)$ vertex independent P_4's.*

While alternating cycles and paths are usually defined as the set difference between two maximal matchings, their usefulness often arises when defined between a single maximal matching M and the edges in $E - M$ to aid in determining if M is a maximum or minimum matching.

In the following, four types of alternating cycles and paths are defined and a brief statement of what their existence implies. The implications are utilized in Section 4. In the following, a cycle will be referred to as a path P with adjacent endvertices.

Definition 16. *For any graph G and maximal matching M, $P = (v_1, v_2, \ldots, v_m)$ is an alternating cycle or path when the edges of P alternate between edges in M and edges in $E - M$, and are of one of the following types:*

Type 1: *a cycle in G, m is even, and v_1 is adjacent to v_m. $M' = \nabla(M, E(P))$ is a maximal matching with $|M'| = |M|$.*

Type 2: *an even length alternating path, m is odd, $v_1 \notin V(M)$, and $N[v_m] \subseteq V(M)$. $M' = \nabla(M, E(P))$ is a maximal matching with $|M'| = |M|$.*

Type 3: *an odd length alternating path, m is even, $v_1 \notin V(M)$, and $v_m \notin V(M)$. $M' = \nabla(M, E(P))$ is a maximal matching with $|M'| = |M| + 1$.*

Type 4: *an odd length alternating path, m is even, $N[v_1] \subseteq V(M)$, $N[_m] \subseteq V(M)$, and v_1 is not adjacent to v_m. $M' = \nabla(M, E(P))$ is a maximal matching with $|M'| = |M| - 1$.*

4. Bipartite Graphs with $\beta_1'(G) = \beta_1(G)$

Let G be a bipartite graph with partite sets A and B, where $|A| \leq |B|$. For a maximal matching M, let $X_M = V - V(M)$, and $M_0 = \{e : e \in M \text{ and } V(e) \cap N(X_M) = \emptyset\}$. The edges in M_0 have no neighbors in X_M, and are critical for graphs for which $\beta_1' = \beta_1$. It is possible for M_0 and X_M to be empty and, when $X_M = \emptyset$, $M_0 = M$.

Definition 17. *Let $\mathcal{B}$ be the set of bipartite graphs G such that for every maximal matching M, with $M_0 = \{e : e \in M$ and $N(V(e)) \subseteq V(M)\}$,*
(1) *each component of $[V(M_0)]$ is a complete bipartite graph, and*
(ii) *$V(M)$ contains an independent vertex cover of G.*

Theorem 18. *$G \in B$ if and only if G is bipartite and $\beta_1'(G) = \beta_1(G)$.*

Proof. Assume $G \in B$ and let M be a minimum matching of G. From Definition 17, G is bipartite and there is an independent vertex cover $A \subseteq V(M)$. Therefore, $\alpha(G) \leq |A|$. Since A is independent and a subset of $V(M)$, A cannot contain two endvertices of any edge, thus, $|A| \leq |M|$. Since M was selected to be a minimum matching, $|M| = \beta_1'(G) \leq \beta_1(G)$. From equation (1), $\beta_1(G) \leq \alpha(G)$. Therefore, $\alpha(G) \leq |A| \leq |M| = \beta_1'(G) \leq \beta_1(G) \leq \alpha(G)$. That is, $\beta_1'(G) = \beta_1(G)$.

Next, assume G is bipartite and that $\beta_1'(G) = \beta_1(G)$. Let M be an arbitrary maximal matching of G and partition M into M_0, M_1, and M_2 as follows.

$M_0 = \{vv' \colon vv' \in M, N(v) \subseteq V(M), \text{ and } N(v') \subseteq V(M)\}$,

$M_1 = \{vv' \colon vv' \in M, N(v) \not\subseteq V(M), \text{ and } N(v') \subseteq V(M)\}$, and

$M_2 = \{vv' \colon vv' \in M, N(v) \not\subseteq V(M), \text{ and } N(v') \not\subseteq V(M)\}$.

We first show that M_2 is empty. If not, there is an edge $vv' \in M_2$ with a vertex $x \in N(v) - V(M)$ and a vertex $y \in N(v') - V(M)$. Since G is bipartite, $x \neq y$ and $P = (x, v, v', y)$ forms an alternating path of Type 3 in Definition 16, a contradiction since $\nabla(M, E(P))$ would be a matching of G with $|M| + 1 > \beta_1(G)$

edges. Therefore, we may assume $M_2 = \emptyset$, and let $G_0 = [V(M_0)]$ and $G_1 = [V - V(M_0) >]$.

