Discussiones Mathematicae
Graph Theory 33 (2013) 395–410
doi:10.7151/dmgt.1677

A CHARACTERIZATION OF TREES FOR A NEW LOWER BOUND ON THE k-INDEPENDENCE NUMBER[1]

Nacéra Meddah and Mostafa Blidia

LAMDA-RO, Department of Mathematics
University of Blida
B.P. 270, Blida, Algeria

e-mail: meddahn11@yahoo.fr
mblidia@hotmail.com

Abstract

Let k be a positive integer and $G = (V, E)$ a graph of order n. A subset S of V is a k-independent set of G if the maximum degree of the subgraph induced by the vertices of S is less or equal to $k-1$. The maximum cardinality of a k-independent set of G is the k-independence number $\beta_k(G)$. In this paper, we show that for every graph G, $\beta_k(G) \geq \left\lceil \left(n + (\chi(G)-1)\sum_{v\in S(G)} \min(|L_v|, k-1)\right)/\chi(G)\right\rceil$, where $\chi(G)$, $s(G)$ and L_v are the chromatic number, the number of supports vertices and the number of leaves neighbors of v, in the graph G, respectively. Moreover, we characterize extremal trees attaining these bounds.

Keywords: domination, independence, k-independence.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

Let G be a simple graph with vertex set V and edge set E. The number of vertices of G is called the *order,* and is denoted by $n = n(G)$. The *open neighborhood* $N(v) = N_G(v)$ of a vertex v consists of all vertices adjacent to v and $d(v) = d_G(v) = |N(v)|$ is the *degree* of v. The *closed neighborhood* of a vertex v is defined by $N[v] = N_G[v] = N_G(v) \cup \{v\}$. By $\Delta = \Delta(G)$, we denote the maximum degree of the graph G. A vertex of *degree* one is called a *leaf* and its neighbor is called a *support vertex.* If v is a *support vertex* then L_v will denote the set of the leaves adjacent to v. If v is support vertex with $|L_v| \geq 2$, then v is called *strong support*, else v is called *weak support.* We denote by $S(G)$ and $L(G)$ the set of

[1]This research was supported by "Programmes Nationaux de Recherche: Code 8/u09/510".

support vertices and the set of leaves, respectively, and we let $s(G) = |S(G)|$ and $\ell(G) = |L(G)|$. For a subset $A \subseteq V(G)$, we denote by $\langle A \rangle$ the *subgraph induced* by the vertices of A.

A *caterpillar* is a tree T with the property that the removal of its leaves results in a path. The resulting path $u_1u_2 \cdots u_s$ is referred to as the *spine* of the caterpillar and the leaves are called the *legs* of the caterpillar. A sequence of nonnegative integers $(t_1, t_2, \ldots, t_s)$ where t_i is the number of leaves (legs) adjacent to u_i for $s \geq 1$ is associated with T. However, we denote a caterpillar by $C(t_1, t_2, \ldots, t_s)$. For example, the star $K_{1,p}$ is the caterpillar $C(p)$ with a spine u_1 and p legs.

We define the following special caterpillars which are used in this paper, we denote by

$G_1(u) = C(t_1, m, t_2)$ where $|L_u| = m$ with $k-2 \leq m \leq k-1$ and $1 \leq t_1$, $t_2 \leq k-2$.

$G_2(u) = C(t_1, t_2, k-1)$ where $|L_u| = k-1, 1 \leq t_1 \leq k-2$ and $1 \leq t_2 \leq k-3$.

$F_1(v) = C(k-1, m, t)$ where $|L_v| = m, 0 \leq m \leq k-1$ and $1 \leq t \leq k-1$.

$F_2(v) = C(t, k-2, m)$ where $|L_v| = m, 0 \leq m \leq k-1$ and $1 \leq t \leq k-2$.

Let k be a positive integer. A subset S of V is k-independent of G if the maximum degree of the subgraph induced by the vertices of S is less or equal to $k-1$. The *k-independence number* $\beta_k(G)$ is the maximum cardinality of a k-independent set of G. Notice that 1-independent sets are the classical independent sets, and so $\beta_1(G) = \beta(G)$. If S is a k-independent set of G of size $\beta_k(G)$, then we call S a $\beta_k(G)$-set. General concept *p-independence*, where p is a hereditary property of graphs, was introduced by S.T. Hedetniemi in 1973, see [1] and is studied for example in [5, 6, 7, 8, 9, 10, 11] and elsewhere. Note that Borowiecki and Michalak [3] gave a generalization of the concept of k-independence by considering other hereditary-induced properties than the property for a subgraph to have maximum degree at most $k-1$. Clearly if $k > \Delta$, then the vertex set $V(G)$ is k-independent and in this case $\beta_k(G) = n$. Therefore in this paper, we will assume that k is an integer with $k \leq \Delta$.

