UNDERLYING GRAPHS OF 3-QUASI-TRANSITIVE DIGRAPHS AND 3-TRANSITIVE DIGRAPHS

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Abstract

A digraph is 3-quasi-transitive (resp. 3-transitive), if for any path \(x_0x_1x_2x_3\) of length 3, \(x_0\) and \(x_3\) are adjacent (resp. \(x_0\) dominates \(x_3\)). César Hernández-Cruz conjectured that if \(D\) is a 3-quasi-transitive digraph, then the underlying graph of \(D, UG(D)\), admits a 3-transitive orientation. In this paper, we shall prove that the conjecture is true.

Keywords: graph orientation, 3-quasi-transitive digraph, 3-transitive digraph.

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1. Terminology and Introduction

We only consider finite graphs and digraphs without loops and multiple edges or multiple arcs. Let \(G\) be a graph with vertex set \(V(G)\) and edge set \(E(G)\). A complete graph is a graph in which any two vertices are adjacent. A complete bipartite graph \(G\) is a graph in which the vertices of \(G\) can be partitioned into two partite sets such that every partite set is an independent set and for every pair \(x, y\) of vertices from distinct partite sets, \((x, y) \in E(G)\).

Let \(D\) be a digraph with vertex set \(V(D)\) and arc set \(A(D)\). For any \(x, y \in V(D)\), we will write \(\overrightarrow{xy}\) or \(x \rightarrow y\) if \(xy \in A(D)\), and also, we will write \(\overleftarrow{xy}\) if \(\overrightarrow{xy}\) or \(\overrightarrow{yx}\). For disjoint subsets \(X\) and \(Y\) of \(V(D)\) or subdigraphs of \(D, X \rightarrow Y\) means that every vertex of \(X\) dominates every vertex of \(Y\), \(X \Rightarrow Y\) means that there is no arc from \(Y\) to \(X\) and \(X \Rightarrow Y\) means that both of \(X \rightarrow Y\) and \(X \Rightarrow Y\) hold. Let \(D'\) be a subdigraph of \(D\) and \(x \in V(D) - V(D')\). We say that \(x\) and \(D'\) are adjacent if \(x\) and some vertex of \(D'\) are adjacent.
For any digraph $D$, we can associate a graph $G$ on the same vertex set simply by replacing each arc by an edge with the same ends. This graph is the underlying graph of $D$, denoted $UG(D)$. By a path of a digraph $D$, we mean a directed path of $D$. The length of a path is the number of its arcs. A path of length $k$ is called a $k$-path; the path is odd or even according to the parity of $k$. A digraph $D$ is said to be strongly connected or just strong, if for every pair $x, y$ of vertices of $D$, there is a path from $x$ to $y$. A strong component of a digraph $D$ is a maximal induced subdigraph of $D$ which is strong. The strong component digraph $SC(D)$ of $D$ is obtained by contracting strong components of $D$ and deleting any parallel arcs obtained in this process.

A digraph $D$ is semicomplete if there is at least one arc between any pair of distinct vertices of $D$. A tournament is a semicomplete digraph with no cycle of length 2. A digraph $D$ is semicomplete bipartite, if the vertices of $D$ can be partitioned into two partite sets such that every partite set is an independent set and for every pair $x, y$ of vertices from distinct partite sets, $xy$ or $yx$ (or both) is in $D$. A bipartite tournament is a semicomplete bipartite digraph with no cycle of length 2. A digraph is $k$-quasi-transitive, where $k \geq 2$, if for any path $x_0x_1x_2\ldots x_k$ of length $k$, $x_0$ and $x_k$ are adjacent. A 2-quasi-transitive digraph is also called a quasi-transitive digraph. A 3-quasi-transitive digraph is also called a quasi-arc-transitive digraph (see [7]). A digraph is $k$-transitive, where $k \geq 2$, if for any path $x_0x_1x_2\ldots x_k$ of length $k$, $x_0$ dominates $x_k$. A 2-transitive digraph is also called a transitive digraph. $k$-transitive digraphs and $k$-quasi-transitive digraphs have been studied by several authors. See [1, 2, 3, 5, 6, 7]. For a graph $G$, a digraph $D$ is called an orientation of $G$ if $D$ is obtained from $G$ by replacing each edge $(x, y)$ of $G$ by either $xy$ or $yx$.