From Corollary 12, $\beta_1'(G_0) = \beta_1(G_0) = |M_0|$ and $\beta_1'(G_1) = \beta_1(G_1) = \beta_1(G) - |M_0|$. Since the number of vertices in G_0 is $2|M_0|$, by Theorem 8, each component of G_0 is a complete bipartite graph as required by (1) of Definition 17.

In G_1, let $A_1 = \{v : vv' \in M_1 \text{ and } N(v) \not\subset V(M)\}$ and $A_1' = V(M_1) - A_1$. Suppose v and w are adjacent members of A_1. Then, $P = (v', v, w, w')$ is a Type 4 path in Definition 16 and is a contradiction, since $\nabla(M, E(P))$ would be a matching of G with $|M| - 1 < \beta_1'(G)$ edges. Now, assume v and w are members of A_1, and v' is adjacent to w'. Then, v and w have neighbors x and y, respectively in $V - V(M)$. If $x = y$, then $\{x, v, v', w', w\}$ forms a C_5, contradicting G being bipartite. Therefore, $x \neq y$ and, since both are members of $V - V(M)$, x and y are not adjacent. Thus, $P = (x, v, v', w', w'y)$ is an alternating path of Type 3 in Definition 16 and, hence, $\nabla(M, E(P))$ would be a matching of G with $|M| + 1 > \beta_1'(G)$ edges, a contradiction. Therefore, A_1 is an independent vertex cover of G_1, and if $M_0 = \emptyset$, $G \in \mathcal{B}$.

When $M_0 \neq \emptyset$, let G_0' be an arbitrary component of G_0. Suppose two adjacent vertices, v and w, in G_0' have, respectively, neighbors x and y in A_1. Assume xx' and yy' are edges in M_1. Then, there is an odd length alternating path $P = (x', x, v, \ldots, w, y, y')$ where $x' \neq y'$, $N(x') \subseteq V(M)$, and $N(y') \subseteq V(M)$. That is, P is an alternating path of Type 4 in Definition 16, a contradiction. Suppose x and y are both members of $V(M_1) - A_1$. Again, we may assume xx' and yy' are edges in M_1. In this case, x' and y' are both members of $V - V(M)$, and x and y are both members of A_1 with distinct neighbors a and b, restrictively. Therefore, $P = (a, x, x', v, \ldots, w, y', y, b)$ is an alternating path of Type 3 in Definition 16. Again, this is a contradiction, and hence, every component in G_0 has a partite set with no neighbors in A_1. These sets, with A_1, is a set A of independent vertices in G and, since every edge has an endpoint in A, A is a vertex cover of G. This final result establishes that $G \in \mathcal{B}$. ■

Note: There are graphs with $\beta_1(G) = |A| > \beta_1'(G)$, such as P_4.

5. Vertex Covers and Edge Dominating Sets

A goal in this section is to characterize extremal bounds on the total vertex cover number as well as relating (total) vertex covers to edge dominating sets. We begin with a definition, and then state a result from [6] in different terms than given in [6]. A *stem* is a vertex that is adjacent to a degree one vertex. These vertices are sometimes called *support vertices* in the literature, but since we will also be referring to such vertices in trees (where degree one vertices are leaves), we shall simply refer to them as stems for notational consistency.

Theorem 19. *Let G be a connected graph with $n \geq 3$ vertices. Then, $\gamma(G) = \alpha_t(G)$ if and only if the set of stems is a total vertex cover.*

Proof. If S is a set of stems that forms a total vertex cover, then $\gamma(G) \leq \alpha_t(G) \leq |S|$. For any graph G, either each stem or all of its degree one neighbors must be in any dominating set. Thus, $|S| \leq \gamma(G)$, and equality holds.

If $\gamma(G) = \alpha_t(G)$ and C is an α_t-set, then C is also a minimum dominating set. Every vertex in a minimum dominating set must have a private neighbor that is not dominated by any other member of C. Since C is a total vertex cover of minimum cardinality, a vertex cannot be its own private neighbor, so the private neighbor must be a vertex in $V - C$ and, hence, has degree one. Thus, C is a set of stems. No stem can be in $V - C$. ■

Theorem 20. *For all graphs G, $\alpha_t(G) > \beta_1'(G)$.*

Proof. Let D be an independent minimum edge dominating set. Let C be a minimum total vertex cover. Suppose by way of contradiction that $|D| = |C| \geq 2$. C consists of one vertex from each edge in D. Then, $V(G) - C$ is an independent set since C is a vertex cover. Since C is a total vertex cover, there must exist two vertices $u, v \in C$ such that $uv \in E(G)$. Then $uv \notin D$ and thus, D is not a minimum edge dominating set, since there exist vertices $a, b \notin C$ such that $D' = (D - \{ua, vb\}) \cup \{uv\}$ is an edge dominating set. That is, uv is incident to any edges that ua and vb are, since a and b are independent. Furthermore, any other vertices adjacent to a or b must be in C, so any other edges besides uv that are incident to a are dominated by D'. ■