A *p-coloring* of a graph G is a function defined on V into a set of colors $\{1, 2, \ldots, p\}$ such that any two adjacent vertices have different colors. Each set of vertices colored with one color is an *independent* set of vertices of G. The minimum cardinality p for which G has *p-coloring* is the *chromatic number* $\chi(G)$ of G. The parameter $\chi(G)$ has been extensively studied by many authors. One of the classical results concerning the *chromatic number* of a graph is due to Brooks [4].

Theorem 1. *For any graph G, $\chi(G) \leq \Delta + 1$, with equality if and only if either $\Delta \neq 2$ and G has a subgraph $K_{\Delta+1}$ as a connected component or $\Delta = 2$ and G has a cycle C_{2k+1} as a connected component.*

It is well known that bipartite graphs satisfy $\beta(G) \geq n(G)/2$. In [12] Volkmann gave a constructive characterization of trees T satisfy $\beta(T) = \lceil n(T)/2 \rceil$. Also, in [2] Blidia *et al.* proved that bipartite connected graphs G of order $n \geq 2$ with $s(G)$ support vertices satisfy $\beta_2(G) \geq (n + s(G))/2$, and gave a constructive characterization of trees attaining this bound.

In this paper, we generalize the above results by giving a new lower bound on $\beta_k(G)$ in terms of the order, the chromatic number and the number of supports vertices. Moreover, we characterize extremal trees attaining this bound.

2. Lower Bound

Theorem 2. *Let G be a graph of order n with a chromatic number $\chi(G)$ and k be an integer with $2 \leq k \leq \Delta$. Then*

$$\beta_k(G) \geq \left\lceil \left(n + (\chi(G) - 1) \textstyle\sum_{v \in S(G)} \min(|L_v|, k-1) \right) / \chi(G) \right\rceil,$$

and this bound is sharp.

Proof. The result can be easily checked if G is a union of cliques. Thus we assume that G is not a union of cliques and let C be a set of leaves defined as follows: for each support vertex v of G we put in C exactly $\min(|L_v|, k-1)$ of its leaves. So $|C| = \sum_{v \in S(G)} \min(|L_v|, k-1)$. Let $A_1, A_2, \ldots, A_{\chi(G)}$ be a $\chi(G)$-coloration of the subgraph induced by the vertices of $V(G) - C$. Without loss of generality, we can assume that $|A_1| \leq |A_2| \leq \cdots \leq |A_{\chi(G)}|$. Note that $\chi(G) = \chi \langle V(G) - C \rangle$. Then we have $n - |C| = |A_1| + |A_2| + \cdots + |A_{\chi(G)}| \leq \chi(G) |A_{\chi(G)}|$, and hence $|A_{\chi(G)}| \geq (n - |C|)/\chi(G)$. Now, since $A_{\chi(G)} \cup C$ is k-independent, $\beta_k(G) \geq |A_{\chi(G)} \cup C| \geq (n - |C|)/\chi(G) + |C|$. It follows that $\beta_k(G) \geq (n + (\chi(G) - 1)|C|)/\chi(G)$, and so

$$\beta_k(G) \geq \left\lceil \left(n + (\chi(G) - 1) \textstyle\sum_{v \in S(G)} \min(|L_v|, k-1) \right) / \chi(G) \right\rceil.$$

That this bound is sharp may be seen for trees by the following characterization and for graphs different from trees by the graph obtained from a clique by attaching $k - 1$ vertices to each vertex of the clique, then for an integer k with $2 \leq k \leq \Delta$ equality holds in the general bound. ■

Note that $\chi(G) = 2$ for every bipartite graphs G having at least one edge. Hence, as immediate consequences to Theorems 1 and 2, we obtain the following corollaries.

Corollary 3. Let G be a graph of order n and maximum degree Δ, and k be an integer with $2 \leq k \leq \Delta$. Then

$$\beta_k(G) \geq \left\lceil \left(n + \Delta \textstyle\sum_{v \in S(G)} \min(|L_v|, k-1) \right) / (\Delta + 1) \right\rceil.$$

Corollary 4. Let T be a tree of order n, and k be an integer with $2 \leq k \leq \Delta$. Then $\beta_k(T) \geq \left\lceil \left(n + \sum_{v \in S(T)} \min(|L_v|, k-1) \right) /2 \right\rceil$.

3. Characterization

Our aim in this section is to gave a characterization of trees attaining the lower bound in Corollary 4. Note that the difference between the two sides of the inequality can be made arbitrarily large even for trees. To see this, we consider a caterpillar $T = C(t_1, t_2, \ldots, t_{2p})$ where $t_i = t > k$ for every $i : 1 \leq i \leq 2p$. It is easy to verify that $\beta_k(T) - \left\lceil \left(n + \sum_{v \in S(T)} \min(|L_v|, k-1) \right) /2 \right\rceil = p(t-k)$.