In [4], see also [2], Ghouila-Houri proved the following theorem.

**Theorem 1** [2, 4]. A graph $G$ has a quasi-transitive orientation if and only if it has a transitive orientation.

It seems natural to consider an analogue of Theorem 1 for 3-quasi-transitive digraphs and 3-transitive digraphs.

In [3], the author proposed the following conjecture.

**Conjecture 2** [3]. Let $D$ be a 3-quasi-transitive oriented graph. Then the underlying graph of $D$, $UG(D)$, admits a 3-transitive orientation.

In Section 2, we will prove that the conjecture is true.

### 2. Main Result

We begin with some useful lemmas. Let $F_n$ be the digraph with vertex set $\{x_0, x_1, \ldots, x_n\}$ and arc set $\{x_0x_1, x_1x_2, x_2x_0\} \cup \{x_0x_{i+3}, x_{i+3}x_1 : i = 0, 1, \ldots, n-3\}$, where $n \geq 3$. 
Lemma 3 [5]. Let $D$ be a strong 3-quasi-transitive digraph of order $n$. Then $D$ is either a semicomplete digraph, a semicomplete bipartite digraph or isomorphic to $F_n$.

Lemma 4 [7]. Let $D'$ be a non-trivial strong induced subdigraph of a 3-quasi-transitive digraph $D$. For any $s \in V(D) - V(D')$, if there exists a directed path between $s$ and $D'$, then $s$ and $D'$ are adjacent.

Lemma 5 [5]. Let $D$ be a 3-quasi-transitive digraph. For a pair $x, y$ of $V(D)$, if there exists an $(x, y)$-path of odd length, then $x$ and $y$ are adjacent.

Lemma 6 [7]. Let $D'$ be a non-trivial strong induced subdigraph of a 3-quasi-transitive digraph $D$ and let $s \in V(D) - V(D')$ with at least one arc from $s$ to $D'$ and $s \Rightarrow D'$. Then each of the following holds:
(a) If $D'$ is a bipartite digraph with bipartition $(X, Y)$ and $s$ dominates a vertex of $X$, then $s \mapsto X$.
(b) If $D'$ is a non-bipartite digraph, then $s \mapsto D'$.

Lemma 7 [7]. Let $D'$ be a non-trivial strong induced subdigraph of a 3-quasi-transitive digraph $D$ and let $s \in V(D) - V(D')$ with at least one arc from $D'$ to $s$ and $D' \Rightarrow s$. Then each of the following holds:
(a) If $D'$ is a bipartite digraph with bipartition $(X, Y)$ and there exists a vertex of $X$ which dominates $s$, then $X \mapsto s$.
(b) If $D'$ is a non-bipartite digraph, then $D' \mapsto s$.

Lemma 8 [7]. Let $D_1$ and $D_2$ be two distinct non-trivial strong components of a 3-quasi-transitive digraph with at least one arc from $D_1$ to $D_2$. Then either $D_1 \Rightarrow D_2$ or $D_1 \cup D_2$ is a semicomplete bipartite digraph.

Lemma 9. If a graph $G$ has a strong 3-quasi-transitive orientation, then it has a transitive orientation.