Theorem 21. *For all graphs G, $\alpha(G) = \beta_1'(G)$ if and only if G is bipartite and $\beta_1'(G) = \beta_1(G)$.*

Proof. When $\alpha(G) = \beta_1'(G)$, by equation (1), $\beta_1'(G) = \beta_1(G) = \alpha(G)$. Let D be a minimum matching and let C be a minimum vertex cover. Then, C consists of one vertex from each edge in D. Since the edges in D are independent, any vertex cover must include at least one vertex from each edge in the matching. Therefore, $V(G) - C$ is an independent set, since C is a vertex cover. Observe that C is an independent set, otherwise there is a smaller edge dominating set (see the argument used in Theorem 20). Since C is an independent set and $V - C$ is also an independent set, G is bipartite.

For the other direction, suppose G is bipartite and $\beta_1'(G) = \beta_1(G)$. Since $\alpha(G) = \beta_1(G)$ for all bipartite graphs, the theorem is true. ■

Definition 17 and the results of Theorems 18 and 21 can be summarized as follows.

Corollary 22. *For graphs G, the following are equivalent statements.*
(1) $G \in \mathcal{B}$.

(2) *$\alpha(G) = \beta_1'(G)$.*

(3) *$\beta_1'(G) = \beta_1(G)$ and G is bipartite.*

Our next goal is to characterize the graphs G satisfying $\alpha_t(G) = 2\alpha(G) - 1$.

Lemma 23. *For any α-set C of a graph G, and any $C' \subseteq C$, let X be a smallest $X \subseteq V - C$ for which X dominates C'. Then, $|X| \leq |C'|$ and further, $|X| = |C'|$ if and only if every vertex in $V - C$ has at most one neighbor in C'.*

Proof. Since C is a minimum vertex cover, every vertex in C has a neighbor in $V - C$. That is, $V - C$ dominates C', hence, there is a smallest subset X that dominates C'. Each vertex in X dominates a vertex in C' that is not dominated by any other vertex of X. Therefore, $|X| \leq |C'|$.

Suppose $v \in V - C$ has $m \geq 2$ neighbors in C'. Then, v and a smallest set of vertices from $V - C$ that dominates $C' - N(v)$ is a dominating set of C'. Thus, $|X| \leq 1 + |C' - N(v)| = |C'| - m + 1 < |C'|$ when $m \geq 2$. Therefore, $|X| = |C'|$ and vertices in $V - C$ have one neighbor in C'.

If every vertex in $V - C$ has at most one neighbor in C', then every dominating set of C' must have $|C'|$ vertices. ■

Definition 24. *A simple star is a $K_{1,m}$ with a degree $m \geq 2$ center vertex. A connected graph G, with $k = \alpha(G) \geq 2$ and α-set $C = \{w_0, w_1, w_2, \ldots, w_{k-1}\}$, is a k-star when the following conditions hold.*

(1) *C is an independent set*

(2) *Vertices in $V - C$ have degree at most two and every pair of degree two vertices in $V - C$ has a common neighbor in C.*

(3) *Assume, of the vertices in C, w_0 has a maximum number of degree two neighbors.*

(3.1) *If $N[w_0]$ contains every degree two vertex in $V - C$, then $G - \{w_0\}$ is a collection of zero or more isolated vertices and $k - 1$ simple stars with respective center vertices $w_1, w_2, \ldots, w_{k-1}$, otherwise*

(3.2) *$k = 3$, every vertex in C has degree at least three, and every pair of vertices in C has a common neighbor in $V - C$.*

(4) *deg(w_0) $\geq k$.*

Lemma 25. *If a graph G is a k-star, then $\alpha_t(G) = 2\alpha(G) - 1$.*

Proof. Let G be a k-star. By Theorem 1, since G is not a star, $\alpha_t(G) \leq 2\alpha(G) - 1$. When G satisfies (3.1) of Definition 24, w_0 is adjacent to every degree two vertex in $V - C$, and each $w_i, 1 \leq i \leq k - 1$, is adjacent to at least one degree two vertex in $V - C$. Let $X = \{x_1, x_2, \ldots, x_{k-1}\} \subseteq V - C$ where, for $1 \leq i \leq k - 1$,