The following result that can be found in [2] is straightforward.

Lemma 5. *For $k \geq 1$, let w be a vertex of a graph G'' such that every neighbor of w has degree at most k, at least w or one of its neighbors has degree k or more, and every vertex in $V(G'')\backslash N[w]$, if any, has degree less than k in G''. Let G' be any graph and G the graph constructed from G' and G'' by adding an edge between w and any vertex of G'. Then $\beta_k(G) = \beta_k(G') + |V(G'')| - 1$.*

For $k \geq 2$, we define the family $\mathcal{H}(k)$ of all nontrivial trees T that can be obtained from a sequence $T_0, T_1, \ldots, T_p$ $(p \geq 1)$ of trees, where T_0 is either $C(t)$ with $1 \leq t \leq k+1$, $C(t, k-1)$ or $C(t, k)$, with $1 \leq t \leq k-2$ and $k \geq 3$, or $G_1(u)$ with $k \geq 3$ or $G_2(u)$ with $k \geq 4$, $T = T_p$, and if $p \geq 2$, then T_{i+1} can be obtained recursively from T_i by one of the following operations.

Operation $\mathcal{H}_1(k)$. Add a copy of a caterpillar $C(k)$ attached by an edge between any vertex of $C(k)$ and a vertex v of T_i, with the condition that if v is a leaf in T_i, then its support vertex z in T_i satisfies $|L_z| \geq k$.

Operation $\mathcal{H}_2(k)$. Add a copy of a caterpillar $C(k+1)$ centred in u, attached by an edge uv at a vertex v of T_i, with the condition that $n(T_i) + \sum_{x \in S(T_i)} \min(|L_x|, k-1)$ is even, and if v is a leaf in T_i, then its support vertex z in T_i satisfies $|L_z| \geq k$.

Operation $\mathcal{H}_3(k)$. For $k \geq 3$, add a copy of a caterpillar $C(t, k-1)$ with $1 \leq t \leq k-2$ and supports vertices u_1, u, where $|L_{u_1}| = t$, attached by an edge uv at a vertex v of T_i, with the condition that if v is a leaf in T_i, then its support vertex z in T_i satisfies $|L_z| \geq k$.

Operation $\mathcal{H}_4(k)$. For $k \geq 3$, add a copy of a caterpillar $C(t, k)$ with $1 \leq t \leq k-2$ and supports vertices u_1, u, where $|L_{u_1}| = t$, attached by an edge uv at a vertex v of T_i, with the condition that $n(T_i) + \sum_{x \in S(T_i)} \min(|L_x|, k-1)$ is even, and if v is a leaf in T_i, then its support vertex z in T_i satisfies $|L_z| \geq k$.

Operation $\mathcal{H}_5(k)$. For $k \geq 3$, add a copy of the caterpillar $G_1(u)$, attached by an edge uv at a vertex v of T_i, with the condition that $n(T_i) + \sum_{x\in S(T_i)} \min(|L_x|, k-1)$ is even, and if v is a leaf in T_i, then its support vertex z in T_i satisfies $|L_z| \geq k$.

Operation $\mathcal{H}_6(k)$. For $k \geq 4$, add a copy of the caterpillar $G_2(u)$, attached by an edge uv at a vertex v of T_i, with the condition that $n(T_i) + \sum_{x\in S(T_i)} \min(|L_x|, k-1)$ is even, and if v is a leaf in T_i, then its support vertex z in T_i satisfies $|L_z| \geq k$.

Operation $\mathcal{H}_7(k)$. For $k \geq 3$, add a copy of a caterpillar $C(t, k-1)$ with $1 \leq t \leq k-2$ and supports vertices u_1, u, where $|L_u| = k-1$, attached by an edge uv at a vertex v of T_i, with the condition that $n(T_i) + \sum_{x\in S(T_i)} \min(|L_x|, k-1)$ is even, and v is a leaf in T_i of support vertex z satisfies $|L_z| \leq k-1$.

Operation $\mathcal{H}_8(k)$. Add a copy of a caterpillar $C(k)$ attached by an edge between any vertex of $C(k)$ and a vertex v of T_i, with the condition that $n(T_i) + \sum_{x\in S(T_i)} \min(|L_x|, k-1)$ is even, and v is a leaf in T_i of support vertex z satisfies $|L_z| \leq k-1$.

Operation $\mathcal{H}_9(k)$. Add a copy of a caterpillar $C(k-1, m)$ with $1 \leq m \leq k-1$ and supports vertices u_1, u, where $|L_u| = m$, attached by an edge uv at a vertex v of T_i, with the condition that if v is a leaf in T_i, then its support vertex z in T_i satisfies $|L_z| \geq k$.

