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UNDERLYING GRAPHS OF 3-QUASI-TRANSITIVE DIGRAPHS AND 3-TRANSITIVE DIGRAPHS

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Abstract

A digraph is 3-quasi-transitive (resp. 3-transitive), if for any path $x_0x_1x_2x_3$ of length 3, x_0 and x_3 are adjacent (resp. x_0 dominates x_3). César Hernández-Cruz conjectured that if D is a 3-quasi-transitive digraph, then the underlying graph of D, $UG(D)$, admits a 3-transitive orientation. In this paper, we shall prove that the conjecture is true.

Keywords: graph orientation, 3-quasi-transitive digraph, 3-transitive digraph.

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1. Terminology and Introduction

We only consider finite graphs and digraphs without loops and multiple edges or multiple arcs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *complete graph* is a graph in which any two vertices are adjacent. A *complete bipartite graph* G is a graph in which the vertices of G can be partitioned into two partite sets such that every partite set is an independent set and for every pair x, y of vertices from distinct partite sets, $(x, y) \in E(G)$.

Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. For any $x, y \in V(D)$, we will write $\overrightarrow{xy}$ or $x \to y$ if $xy \in A(D)$, and also, we will write $\overline{xy}$ if $\overrightarrow{xy}$ or $\overrightarrow{yx}$. For disjoint subsets X and Y of $V(D)$ or subdigraphs of D, $X \to Y$ means that every vertex of X dominates every vertex of Y, $X \Rightarrow Y$ means that there is no arc from Y to X and $X \mapsto Y$ means that both of $X \to Y$ and $X \Rightarrow Y$ hold. Let D' be a subdigraph of D and $x \in V(D) - V(D')$. We say that x and D' are *adjacent* if x and some vertex of D' are adjacent.

For any digraph D, we can associate a graph G on the same vertex set simply by replacing each arc by an edge with the same ends. This graph is the *underlying graph* of D, denoted $UG(D)$. By a *path* of a digraph D, we mean a directed path of D. The length of a path is the number of its arcs. A path of length k is called a *k-path*; the path is odd or even according to the parity of k. A digraph D is said to be *strongly connected* or just *strong*, if for every pair x, y of vertices of D, there is a path from x to y. A *strong component* of a digraph D is a maximal induced subdigraph of D which is strong. The *strong component digraph $SC(D)$* of D is obtained by contracting strong components of D and deleting any parallel arcs obtained in this process.

A digraph D is *semicomplete* if there is at least one arc between any pair of distinct vertices of D. A *tournament* is a semicomplete digraph with no cycle of length 2. A digraph D is *semicomplete bipartite*, if the vertices of D can be partitioned into two partite sets such that every partite set is an independent set and for every pair x, y of vertices from distinct partite sets, xy or yx (or both) is in D. A *bipartite tournament* is a semicomplete bipartite digraph with no cycle of length 2. A digraph is *k-quasi-transitive*, where $k \geq 2$, if for any path $x_0x_1x_2 \dots x_k$ of length k, x_0 and x_k are adjacent. A 2-quasi-transitive digraph is also called a *quasi-transitive digraph.* A 3-quasi-transitive digraph is also called a quasi-arc-transitive digraph (see [7]). A digraph is *k-transitive*, where $k \geq 2$, if for any path $x_0x_1x_2 \dots x_k$ of length k, x_0 dominates x_k. A 2-transitive digraph is also called a *transitive digraph.* k-transitive digraphs and k-quasi-transitive digraphs have been studied by several authors. See [1, 2, 3, 5, 6, 7]. For a graph G, a digraph D is called an *orientation* of G if D is obtained from G by replacing each edge (x, y) of G by either xy or yx.

In [4], see also [2], Ghouila-Houri proved the following theorem.

Theorem 1 [2, 4]**.** *A graph G has a quasi-transitive orientation if and only if it has a transitive orientation.*

It seems natural to consider an analogue of Theorem 1 for 3-quasi-transitive digraphs and 3-transitive digraphs.

In [3], the author proposed the following conjecture.

Conjecture 2 [3]**.** *Let D be a 3-quasi-transitive oriented graph. Then the underlying graph of D, $UG(D)$, admits a 3-transitive orientation.*

In Section 2, we will prove that the conjecture is true.