$N(x_i) = \{w_0, w_i\}$. Suppose $\alpha_t(G) < 2\alpha(G) - 1$ and let $C' \subseteq V$ be an α_t-set of G, where $C' \cap C$ is as large as possible. Assume, for some $i > 0$, that $w_i \notin C'$. Then, $N(w_i) \subseteq C'$ and, since x_i has only w_i and w_0 as neighbors, in order that x_i have a neighbor in C', $w_0 \in C'$. Further, w_i cannot have a degree one neighbor, since it would be in C' without a neighbor in C', a contradiction. Since w_i is the center vertex of a simple star, $\deg(w_i) \geq 2$ and, hence, w_i must have a second degree two neighbor, say y, in C'. The only edge uniquely covered by y is yw_i. Since w_i also covers yw_i and w_i has x_i as a neighbor in C', $(C' - \{y\}) \cup \{w_i\}$ is a total vertex cover and also has $|C'|$ vertices, but with one more vertex from C than C' has from C, contradicting the assumption that C' contains the maximum possible. Therefore, we may assume C' contains $\{w_1, w_2, \ldots, w_{k-1}\} = C - \{w_0\}$.

For C' to be a total vertex cover, each w_i must have a neighbor in C'. Since w_0 is adjacent to every degree two vertex in $V - C$, vertices in $V - C$ have at most one neighbor in $C - \{w_0\}$. Therefore, from Lemma 23, all dominating sets of $C - \{w_0\}$ must contain exactly $k-1$ vertices from $V - C$, for example and without loss of generality, the set X formed above. Therefore, $C' \supseteq \{w_1, w_2, \ldots, w_{k-1}\} \cup \{x_1, x_2, \ldots, x_{k-1}\}$. That is, $|C'| \geq 2(k-1)$. Since $|C'| < 2k - 1$, $|C'| = 2k - 2$. Thus, C' cannot contain additional vertices and, in this case, $w_0 \notin C'$. Since $N(w_0) \cap \{w_1, w_2, \ldots, w_{k-1}\} = \emptyset$, w_0 has exactly $k - 1$ neighbors in C'. From Condition 4, $\deg(w_0) \geq k$, so w_0 must have at least one neighbor that is not in C', a contradiction, since C' would not be a total vertex cover.

Therefore, when G is a k-star and there is a vertex in an α-set that is adjacent to every degree two vertex not in the α-set, $\alpha_t(G) = 2\alpha(G) - 1$.

Now assume G is a k-star and Condition 3.1 of Definition 24 is not met. That is, G is a 3-star and, for $0 \leq i \leq 2$, $N(w_i)$ does not contain every degree two vertex of $V - C$, and every pair of vertices in C possess a common neighbor in $V - C$.

If $\alpha_t(G) < 2\alpha(G) - 1 = 5$, there exists a total vertex cover, C', with four vertices. If $C \subseteq C'$, then C' contains exactly one vertex $x \in V - C$. Since C is an independent set, to make C' a total vertex cover requires x to be adjacent to all three members of C, contradicting that vertices in $V - C$ have degree at most two.

Therefore, without loss of generality, we may assume $w_2 \notin C'$. Similar to previous arguments for vertices not in a total vertex cover, w_2 cannot have degree one neighbors, implying it must have exactly three degree two neighbors in C', which also must be members of $V - C$. Thus, C' contains exactly one member of C, say w_0. Therefore, w_1 is also not in C' and, hence, has no degree one neighbor, but has three neighbors in C', which must be the same set of neighbors as w_2. Then, one of these neighbors must also be adjacent to w_0, contradicting that vertices of $V - C$ have degree at most two. Thus, for this case, every total vertex cover of G requires at least five vertices and, again, $\alpha_t(G) = 2\alpha(G) - 1$, and completes the proof of the Lemma. ■

Theorem 26. *Let G be a connected graph. Then, G is a k-star if and only if $\alpha_t(G) = 2\alpha(G) - 1$.*

Proof. If G is a k-star, the conclusion follows from Lemma 25. Therefore, assume G is connected and that $\alpha_t(G) = 2\alpha(G) - 1$. With $k = \alpha(G)$, let $C = \{w_0, w_1, w_2, \dots, w_{k-1}\}$ be an arbitrary α-set of G. We show each condition of Definition 24 holds.

Condition 1. Suppose w_i and w_j are adjacent. A total vertex cover of G can be formed with $\{w_i, w_j\}$, $C - \{w_i, w_j\}$, and a smallest set $X \subseteq V - C$, where X dominates $C - \{w_i, w_j\}$. From Lemma 23, $|X| \leq |C - \{w_i, w_j\}| = k - 2$. Then, $\alpha_t(G) \leq 2+2(k-2) = 2k-2 = 2\alpha(G)-2 < 2(G)-1$, a contradiction. Therefore, C is an independent α-set.