Operation $\mathcal{H}_{10}(k)$. Add a copy of a caterpillar $C(k-1, k)$ of supports vertices u_1, u, where $|L_u| = k$, attached by an edge uv at a vertex v of T_i, with the condition that $n(T_i) + \sum_{x\in S(T_i)} \min(|L_x|, k-1)$ is even, and if v is a leaf in T_i, then its support vertex z in T_i satisfies $|L_z| \geq k$.

Operation $\mathcal{H}_{11}(k)$. Add a copy of the caterpillar $F_1(u)$, attached by an edge uv at a vertex v of T_i, with the condition that $n(T_i) + \sum_{x\in S(T_i)} \min(|L_x|, k-1)$ is even, and if v is a leaf in T_i, then its support vertex z in T_i satisfies $|L_z| \geq k$.

Operation $\mathcal{H}_{12}(k)$. For $k \geq 3$, add a copy of the caterpillar $F_2(u)$, attached by an edge uv at a vertex v of T_i, with the condition that $n(T_i) + \sum_{x\in S(T_i)} \min(|L_x|, k-1)$ is even, and if v is a leaf in T_i, then its support vertex z in T_i satisfies $|L_z| \geq k$.

Operation $\mathcal{H}_{13}(k)$. Add a copy of a caterpillar $C(k-1, m)$ with $1 \leq m \leq k-1$ and supports vertices u_1, u, where $|L_u| = m$, attached by an edge uv at a vertex v of T_i, with the condition that $n(T_i) + \sum_{x\in S(T_i)} \min(|L_x|, k-1)$ is even, and v is a leaf in T_i of support vertex z satisfies $|L_z| \leq k-1$.

Lemma 6. *Let T be a nontrivial tree and k be an integer with $2 \leq k \leq \Delta$. If $T \in \mathcal{H}(k)$, then* $\beta_k(T) = \left\lceil \left(n(T) + \sum_{x\in S(T)} \min(|L_x|, k-1)\right)/2 \right\rceil$.

Proof. For $2 \leq k \leq \Delta(T)$, let T be any tree of $\mathcal{H}(k)$. We proceed by induction on the number of operations $\mathcal{H}_i$ where $1 \leq i \leq 13$, performed to construct T. The property is true for $T_0 = C(t)$ with $1 \leq t \leq k+1$, $C(t, k-1)$ or $C(t,k)$ with $1 \leq t \leq k-2$ and $k \geq 3$, $G_1(u)$ with $k \geq 3$ and $G_2(u)$ with $k \geq 4$. Suppose the property is true for all trees of $\mathcal{H}(k)$ constructed with $j-1 \geq 0$ operations and let T be a tree constructed with j operations. Consider the following cases depending on whether T is obtained by performing Operation $\mathcal{H}_1(k)$ or Operation $\mathcal{H}_7(k)$. We omit the proof for the remaining operations since it is similar to that used in these two cases.

Suppose that the last operation performed on a tree T' already obtained by $j-1$ operations, is $\mathcal{H}_1(k)$. Then $n(T) = n(T') + k + 1$ and $s(T) = s(T') + 1$ because every support of T' is a support of T (since if v is a leaf in T_i, then its support vertex z in T_i satisfies $|L_z| \geq k \geq 2$). Thus

$$\text{(1)} \qquad \sum\nolimits_{x \in S(T)} \min(|L_x|, k-1) = \sum\nolimits_{x \in S(T')} \min(|L_x|, k-1) + k - 1.$$

By Lemma 5 and the inductive hypothesis applied on T' we have, $\beta_k(T) = \beta_k(T') + k = \left\lceil \left(n(T') + \sum_{x \in S(T')} \min(|L_x|, k-1) \right) /2 \right\rceil + k$.

Now, if $n(T') + \sum_{x \in S(T')} \min(|L_x|, k-1)$ is even, then by using $n(T) = n(T') + k + 1$ and formula (1), we obtain

$$\begin{aligned} \beta_k(T) &= \left(n(T') + 2k + \sum\nolimits_{x \in S(T')} \min(|L_x|, k-1) \right) /2 \\ &= \left(n(T) + \sum\nolimits_{x \in S(T)} \min(|L_x|, k-1) \right) /2 \\ &= \left\lceil \left(n + \sum\nolimits_{x \in S(T)} \min(|L_x|, k-1) \right) /2 \right\rceil . \end{aligned}$$

Also, if $n(T') + \sum_{x \in S(T')} \min(|L_x|, k-1)$ is odd, then by using $n(T) = n(T') + k + 1$ and formula (1), we obtain :

$$\begin{aligned} \beta_k(T) &= \left(n(T') + 1 + 2k + \sum\nolimits_{x \in S(T')} \min(|L_x|, k-1) \right) /2 \\ &= \left(n(T) + 1 + \sum\nolimits_{x \in S(T)} \min(|L_x|, k-1) \right) /2 \\ &= \left\lceil \left(n + \sum\nolimits_{x \in S(T)} \min(|L_x|, k-1) \right) /2 \right\rceil . \end{aligned}$$

In both cases, $\beta_k(T) = \left\lceil \left(n + \sum_{x \in S(T)} \min(|L_x|, k-1) \right) /2 \right\rceil$.