2. Main Result

We begin with some useful lemmas. Let F_n be the digraph with vertex set $\{x_0, x_1, \dots, x_n\}$ and arc set $\{x_0x_1, x_1x_2, x_2x_0\} \cup \{x_0x_{i+3}, x_{i+3}x_1 : i = 0, 1, \dots, n-3\}$, where $n \geq 3$.

Lemma 3 [5]**.** *Let D be a strong 3-quasi-transitive digraph of order n. Then D is either a semicomplete digraph, a semicomplete bipartite digraph or isomorphic to F_n.*

Lemma 4 [7]**.** *Let D' be a non-trivial strong induced subdigraph of a 3-quasi-transitive digraph D. For any $s \in V(D) - V(D')$, if there exists a directed path between s and D', then s and D' are adjacent.*

Lemma 5 [5]**.** *Let D be a 3-quasi-transitive digraph. For a pair x, y of $V(D)$, if there exists an (x, y)-path of odd length, then x and y are adjacent.*

Lemma 6 [7]**.** *Let D' be a non-trivial strong induced subdigraph of a 3-quasi-transitive digraph D and let $s \in V(D) - V(D')$ with at least one arc from s to D' and $s \Rightarrow D'$. Then each of the following holds:*

(a) *If D' is a bipartite digraph with bipartition $(X; Y)$ and s dominates a vertex of X, then $s \mapsto X$.*

(b) *If D' is a non-bipartite digraph, then $s \mapsto D'$.*

Lemma 7 [7]**.** *Let D' be a non-trivial strong induced subdigraph of a 3-quasi-transitive digraph D and let $s \in V(D) - V(D')$ with at least one arc from D' to s and $D' \Rightarrow s$. Then each of the following holds:*

(a) *If D' is a bipartite digraph with bipartition (X, Y) and there exists a vertex of X which dominates s, then $X \mapsto s$.*

(b) *If D' is a non-bipartite digraph, then $D' \mapsto s$.*

Lemma 8 [7]**.** *Let D_1 and D_2 be two distinct non-trivial strong components of a 3-quasi-transitive digraph with at least one arc from D_1 to D_2. Then either $D_1 \mapsto D_2$ or $D_1 \cup D_2$ is a semicomplete bipartite digraph.*

Lemma 9. *If a graph G has a strong 3-quasi-transitive orientation, then it has a transitive orientation.*

Proof. Let D be a strong 3-quasi-transitive orientation of G and let the order of D be n. By Lemma 3, D is either a tournament, a bipartite tournament or isomorphic to F_n because D has no cycle of length 2. If D is a tournament (resp. a bipartite tournament), then G is a complete graph (resp. a complete bipartite graph). Any acyclic orientation of a complete graph is transitive and orienting the edges of a complete bipartite graph from one side of the bipartition to the other results in a transitive orientation. Now suppose that D is isomorphic to F_n with vertex set $\{x_0, x_1, \ldots, x_n\}$ and arc set $\{x_0x_1, x_1x_2, x_2x_0\} \cup \{x_0x_{i+3}, x_{i+3}x_1 : i = 0, 1, \ldots, n-3\}$, where $n \geq 3$. We reorient D as a digraph D' with arc set $\{x_0x_1\} \cup \{x_{i+2} \to x_0, x_{i+2} \to x_1 : i = 0, 1, \ldots, n-2\}$. Clearly, D' is a transitive digraph. The proof of Lemma 9 is complete. ■

Lemma 10 [3]**.** *If D is a k-transitive digraph with $k \geq 2$, then D is $(k+n(k-1))$-transitive for any $n \geq 1$ such that $k+n(k-1) \leq diam(D)$, where $diam(D)$ is the diameter of D.*

The following theorem is our main result.

Theorem 11. *A graph G has a 3-quasi-transitive orientation if and only if it has a 3-transitive orientation.*

Proof. Since every 3-transitive digraph is also a 3-quasi-transitive digraph, the sufficiency is trivial.