Proof. Let $D$ be a strong 3-quasi-transitive orientation of $G$ and let the order of $D$ be $n$. By Lemma 3, $D$ is either a tournament, a bipartite tournament or isomorphic to $F_n$, because $D$ has no cycle of length 2. If $D$ is a tournament (resp. a bipartite tournament), then $G$ is a complete graph (resp. a complete bipartite graph). Any acyclic orientation of a complete graph is transitive and orienting the edges of a complete bipartite graph from one side of the bipartition to the other results in a transitive orientation. Now suppose that $D$ is isomorphic to $F_n$ with vertex set $\{x_0, x_1, \ldots, x_n\}$ and arc set $\{x_0 x_1, x_1 x_2, x_2 x_0\} \cup \{x_0 x_{i+3}, x_{i+3} x_1 : i = 0, 1, \ldots, n - 3\}$, where $n \geq 3$. We reorient $D$ as a digraph $D'$ with arc set $\{x_0 x_1\} \cup \{x_{i+2} \rightarrow x_0, x_{i+2} \rightarrow x_1 : i = 0, 1, \ldots, n - 2\}$. Clearly, $D'$ is a transitive digraph. The proof of Lemma 9 is complete. \[\square\]
Lemma 10 [3]. If \( D \) is a \( k \)-transitive digraph with \( k \geq 2 \), then \( D \) is \((k+n(k-1))\)-transitive for any \( n \geq 1 \) such that \( k+n(k-1) \leq \text{diam}(D) \), where \( \text{diam}(D) \) is the diameter of \( D \).

The following theorem is our main result.

Theorem 11. A graph \( G \) has a 3-quasi-transitive orientation if and only if it has a 3-transitive orientation.

Proof. Since every 3-transitive digraph is also a 3-quasi-transitive digraph, the sufficiency is trivial.

We shall prove the necessity below. Suppose that \( D \) is a 3-quasi-transitive orientation of \( G \) and \( D \) is not a 3-transitive orientation. If \( D \) is strong, then, by Lemmas 9 and 10, we are done. Suppose now that \( D \) is non-strong and \( D_1, D_2, \ldots, D_t \) are its strong components. Note that every \( D_i \), \( i = 1, 2, \ldots, t \), is also a strong 3-quasi-transitive digraph. Hence, according to Lemma 9, every \( D_i \), for \( i = 1, 2, \ldots, t \), can be reoriented as a transitive digraph. Now, we reorient every \( D_i \) as a transitive digraph \( D_i' \) as in the proof of Lemma 9 and keep the directions of remaining arcs in \( D \). Denote the resulting digraph by \( D' \). From Lemma 10, we know that if a digraph is transitive, then it must be 3-transitive. Hence \( D_i' \) is 3-transitive, \( i = 1, 2, \ldots, t \). Now we shall show that \( D' \) is 3-transitive.

It suffices to prove that for any path \( x_0x_1x_2x_3 \) in \( D' \), \( x_0 \rightarrow x_3 \) in \( D' \). By the definition of \( D' \), we can see that \( D' \) is acyclic. Hence it is sufficient to show that \( x_0x_3 \) in \( D' \). Observe that \( x_0x_3 \) in \( D' \) if and only if \( x_0x_3 \) in \( D \). Hence we shall prove that \( x_0x_3 \) in \( D \) or \( x_0x_3 \) in \( D' \). Furthermore, in order to show that \( x_0x_3 \) in \( D \), by Lemma 5, we only need to prove that there is an odd path from \( x_0 \) to \( x_3 \) in \( D \).

If \( x_0 \) and \( x_3 \) belong to the same strong component in \( D \), say \( D_i \), then \( x_1 \) and \( x_2 \) both belong to \( D_i \), otherwise, assume, without loss of generality, that \( x_1 \in V(D_j) \) where \( i \neq j \). Because the arcs of \( D \) between distinct strong components are not reoriented, it would be the case that \( D_i \) can reach \( D_j \) and \( D_j \) also can reach \( D_i \) in \( D \), contradicting that they are distinct strong components. Since \( x_k \in V(D_i) \), for \( k = 0, 1, 2, 3 \) and \( D_i' \) is 3-transitive, we have \( x_0 \rightarrow x_3 \).

Now assume that \( x_0 \) and \( x_3 \) belong to distinct strong components and assume, without loss of generality, that \( x_0 \in V(D_i) \) and \( x_3 \in V(D_j) \) with \( 1 \leq i \neq j \leq t \). The following two claims will be useful.