Suppose now that the last operation performed on a tree T' already obtained by $j-1$ operations, is $\mathcal{H}_7(k)$. Then $n(T) = n(T') + k + t + 1$ with $1 \leq t \leq k-2$ and $s(T') + 1 \leq s(T) \leq s(T') + 2$ (since v is a leaf in T_i whose support vertex z satisfies $|L_z| \leq k-1$). Thus

$$\text{(2)} \quad \sum\nolimits_{x \in S(T)} \min(|L_x|, k-1) = \sum\nolimits_{x \in S(T')} \min(|L_x|, k-1) + k - 1 + t - 1.$$

By Lemma 5 and the inductive hypothesis applied on T',

$$\beta_k(T) = \beta_k(T') + k + t = \left\lceil \left(n(T') + \sum\nolimits_{x \in S(T')} \min(|L_x|, k-1) \right) /2 \right\rceil + k + t.$$

Since $n(T') + \sum_{x\in S(T')} \min(|L_x|, k-1)$ is even and by using $n(T) = n(T') + k + t + 1$ and formula (2), we obtain

$$\begin{aligned} \beta_k(T) &= \left(n(T') + 2k + 2t + \sum\nolimits_{x \in S(T')} \min(|L_x|, k-1) \right) /2 \\ &= \left(n(T) + 1 + \sum\nolimits_{x \in S(T)} \min(|L_x|, k-1) \right) /2 \\ &= \left\lceil \left(n + \sum\nolimits_{x \in S(T)} \min(|L_x|, k-1) \right) /2 \right\rceil . \end{aligned}$$

■

Now we are ready to give a characterization of trees achieving equality in Corollary 4.

Theorem 7. *Let T be a nontrivial tree and k be an integer with $2 \le k \le \Delta$. Then $\beta_k(T) = \left\lceil \left(n(T) + \sum_{x\in S(T)} \min(|L_x|, k-1) \right) /2 \right\rceil$ if and only if $T \in \mathcal{H}(k)$.*

Proof. The sufficient condition follows from Lemma 6.

Conversely, for $2 \le k \le \Delta$, let T be a tree of order n with $\beta_k(T) = \left\lceil \left(n + \sum_{x\in S(T)} \min(|L_x|, k-1) \right) /2 \right\rceil$. Let $Y(T)$ be the set of vertices of degree at least k of the tree T. We proceed by induction on $|Y(T)|$.

If $|Y(T)| = 0$, then $\beta_k(T) = n = \left\lceil \left(n + \sum_{x\in S(T)} \min(|L_x|, k-1) \right) /2 \right\rceil$ if and only if $n + \sum_{x\in S(T)} \min(|L_x|, k-1)$ is odd. Since if $n + \sum_{x\in S(T)} \min(|L_x|, k-1)$ is even, then $n = \sum_{x\in S(T)} \min(|L_x|, k-1)$, which is impossible. Thus $n = \left(n + 1 + \sum_{x\in S(T)} \min(|L_x|, k-1) \right) /2$, and so $n - 1 = \sum_{x\in S(T)} \min(|L_x|, k-1)$. In this case, if $|S(T)| \ge 2$, then $n - 1 > \sum_{x\in S(T)} \min(|L_x|, k-1)$, and so $|S(T)| = 1 = |\{u\}|$. Hence $n-1 = |L_u| = \sum_{x\in S(T)} \min(|L_x|, k-1)$ with $|L_u| \le k-1$. Thus all stars $C(t)$ with $1 \le t \le k-1$ satisfy $\beta_k(T) = \left\lceil \left(n + \sum_{x\in S(T)} \min(|L_x|, k-1) \right) /2 \right\rceil$.

If $|Y(T)| = 1 = |\{u\}|$, then

$$\beta_k(T) = \left\lceil \left(n + \sum\nolimits_{x \in S(T)} \min(|L_x|, k-1) \right) /2 \right\rceil = n - 1.$$

First we assume that $n + \sum_{x\in S(T)} \min(|L_x|, k-1)$ is even, hence we obtain $n = \sum_{x\in S(T)} \min(|L_x|, k-1) + 2$. Now, if $|S(T)| = 1$, then $n = k+1$ and so $T = C(k)$ of center u, and if $|S(T)| \ge 2$, then let $\{u_1, u_2, \ldots, u_p\}$: $p \ge 1$, be the set of support vertices descendant from u with $1 \leq |L_{u_i}| = t_i \leq k-2$; $1 \leq i \leq p$, $k \geq 3$, and $|L_u| = m$. Since $d_T(u) \geq k \geq 2$, if $m = 0$, then $\sum_{x \in S(T)} \min(|L_x|, k-1) < n-2$, a contradiction. Hence $m \geq 1$.