We shall prove the necessity below. Suppose that D is a 3-quasi-transitive orientation of G and D is not a 3-transitive orientation. If D is strong, then, by Lemmas 9 and 10, we are done. Suppose now that D is non-strong and $D_1, D_2, \ldots, D_t$ are its strong components. Note that every D_i, $i = 1, 2, \ldots, t$, is also a strong 3-quasi-transitive digraph. Hence, according to Lemma 9, every D_i, for $i = 1, 2, \ldots, t$, can be reoriented as a transitive digraph. Now, we reorient every D_i as a transitive digraph D_i' as in the proof of Lemma 9 and keep the directions of remaining arcs in D. Denote the resulting digraph by D'. From Lemma 10, we know that if a digraph is transitive, then it must be 3-transitive. Hence D_i' is 3-transitive, $i = 1, 2, \ldots, t$. Now we shall show that D' is 3-transitive. It suffices to prove that for any path $x_0x_1x_2x_3$ in D', $x_0 \to x_3$ in D'. By the definition of D', we can see that D' is acyclic. Hence it is sufficient to show that $\overline{x_0x_3}$ in D'. Observe that $\overline{x_0x_3}$ in D' if and only if $\overline{x_0x_3}$ in D. Hence we shall prove that $\overline{x_0x_3}$ in D or $\overline{x_0x_3}$ in D'. Furthermore, in order to show that $\overline{x_0x_3}$ in D, by Lemma 5, we only need to prove that there is an odd path from x_0 to x_3 in D.

If x_0 and x_3 belong to the same strong component in D, say D_i, then x_1 and x_2 both belong to D_i, otherwise, assume, without loss of generality, that $x_1 \in V(D_j)$ where $i \neq j$. Because the arcs of D between distinct strong components are not reoriented, it would be the case that D_i can reach D_j and D_j also can reach D_i in D, contradicting that they are distinct strong components. Since $x_k \in V(D_i)$, for $k = 0, 1, 2, 3$ and D_i' is 3-transitive, we have $x_0 \to x_3$.

Now assume that x_0 and x_3 belong to distinct strong components and assume, without loss of generality, that $x_0 \in V(D_i)$ and $x_3 \in V(D_j)$ with $1 \leq i \neq j \leq t$. The following two claims will be useful.

Claim 1. *D_j is reachable from D_i in D.*

Proof. It suffices to show that there exists a path from D_i to D_j in D. If $x_2 \in V(D_i)$ ($x_1 \in V(D_j)$), then x_2x_3 (x_0x_1) is the desired path. So suppose $x_2 \notin V(D_i)$ and $x_1 \notin V(D_j)$. If $x_2 \in V(D_j)$, then since $x_1 \notin V(D_j)$, $x_1x_2 \in A(D)$. If $x_0x_1 \in A(D)$, then $x_0x_1x_2$ is the desired path; if not, then $x_1 \in V(D_i)$ and so

x_1x_2 is the desired path. Thus suppose $x_2 \in V(D_s)$, with $1 \leq s \leq t$ and $s \neq i, j$. So $x_2x_3 \in A(D)$. If $x_1 \in V(D_i)$, then $x_1x_2x_3$ is a path from D_i to D_j in D. Thus we may assume that $x_1 \notin V(D_i)$ which implies $x_0x_1 \in A(D)$. If $x_1x_2 \in A(D)$, then $x_0x_1x_2x_3$ is the desired path; if not, then x_1 and x_2 both belong to D_s. Since D_s is strong, there exists a path P from x_1 to x_2 in D and then $x_0x_1Px_2x_3$ is the desired path. □

Claim 2. *If $x_1, x_2 \notin V(D_i) \cup V(D_j)$, then there exists an odd path from x_0 to x_3 in D.*

Proof. Since $x_1, x_2 \notin V(D_i) \cup V(D_j)$, $x_0x_1, x_2x_3 \in A(D)$. If $x_1x_2 \in A(D)$, then $x_0x_1x_2x_3$ is the desired path. Now assume that $x_1x_2 \notin A(D)$. Then by the definition of D', we have that $x_2x_1 \in A(D)$ and x_1, x_2 belong to the same strong component in D, say D_k. If D_k is a non-bipartite digraph, then by Lemmas 6 and 7, $x_0 \mapsto D_k$ and $D_k \mapsto x_3$ and in particular, $x_0 \to x_2$ and $x_1 \to x_3$ in D. Note that $x_0x_2x_1x_3$ is a path length 3 in D. If D_k is a bipartite digraph, then x_1 and x_2 belong to different partite sets. Again since D_k is a strong bipartite digraph, there exists an odd path P from x_1 to x_2 in D. Then we have that $x_0x_1Px_2x_3$ is an odd path from x_0 to x_3 in D. Thus the claim holds. The proof of Claim 2 is complete. □

We consider two cases.