Condition 2. Let x be an arbitrary largest degree vertex in $V - C$. A total vertex cover can be formed with $N[x]$, $C - N(x)$, and a smallest set $X \subseteq V - C$ that dominates $C-N(x)$. Again, from Lemma 23, $|X| \leq |C-N(x)| = k-\deg(x)$. Then, $2\alpha(G) - 1 = \alpha_t(G) \leq (1 + \deg(x)) + 2(k - \deg(x)) = 2k - \deg(x) + 1 = 2\alpha(G) - \deg(x) + 1$. Therefore, $\deg(x) \leq 2$. Since $\deg(x) \geq 2$, $\deg(x) = 2$.

In the previous argument, a contradiction also occurs when $|X| < |C-N(x)|$. Therefore, $|X| = |C-N(x)|$. Thus, no degree two vertex in $V-C$ can have both of its neighbors in $C - N(x)$, otherwise a smaller set X would exist. Therefore, x and every degree two vertex in $V - C$ have a common neighbor. Since x was selected arbitrarily from the set of degree two vertices in $V - C$, every pair of degree two vertices in $V - C$ has a common neighbor in C.

Condition 3. Suppose $x \in V - C$ has neighbors w_0 and w_1, appropriately relabeled so that w_0 has a maximum number of degree two neighbors. Then, every other degree two vertex in $V - C$ is adjacent to either w_0 or w_1. Then, either (Condition 3.1) every degree two vertex in $V - C$ is adjacent to w_0, or (Condition 3.2) there is a degree two vertex in $V - C$ that is not adjacent to w_0.

Condition 3.1. When w_0 is the common neighbor of every degree two vertex in $V-C$, in $G-\{w_0\}$ the neighbors of each w_i are degree one. That is, $N[w_i]$ induces a simple star with center vertex w_i. All other vertices (degree one neighbors of w_0 in G) are isolated vertices in $G - \{w_0\}$.

Condition 3.2. No vertex of C dominates every degree two vertex in $V - C$. If $k = 2$, every degree two vertex in $V - C$ has w_0 as a neighbor, Thus, $k \geq 3$. Every vertex in C has a degree two neighbor in $V - C$ and each pair of degree two vertices in $V - C$ has a common neighbor. Without loss of generality, we may assume, for $0 \leq i \neq j \leq 2$, that $x_{i,j}$ is a degree two vertex in $V - C$ with $N(x_{i,j}) = \{w_i, w_j\}$. Suppose $k \geq 4$. Then, w_3 must have a neighbor x other than

$x_{0,1}, x_{0,2}$, and $x_{1,2}$, but x must have a common neighbor with each of $x_{0,1}, x_{0,2}$ and $x_{1,2}$, an impossibility. Therefore, $k = 3$ and every pair of vertices in C has a common neighbor in $V - C$. Suppose, say, $\deg(w_2) = 2$. Then, $\{w_0, w_1, x_{0,2}, x_{1,2}\}$ is a total vertex cover, since the only edges not covered by w_0 and w_1 are the two that are incident to w_2, and these two are covered by $x_{0,2}$ and $x_{1,2}$. Hence, $\alpha_t(G) \leq 4 < 2\alpha(G) - 1 = 5$, a contradiction.

Condition 4. If $N(w_0)$ contains every degree two vertex in $V - C$ (Condition 3.1), then an α_t-set consists of $C \cup X$, where each $x_i \in X$ is adjacent to both x_0 and w_i. If $\deg(w_0) = k - 1$, then every edge incident to w_0 is covered by one of the $x_i \in X$. Therefore, $(C - \{w_0\}) \cup X$ is a total vertex cover with $2\alpha(G) - 2$ vertices, a contradiction.

In the proof for Condition 3.2, every vertex in C was shown to have at least three neighbors. Thus, $\deg(w_0) \geq 3 = k$. Since all conditions are satisfied, G is a k-star. ■

It would be of interest to characterize graphs having $\alpha_t(G) = 2\beta_1'(G)$ in a way that lends itself to an efficient algorithm, as opposed to Proposition 6. For instance, we state the following, where a subdivision of $K_{1,m}$ consists of m paths having $p_1, p_2, \ldots, p_m$ vertices, respectively, all sharing a common vertex v.

Proposition 27. *Let G be a subdivision of $K_{1,m}$. Then $\alpha_t(G) = 2\beta_1'$ if and only if at most one path has length congruent to* 2 (mod 3).