(i) Now, if $m \leq k-1$, then $n = \sum_{i=1}^{p} t_i + m + 2 = (\sum_{i=1}^{p} t_i + 1) + m + 1$, and so $|S(T)| = 2$. Since $d_T(u) \geq k$, $m \geq k-1$. Thus $m = k-1$ and $T = C(t, k-1)$ with $1 \leq t \leq k-2$ and $k \geq 3$.

(ii) If $m \geq k$, then $n = \sum_{i=1}^{p} t_i + (k-1) + 2$, and so $|S(T)| = 2$. If $m = k$, then $n + \sum_{x \in S(T)} \min(|L_x|, k-1) = 2k + 2t_1 + 1$ which is odd, and if $m > k$, then $\beta_k(T) > \left\lceil \left(n + \sum_{x \in S(T)} \min(|L_x|, k-1) \right) / 2 \right\rceil$. In both cases we have a contradiction.

Next we assume that $n + \sum_{x \in S(T)} \min(|L_x|, k-1)$ is odd. Hence $\beta_k(T) = n - 1 = \left(n + 1 + \sum_{x \in S(T)} \min(|L_x|, k-1) \right) / 2$, implying that $n = \sum_{x \in S(T)} \min(|L_x|, k-1) + 3$. Now, if $|S(T)| = 1$, then $n = k+2$ and so $T = C(k+1)$ of center u, and if $|S(T)| \geq 2$, then as previously, let $\{u_1, u_2, \ldots, u_p\}$: $p \geq 1$, be the set of support vertices descendant from u with $1 \leq t_i = |L_{u_i}| \leq k-2$; $1 \leq i \leq p$, $k \geq 3$, and $|L_u| = m$. Since $d_T(u) \geq k \geq 2$, if $m = 0$, then $n = \sum_{i=1}^{p} t_i + 3 = (\sum_{i=1}^{p} t_i + 2) + |\{u\}|$, and so $|S(T)| = 2$. Hence either $2 = d_T(u) \geq k \geq 2$, and so $k = 2$ and $|L_{u_i}| = t_i \leq 0$ for $1 \leq i \leq 2$, or $1 = d_T(u) \geq k \geq 2$. In both cases we have a contradiction. Hence $m \geq 1$.

(i) Now, if $m \leq k-1$, then $n = \sum_{i=1}^{p} t_i + m + 3$. Either $n = (\sum_{i=1}^{p} t_i + 1) + m + 2$, and so $|S(T)| = 2$ and $p = 1$. Since $d_T(u) \geq k$, $m = k-1$, and T is obtained from $T = C(t)$ with $1 \leq t \leq k-2$ by using Operation $\mathcal{H}_1(k)$. Or $n = (\sum_{i=1}^{p} t_i + 2) + m + 1$, hence $|S(T)| = 3$, $p = 2$. So $d_T(u) = m + 2 \geq k$, implying that $k-2 \leq m \leq k-1$, and thus T is the caterpillar $G_1(u)$ with $k \geq 3$, or $d_T(u) = m + 1 \geq k$, implying that $m = k-1$, and thus T is the caterpillar $G_2(u)$ with $k \geq 4$.

(ii) If $m \geq k$, then $n = \sum_{i=1}^{p} t_i + (k-1) + 3 = (\sum_{i=1}^{p} t_i + 1) + k + 1$. Hence $|S(T)| = 2$, $p = 1$ and $T = C(t, k)$ with $1 \leq t \leq k-2$ for $k \geq 3$.

Now assume that the assertion of the theorem is true for all trees with $|Y(T)| < \lambda$ with $\lambda \geq 2$ and let T be a tree of order n such that $|Y(T)| = \lambda$. Root T at a leaf r of maximum eccentricity. Let u be a vertex of degree at least k, at maximum distance from r and under this condition, of maximum degree. Since $|Y(T)| \geq 2$, u is at distance at least two from r. Let v, z be the parents of u and v in the rooted tree, respectively and w the parent of z if there exists. We distinguish between the following two cases.