Case 1. At least one of D_i and D_j is trivial, say D_i. Since D_i is trivial, by the definition of D', $x_0x_1 \in V(D)$. If $V(D_j)$ is also trivial, then $x_1, x_2 \notin V(D_i) \cup V(D_j)$. By Claim 2, we are done. Now assume that D_j is non-trivial. By Claim 1, Lemma 4 and the definition of strong components, there exists at least an arc from x_0 to D_j. If D_j is a non-bipartite digraph, then by Lemma 6, $x_0 \mapsto D_j$. In particular, $x_0 \to x_3$ and so we are done.

Now suppose that D_j is a bipartite digraph. Assume that (X_j, Y_j) is the bipartition of D_j and assume, without loss of generality, that $x_3 \in X_j$. By Lemma 6, $x_0 \mapsto X_j$ or $x_0 \mapsto Y_j$. If $x_0 \mapsto X_j$, then $x_0 \to x_3$ and so we are done. Suppose that $x_0 \mapsto Y_j$.

Subcase 1.1. $x_2 \in V(D_j)$. Since x_2 and x_3 are adjacent, we have $x_2 \in Y_j$. Since x_1 and x_2 are adjacent, we have that $x_1 \notin Y_j$. If $x_1 \in X_j$, then by $x_0x_1 \in A(D)$ and Lemma 6, $x_0 \mapsto X_j$. In particular, $x_0 \to x_3$ and so we are done. Now assume that $x_1 \notin V(D_j)$ and so $x_1x_2 \in A(D)$. Since D_j is a strong bipartite digraph, there exists an odd path P from x_2 to x_3 in D_j. Then $x_0x_1x_2Px_3$ is an odd path from x_0 to x_3.

Subcase 1.2. $x_2 \notin V(D_j)$. Since $x_2 \notin V(D_j)$, we have $x_2x_3 \in A(D)$. By the definition of strong components, $x_1 \notin V(D_j)$. Combining this with Claim 2, there exists an odd path from x_0 to x_3 and so we are done.

Case 2. D_i and D_j are both non-trivial. By Claim 1 and Lemma 4, there exists at least an arc from D_i to D_j. By Lemma 8, we have $D_i \mapsto D_j$ or $D_i \cup D_j$ is a bipartite tournament. If $D_i \mapsto D_j$, then $x_0 \to x_3$ in D and so we are done. If $D_i \cup D_j$ is a bipartite tournament, then D_i and D_j are both bipartite. Assume that the bipartitions of D_i and D_j are (X_i, Y_i) and (X_j, Y_j), respectively and the bipartition of $D_i \cup D_j$ is $(X_i \cup X_j,\, Y_i \cup Y_j)$. Assume, without loss of generality, that $x_0 \in X_i$. If $x_3 \in Y_j$, then $x_0x_3 \in A(D)$ and so we are done. Suppose that $x_3 \in X_j$.

Subcase 2.1. $x_2 \in V(D_j)$. Since x_2 and x_3 are adjacent, $x_2 \in Y_j$. This implies that $x_1 \notin V(D_j) \cup V(D_i)$, which follows from the fact that $D_i \cup D_j$ is bipartite. Thus we have $x_0x_1, x_1x_2 \in A(D)$. Since D_j is a strong bipartite digraph, there is an odd path P from x_2 to x_3 in D_j. Then $x_0x_1x_2Px_3$ is an odd path from x_0 to x_3 in D and so we are done.

Subcase 2.2. $x_2 \notin V(D_j)$. So $x_2x_3 \in A(D)$. By the definition of strong components, $x_1 \notin V(D_j)$.

If $x_2 \in V(D_i)$, then since x_2 and x_3 are adjacent, $x_2 \in Y_i$. Since x_1 and x_0, x_2 are both adjacent, $x_1 \notin V(D_i)$, which implies that $x_0x_1, x_1x_2 \in A(D)$, a contradiction to the definition of strong components. Thus $x_2 \notin V(D_i)$.

If $x_1 \in V(D_i)$, then since x_0 and x_1 are adjacent, $x_1 \in Y_i$ and $x_1x_2 \in A(D)$. Since D_i is a strong bipartite digraph, there is an odd path P from x_0 to x_1 in D_i. Then $x_0Px_1x_2x_3$ is an odd path from x_0 to x_3 in D and so we are done. Assume that $x_1 \notin A(D_i)$.

Note that now $x_1, x_2 \notin V(D_i) \cup V(D_j)$. By Claim 2, there exists an odd path from x_0 to x_3 in D and so we are done.

We have considered all the cases. The proof of Theorem 11 is complete. ■

Conjecture 2 then is an immediate consequence of Theorem 11.

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