Claim 1. \( D_j \) is reachable from \( D_i \) in \( D \).

Proof. It suffices to show that there exists a path from \( D_i \) to \( D_j \) in \( D \). If \( x_2 \in V(D_i) \) \( (x_1 \in V(D_j)) \), then \( x_2x_3(x_0x_1) \) is the desired path. So suppose \( x_2 \notin V(D_i) \) and \( x_1 \notin V(D_j) \). If \( x_2 \in V(D_j) \), then since \( x_1 \notin V(D_j) \), \( x_1x_2 \in A(D) \). If \( x_0x_1 \in A(D) \), then \( x_0x_1x_2 \) is the desired path; if not, then \( x_1 \in V(D_i) \) and so
$x_1x_2$ is the desired path. Thus suppose $x_2 \in V(D_s)$, with $1 \leq s \leq t$ and $s \neq i, j$. So $x_2x_3 \in A(D)$. If $x_1 \in V(D_i)$, then $x_1x_2x_3$ is a path from $D_i$ to $D_j$ in $D$. Thus we may assume that $x_1 \notin V(D_i)$ which implies $x_0x_1 \in A(D)$. If $x_1x_2 \in A(D)$, then $x_0x_1x_2x_3$ is the desired path; if not, then $x_1$ and $x_2$ both belong to $D_s$. Since $D_s$ is strong, there exists a path $P$ from $x_1$ to $x_2$ in $D$ and then $x_0x_1Px_2x_3$ is the desired path.

\[\Box\]

Claim 2. If $x_1, x_2 \notin V(D_i) \cup V(D_j)$, then there exists an odd path from $x_0$ to $x_3$ in $D$.

**Proof.** Since $x_1, x_2 \notin V(D_i) \cup V(D_j)$, $x_0x_1, x_2x_3 \in A(D)$. If $x_1x_2 \in A(D)$, then $x_0x_1x_2x_3$ is the desired path. Now assume that $x_1x_2 \notin A(D)$. Then by the definition of $D'$, we have that $x_2x_1 \in A(D)$ and $x_1, x_2$ belong to the same strong component in $D$, say $D_k$. If $D_k$ is a non-bipartite digraph, then by Lemmas 6 and 7, $x_0 \rightarrow D_k$ and $D_k \rightarrow x_3$ and in particular, $x_0 \rightarrow x_2$ and $x_1 \rightarrow x_3$ in $D$. Note that $x_0x_2x_1x_3$ is a path length 3 in $D$. If $D_k$ is a bipartite digraph, then $x_1$ and $x_2$ belong to different partite sets. Again since $D_k$ is a strong bipartite digraph, there exists an odd path $P$ from $x_1$ to $x_2$ in $D$. Then we have that $x_0x_1Px_2x_3$ is an odd path from $x_0$ to $x_3$ in $D$. Thus the claim holds. The proof of Claim 2 is complete. \[\Box\]

We consider two cases.

**Case 1.** At least one of $D_i$ and $D_j$ is trivial, say $D_i$. Since $D_i$ is trivial, by the definition of $D'$, $x_0x_1 \in V(D)$. If $V(D_j)$ is also trivial, then $x_1, x_2 \notin V(D_i) \cup V(D_j)$. By Claim 2, we are done. Now assume that $D_j$ is non-trivial. By Claim 1, Lemma 4 and the definition of strong components, there exists at least an arc from $x_0$ to $D_j$. If $D_j$ is a non-bipartite digraph, then by Lemma 6, $x_0 \rightarrow D_j$. In particular, $x_0 \rightarrow x_3$ and so we are done.

Now suppose that $D_j$ is a bipartite digraph. Assume that $(X_j, Y_j)$ is the bipartition of $D_j$ and assume, without loss of generality, that $x_3 \in X_j$. By Lemma 6, $x_0 \rightarrow X_j$ or $x_0 \rightarrow Y_j$. If $x_0 \rightarrow X_j$, then $x_0 \rightarrow x_3$ and so we are done. Suppose that $x_0 \rightarrow Y_j$.