Case 1. $d_T(u) \geq k+1$. Then let T' and T'' be the components of $T - uv$, containing v and u, respectively. Then $d_{T''}(u) \geq k$ and from the choice of u, all vertices of $V(T'') \setminus \{u\}$ have degree less than k. Hence u fulfills the conditions on the vertex w in Lemma 5. Apply the inductive hypothesis to T' since $|Y(T')| <$

$|Y(T)|$. Moreover $|V(T'')| = n'' \geq k+1$, $n(T') = n' = n - n''$ and by Lemma 5 and Theorem 2,

$$\begin{aligned}\beta_k(T) &= \beta_k(T') + n'' - 1 \\ &\geq \left\lceil \left(n' + \sum\nolimits_{x\in S(T')} \min(|L_x|, k-1)\right)/2 \right\rceil + n'' - 1.\end{aligned}$$

Suppose that $|L_u| = m \geq 0$ and that u has $\{u_1, \ldots, u_p\}$ (possible empty) as support vertices descendant from u, with $1 \leq t_i = |L_{u_i}| \leq k-2 : 1 \leq i \leq p$ with $k \geq 3$, so $|S(T'' - \{u\})| = p \geq 0$. Hence, $n'' \geq \sum_{i=1}^{p} t_i + m + p + 1$. Now, if $p = 0$, then $\sum_{i=1}^{p} t_i = 0$ and $m \geq k$, hence $m - \min(m, k-1) \geq 1$ and $p + m - \min(m, k-1) - 1 \geq 0$. Also if $p \geq 1$, then $p + m - \min(m, k-1) - 1 \geq 0$. In both cases $p + m - \min(m, k-1) - 1 \geq 0$. We consider two subcases.

Subcase 1.1. v is not a leaf, or v is a leaf of support vertex z in T', with $|L_z| \geq k$. Then $\sum_{x\in S(T)} \min(|L_x|, k-1) = \sum_{x\in S(T')} \min(|L_x|, k-1) + \min(m, k-1) + \sum_{i=1}^{p} t_i$.

(a) $n' + \sum_{x\in S(T')} \min(|L_x|, k-1)$ is even. By using $n'' \geq \sum_{i=1}^{p} t_i + m + p + 1$,

$$\begin{aligned}\beta_k(T) &= \beta_k(T') + n'' - 1 \\ &\geq \left(n' + \sum\nolimits_{x\in S(T')} \min(|L_x|, k-1)\right)/2 + n'' - 1 \\ &= \left(n' + 2n'' - 2 + \sum\nolimits_{x\in S(T')} \min(|L_x|, k-1)\right)/2 \\ &\geq \left(p + m - \min(m, k-1) - 1 + n + \sum\nolimits_{x\in S(T)} \min(|L_x|, k-1)\right)/2 \\ &\geq \left(n + \sum\nolimits_{x\in S(T)} \min(|L_x|, k-1)\right)/2.\end{aligned}$$

Indeed, if $n + \sum_{x\in S(T)} \min(|L_x|, k-1)$ is even, then

$$\begin{aligned}\left(n + \sum\nolimits_{x\in S(T)} \min(|L_x|, k-1)\right)/2 &= \left\lceil \left(n + \sum\nolimits_{x\in S(T)} \min(|L_x|, k-1)\right)/2 \right\rceil \\ &= \beta_k(T),\end{aligned}$$

hence we have equality througouth the previous inequality chain, and so $p + m - \min(m, k-1) - 1 = 0$. Also, if $n + \sum_{x\in S(T)} \min(|L_x|, k-1)$ is odd, then

$$\begin{aligned}\beta_k(T) &= \left\lceil \left(n + \sum\nolimits_{x\in S(T)} \min(|L_x|, k-1)\right)/2 \right\rceil \\ &\geq \left(n + 1 + \sum\nolimits_{x\in S(T)} \min(|L_x|, k-1)\right)/2 \\ &= \left\lceil \left(n + \sum\nolimits_{x\in S(T)} \min(|L_x|, k-1)\right)/2 \right\rceil = \beta_k(T),\end{aligned}$$

hence we have equality througouth the previous inequality chain, and so $p+m-\min(m,k-1)-1=1$. So, $\beta_k(T)=\left\lceil\left(n+\sum_{x\in S(T)}\min(|L_x|,k-1)\right)/2\right\rceil$ if and only if $\beta_k(T')=\left\lceil\left(n'+\sum_{x\in S(T')}\min(|L_x|,k-1)\right)/2\right\rceil$ and $0\le p+m-\min(m,k-1)-1\le 1$. By the inductive hypothesis on T', $T'\in\mathcal{H}(k)$ and $0\le p\le 2$.

(i) Now, if $m=0$, then $1\le p\le 2$. If $p=1$, then $1=d_T(u)\ge k\ge 2$, which is impossible. If $p=2$, then either $1=d_T(u)\ge k\ge 2$, a contradiction, or $2=d_T(u)\ge k\ge 2$, and so $k=2$ and the two descendants support vertices u_1 and u_2, have $t_1\le 0$ and $t_2\le 0$, respectively, a contradiction with $t_1,t_2\ge 1$.