Proof. Number the edges of each path from 1 to p_i starting with v. If two or more paths have length 2 (mod 3), then we claim $\alpha_t(G) < 2\beta_1'$. To see this, form a total dominating set as follows: include both vertices of every third edge from the length 2 (mod 3) paths starting with the first edge of one of the paths (so this includes v) and starting with the third edge from the other length 2 (mod 3) paths, and include the stems of each of the length 2 (mod 3) paths. For all the other paths, we include both vertices from each edge of a minimum edge dominating set.

Otherwise, if G does not have two paths of length 2 (mod 3), we can find a minimum edge dominating set D such that no two edges in D have endvertices with a common neighbor. There are three cases. If there is one path of length 2 (mod 3), include the edge of that path containing v (i.e., the first edge of the path) and then every third edge of that path (starting with the fourth). For each of the other paths, include the third edge and every subsequent third edge. If there are no length 2 (mod 3) length paths, but there exists a path of length 1 (mod 3), include the first edge of the length 0 (mod 3) path, and then every third edge (starting with the fourth) and every third edge (starting with the third edge) of each path of length 0 (mod 3). If all the paths have length 0 (mod 3), include the second edge of each path and then every third edge after that (starting with the fifth). ■

It would be interesting to characterize G having $\alpha(G) = \alpha_t(G)$. An elegant characterization of trees with this property seems difficult, however. We mention two special cases. Let T be a tree with stem set S such that every stem is adjacent to another stem. Then, $\alpha(T) = \alpha_t(T)$ if and only if the subgraph induced by $G - N[S]$ consists of zero or more independent edges. To see this, note that in such cases the subgraph induced by $G - S$ contains leaves and P_4's, each P_4 can be covered by two adjacent vertices, and the edges between the stems and the leaves or P_4's are covered by the stems.

On the other hand, if T contains no adjacent stems, $\alpha(T) < \alpha_t(T)$, since in order for $\alpha(T) = \alpha_t(T)$, each stem would have to be adjacent to a leaf or an interior vertex (i.e., a vertex that is not a leaf or a stem) in a minimum vertex cover. We can assume there are no leaves in a minimum vertex cover (unless $T = K_2$) and any such interior vertex would have to be the endvertex of a path in the subgraph induced by $G - S$, which cannot be the case.

As a result, subdivisions of stars satisfy $\alpha(G) = \alpha_t(G)$ only when every p_i is of length one or two with at least one of each length.

6. Total Domination

The following was asked in [6]:

Question 2. *For which graphs G is $\gamma_t(G) = \alpha_t(G)$?*

As pointed out in [6], it is not true that $\alpha(G) = \gamma(G)$ implies $\gamma_t = \alpha_t(G)$. To see this, consider C_4 with one pendant vertex v attached to one of the vertices of the cycle. Thus graph G has $\alpha(G) = \gamma(G) = 2$, but $\gamma_t(G) = 2$ and $\alpha_t(G) = 3$. On the other hand, $\gamma(K_3) < \alpha(K_3)$ and $\gamma_t(K_3) = \alpha_t(K_3)$.

Theorem 28. *If $\delta(G) \geq 2$ and $\gamma_t(G) = \alpha_t(G)$, then $\alpha_t(G) = 2\beta_1'(G)$.*

Proof. Suppose $\delta(G) \geq 2$ and $\gamma_t(G) = \alpha_t(G)$. Let C be an α_t-set, so C is also a γ_t-set. Then $V - C$ is an independent set. Suppose by way of contradiction that $[C]$ does not contain a perfect matching. Let M be a maximum matching in $[C]$. Let v be a vertex in C such that $v \notin V(M)$. We claim $C - v$ is a total dominating set. Since every vertex in $V - C$ has degree at least two, $C - v$ dominates $V - C$. Vertex v must have a neighbor in C, since C is a total dominating set. If u is a vertex in C and v was its only neighbor in C, then M would not be a maximum matching in $[C]$, as $M \cup uv$ would be a larger matching. Thus, $C - v$ is a total dominating set smaller than C, a contradiction. Therefore, $[C]$ contains a perfect matching and by Corollary 5, $\alpha_t(G) = 2\beta_1'(G)$. ∎

Theorem 29. *For any tree T, $\alpha_t(T) \leq \frac{3}{2}\gamma_t(T) - 1$.*

Proof. Consider a tree T on n vertices with minimum a total dominating set D. Let $D_1, D_2, \ldots, D_k,\ k \geq 1$, be the distinct maximal connected subgraphs of T induced by D. Note that each such D_i contains at least two vertices of D, since D is a total dominating set. This $1 \leq k \leq n/2$.