**Subcase 1.1.** $x_2 \in V(D_j)$. Since $x_2$ and $x_3$ are adjacent, we have $x_2 \in Y_j$. Since $x_1$ and $x_2$ are adjacent, we have that $x_1 \notin Y_j$. If $x_1 \in X_j$, then by $x_0x_1 \in A(D)$ and Lemma 6, $x_0 \rightarrow X_j$. In particular, $x_0 \rightarrow x_3$ and so we are done. Now assume that $x_1 \notin V(D_j)$ and so $x_1x_2 \in A(D)$. Since $D_j$ is a strong bipartite digraph, there exists an odd path $P$ from $x_2$ to $x_3$ in $D_j$. Then $x_0x_2x_1x_3$ is an odd path from $x_0$ to $x_3$.

**Subcase 1.2.** $x_2 \notin V(D_j)$. Since $x_2 \notin V(D_j)$, we have $x_2x_3 \in A(D)$. By the definition of strong components, $x_1 \notin V(D_j)$. Combining this with Claim 2, there exists an odd path from $x_0$ to $x_3$ and so we are done.
Case 2. $D_i$ and $D_j$ are both non-trivial. By Claim 1 and Lemma 4, there exists at least an arc from $D_i$ to $D_j$. By Lemma 8, we have $D_i \rightarrow D_j$ or $D_i \cup D_j$ is a bipartite tournament. If $D_i \rightarrow D_j$, then $x_0 \rightarrow x_3$ in $D$ and so we are done. If $D_i \cup D_j$ is a bipartite tournament, then $D_i$ and $D_j$ are both bipartite. Assume that the bipartitions of $D_i$ and $D_j$ are $(X_i, Y_i)$ and $(X_j, Y_j)$, respectively and the bipartition of $D_i \cup D_j$ is $(X_i \cup X_j, Y_i \cup Y_j)$. Assume, without loss of generality, that $x_0 \in X_i$. If $x_3 \in Y_j$, then $x_0x_3 \in A(D)$ and so we are done. Suppose that $x_3 \in X_j$.

Subcase 2.1. $x_2 \in V(D_j)$. Since $x_2$ and $x_3$ are adjacent, $x_2 \in Y_j$. This implies that $x_1 \notin V(D_j) \cup V(D_i)$, which follows from the fact that $D_i \cup D_j$ is bipartite. Thus we have $x_0x_1, x_1x_2 \in A(D)$. Since $D_j$ is a strong bipartite digraph, there is an odd path $P$ from $x_2$ to $x_3$ in $D_j$. Then $x_0x_1x_2Px_3$ is an odd path from $x_0$ to $x_3$ in $D$ and so we are done.

Subcase 2.2. $x_2 \notin V(D_j)$. So $x_2x_3 \in A(D)$. By the definition of strong components, $x_1 \notin V(D_j)$.

If $x_2 \in V(D_i)$, then since $x_2$ and $x_3$ are adjacent, $x_2 \in Y_i$. Since $x_1$ and $x_0$, $x_2$ are both adjacent, $x_1 \notin V(D_i)$, which implies that $x_0x_1, x_1x_2 \in A(D)$, a contradiction to the definition of strong components. Thus $x_2 \notin V(D_i)$.

If $x_1 \in V(D_i)$, then since $x_0$ and $x_1$ are adjacent, $x_1 \in Y_i$ and $x_1x_2 \in A(D)$. Since $D_i$ is a strong bipartite digraph, there is an odd path $P$ from $x_0$ to $x_1$ in $D_i$. Then $x_0Px_1x_2x_3$ is an odd path from $x_0$ to $x_3$ in $D$ and so we are done. Assume that $x_1 \notin A(D_i)$.

Note that now $x_1, x_2 \notin V(D_i) \cup V(D_j)$. By Claim 2, there exists an odd path from $x_0$ to $x_3$ in $D$ and so we are done.

We have considered all the cases. The proof of Theorem 11 is complete.

Conjecture 2 then is an immediate consequence of Theorem 11.

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