Each $[D_i]$ contains exactly $|D_i| - 1$ edges, since it is a subtree of T with $|D_i|$ vertices. Thus, the set D covers exactly $|D| - k$ edges. There are $n - |D|$ vertices not in D, and each is adjacent to at least one vertex contained in some D_i set. That is, D covers at least another $n - |D|$ edges, for a total of $n - |D| + |D| - k = n - k$ covered edges. Therefore, there are at most $|E| - (n-k) = n-1-n+k = k-1$ uncovered edges. Thus, D and at most $k - 1$ additional vertices (one endvertex of each uncovered edge) from $V - D$ is sufficient to cover all edges of T, that is, a vertex cover of T. Since D is a total dominating set, D and any subset of $V - D$ is a total vertex cover. Therefore, $\alpha_t(T) \leq |D| + k - 1 \leq |D| + |D|/2 - 1 = 3|D|/2 - 1$. ■

A path P_8 shows the bound of Theorem 29 is sharp. The proof of Theorem 29 shows that any tree having $\alpha_t(T) = \frac{3}{2}\gamma_t(T) - 1$ must have each D_i with exactly two vertices and each vertex in $V - D$ has exactly one neighbor in D. This implies that P_8 is the smallest tree with $\alpha_t(T) = \frac{3}{2}\gamma_t(T) - 1$ and $\alpha_t(T) > \gamma_t(T)$. In general, these trees can be formed by taking any number of trees of diameter three and connecting them together until a tree is formed by repeatedly adding an edge between the leaves of two distinct components.

There exist bipartite graphs where $\frac{\alpha_t}{\gamma_t}$ is arbitrarily large: $K_{m,n}$, when m and n are sufficiently large for one example. For another, take a P_4 and combine it with itself using a parallel construction a sufficient number of times to make the ratio large.

Let $P_m \times P_n$ denote the $m \times n$ grid graph. Let us first show that $\alpha_t(P_2 \times P_n) \leq \lceil \frac{4n}{3} \rceil$. Number the columns of $P_2 \times P_n$ from 1 to n from left to right. If $n \equiv 0 \pmod 3$, the following pattern can be used. ("x" indicates a vertex in the total vertex cover).

```
0--x--0--x--x--x--0--x--0
x--x--x--0--x--0--x--x--x
```

If $n \equiv 1 \pmod 3$, we use the same pattern for the first $n - 1$ rows, but add both vertices from column n. If $n \equiv 2 \pmod 3$, we use the pattern for $n \equiv 1 \pmod 3$ (i.e., two vertices in column $n - 1$) and add one vertex from column n. This shows that $\alpha_t(P_2 \times P_n) \leq \frac{4n+2}{3} = \lceil \frac{4n}{3} \rceil$.

Proposition 30. $\alpha_t(P_2 \times P_n) = \lceil \frac{4n}{3} \rceil$.

Proof. The argument above gives the upper bound. For the lower bound, it is now proved that $\alpha_t(P_2 \times P_n) \geq \lceil \frac{4n}{3} \rceil$. The proof is by induction on n. We establish four base cases: $n = 1$, is trivial, as $\lceil \frac{4}{3} \rceil = 2$. For $n = 2$, three vertices are needed and $\lceil \frac{8}{3} \rceil = 3$. For $n = 3$, four vertices are needed, and $\lceil \frac{12}{3} \rceil = 4$. Lastly, for $n = 4$, six vertices are needed and $\lceil \frac{16}{3} \rceil = 6$.

Now consider $P_2 \times P_n$ and assume that for any $m < n$, a total vertex cover of $P_2 \times P_m$ requires at least $\frac{4m}{3}$ vertices. Let C be a total vertex cover of $P_2 \times P_n$. First suppose there are two columns, k and $k+1$, each containing two vertices of C. Then, the first k columns of C are a total vertex cover of a $P_2 \times P_k$ and the last $n - k$ columns of C are a total vertex of a $P_2 \times P_{n-k}$. Therefore $|C| \geq \frac{4k}{3} + \frac{4(n-k)}{3} \geq \frac{4n}{3}$.

Next suppose no two adjacent columns each contain two vertices of C. So, suppose two adjacent columns, k and $k+1$, contain one vertex each from C. Since C is a total vertex cover, that means columns $k-1$ and $k+2$ must each have two vertices from C. This is because the vertices from C in columns k and $k+1$ must lie in different rows. Again, this implies the first k columns of C form a total vertex cover for a $P_2 \times P_k$ and the last $n-k$ columns form a total vertex cover for a $P_2 \times P_{n-k}$. Thus, $|C| \geq \frac{4k}{3} + \frac{4(n-k)}{3} \geq \frac{4n}{3}$.

Lastly, if neither of the first two possibilities occur, then the configuration of vertices in C alternates between columns with one vertex from C and columns with two vertices from C. Then, $|C| \geq n + \lfloor \frac{n}{2} \rfloor \geq \frac{4n}{3}$, as $n \geq 3$. ■

We now examine $P_3 \times P_n$. If $n \in \{1, 2, 4\}$, it is easy to see that $\alpha_t(P_3 \times P_n) = 2n$. For $n = 3$ or $n > 4$, it is now shown that $\alpha_t(P_3 \times P_n) \leq 2n - 1$. When $n \geq 3$ and n odd, use the following pattern $(1, 3, 1, 3, \ldots, 1, 3, 1)$, where each value indicates the number of vertices in a total vertex cover from the respective columns. A "1" means the middle vertex from the column is chosen and a "2" means the upper and lower vertices are chosen. For $n \geq 6$ and n even, use the following pattern $(1, 3, 1, 3, ..., 1, 3, 1, 2, 3, 1)$. The following diagram illustrates when $n = 12$. ("x" indicates a vertex in the total vertex cover).

```
0--x--0--x--0--x--0--x--0--x--x--0
x--x--x--x--x--x--x--x--x--0--x--x
0--x--0--x--0--x--0--x--0--x--x--0
```

Proposition 31. *For $n = 3$ or $n > 4$, $\alpha_t(P_3 \times P_n) = 2n - 1$.*

Proof. The argument above gives the upper bound and also establishes that $\alpha_t(P_3 X \times P_n) = 2n - 1$ for $1 \leq n \leq 4$.

Hence, we may assume $n \geq 5$ and that $\alpha_t(P_3 \times P_m) \geq 2m - 1$, for $1 \leq m < n$. Let C be an α_t-set of $P_3 \times P_n$.

Suppose there are two columns, k and $k+1$, where one column contains one vertex from C and the other contains two vertices from C. Notice that $2 \leq k \leq n - 1$ and we may assume without loss of generality that column k

contains one vertex from C. The configuration of C vertices in the first k columns is a total vertex cover of the first k columns, that is, of a $P_3 \times P_k$. This is because the one vertex from C in column k cannot be adjacent to either vertex from C in column $k+1$, otherwise the edges between columns k and $k+1$ are not covered. Thus, there are at least $2k-1$ vertices in C in the leftmost k columns.

Observe that a column with only one vertex in C must have that vertex in the middle row. The configuration of vertices from C in the rightmost $n-k$ columns is a total vertex cover for the rightmost $n-k$ columns, but the two vertices in column $k+1$, when considering these columns as a $P_3 \times P_{n-k}$, can be replaced with a single vertex in the middle row of the first column of this $P_3 \times P_{n-k}$ (since in this case, the two vertices in column $k+1$ must be in the top and bottom rows). That is, there are at least $2(n-k)$ entries of C in columns $k+1, k+2, \ldots, n$ of the $P_3 \times P_n$. Therefore, C contains at least $(2k-1)+2(n-k) = 2n-1$ vertices.

Suppose no column with exactly one vertex from C is next to a column with just two entries from C. First, assume there is a column k, $1 < k < n$, with only one vertex in C. Then, columns $k-1$ and $k+1$ both have three vertices from C. As above, the first k columns contain a total vertex cover for a $P_3 \times P_k$ and the last $n-(k-1)$ columns contain a total vertex cover of a $P_3 \times P_{n-k+1}$. Notice, the vertex in C in column k appears in both configurations and is "counted" twice. Therefore, C must have at least $(2k-1)+(2(n-k+1)-1)-1 = 2n-1$ vertices.

If the only column(s) with a single vertex from C is(are) in columns 1 and/or n, we may assume the top and bottom rows each contain at least $n-2$ vertices of C, and the middle row must contain at least four vertices from C. That is, at least $2(n-2)+4 = 2n \geq 2n-1$.

Finally, if there are no columns with just one C vertices, all columns must contain either two or three vertices from C. That is, C contains at least $2n$ vertices. ■

Let $m \leq n$. When m or n is a multiple of 3 and the other is odd, it seems that $\alpha_t(P_m \times P_n) \leq \frac{2mn}{3} - 1$. If one of m, n is a multiple of three and the other is even, it seems that $\alpha_t(P_m \times P_n) \leq \frac{2mn}{3}$. Notice, this formula gives 23 when $m = 3$ and $n = 12$, agreeing with the diagram above. Otherwise, we think that $\alpha_t(P_m \times P_n) \leq \frac{2mn+m-3}{3}$, but do not necessarily believe this is optimal in all cases. For example, if neither m nor n is a multiple of 3 and $n \equiv 2 (\bmod 3)$, then we think that $\alpha_t(P_m \times P_n) \leq \frac{2mn-m+3\lfloor \frac{m}{2} \rfloor}{3}$.

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