(ii) If $m\ge 1$, then we have to consider three situations.

If $p=0$, then $1\le m-\min(m,k-1)\le 2$. Hence $m=k$ if $m-\min(m,k-1)=1$ and $m=k+1$ if $m-\min(m,k-1)=2$. So $T''=C(k)$ or $T''=C(k+1)$, respectively, and T'' is attached to T' by the support vertex u. Thus T is obtained from T' by using Operation $\mathcal{H}_1(k)$ or Operation $\mathcal{H}_2(k)$, respectively. It follows that $T\in\mathcal{H}(k)$.

If $p=1$, then $0\le m-\min(m,k-1)\le 1$. Hence $m=k-1$ if $m-\min(m,k-1)=0$ and $m=k$ if $m-\min(m,k-1)=1$. So $T''=C(t,k-1)$ or $T''=C(t,k)$ with $1\le t\le k-2$ and $k\ge 3$, respectively, and T'' is attached to T' by the support vertex u. Thus T is obtained from T' by using Operation $\mathcal{H}_3(k)$ or Operation $\mathcal{H}_4(k)$, respectively. It follows that $T\in\mathcal{H}(k)$.

If $p=2$, then $m-\min(m,k-1)=0$, and so $m\le k-1$. Thus either $d_{T''}(u)=m+2\ge k$, and so $k-2\le m\le k-1$, or $d_{T''}(u)=m+1\ge k$, and so $m=k-1$. Hence $T''=G_1(u)$ with $k\ge 3$ or $T''=G_2(u)$ with $k\ge 4$, respectively, and T'' is attached to T' by the support vertex u. Thus T is obtained from T' by using Operation $\mathcal{H}_5(k)$ or Operation $\mathcal{H}_6(k)$, respectively. It follows that $T\in\mathcal{H}(k)$.

(b) $n'+\sum_{x\in S(T')}\min(|L_x|,k-1)$ is odd. By using $n''\ge\sum_{i=1}^{p}t_i+m+p+1$,

$$\begin{aligned}
\beta_k(T) &= \beta_k(T')+n''-1\\
&\ge \left\lceil\left(n'+\textstyle\sum_{x\in S(T')}\min(|L_x|,k-1)\right)/2\right\rceil+n''-1\\
&= \left(n'+1+2n''-2+\textstyle\sum_{x\in S(T')}\min(|L_x|,k-1)\right)/2\\
&\ge \left(p+m-\min(m,k-1)+n+\textstyle\sum_{x\in S(T)}\min(|L_x|,k-1)\right)/2\\
&\ge \left(n+\textstyle\sum_{x\in S(T)}\min(|L_x|,k-1)+1\right)/2\\
&\ge \left\lceil\left(n+\textstyle\sum_{x\in S(T)}\min(|L_x|,k-1)\right)/2\right\rceil=\beta_k(T).
\end{aligned}$$

So, $\beta_k(T)=\left\lceil\left(n+\sum_{x\in S(T)}\min(|L_x|,k-1)\right)/2\right\rceil$ if and only if we have equality througouth the previous inequality chain, that is, if and only if $\beta_k(T')=$

$\left\lceil \left(n' + \sum_{x\in S(T')} \min(|L_x|, k-1)\right)/2 \right\rceil$ and $p + m - \min(m, k-1) = 1$. By the inductive hypothesis on T', $T' \in \mathcal{H}(k)$. Also $m \geq 1$, for otherwise $p = 1$ and we obtain a contradiction.

(i) Now, if $m \leq k-1$, then $m - \min(m, k-1) = 0$ and $p = 1$. Hence $m = k-1$ (because $d_{T''}(u) = m+1 \geq k$). So $T'' = C(t, k-1)$ with $1 \leq t \leq k-2$ for $k \geq 3$, attached to T' by the support vertex u. Thus T is obtained from T' by using Operation $\mathcal{H}_3(k)$. It follows that $T \in \mathcal{H}(k)$.

(ii) If $m \geq k$, then $m - \min(m, k-1) = 1$ and $p = 0$. Hence $m = k$, and so $T'' = C(k)$, attached to T' by the support vertex u. Thus T is obtained from T' by using Operation $\mathcal{H}_1(k)$. It follows that $T \in \mathcal{H}(k)$.

Subcase 1.2. v is a leaf of support vertex z in T', with $|L_z| \leq k-1$. Then

$$\sum_{x\in S(T)} \min(|L_x|, k-1) = \sum_{x\in S(T')} \min(|L_x|, k-1) + \min(m, k-1) + \sum_{i=1}^{p} t_i - 1.$$

(a) $n' + \sum_{x\in S(T')} \min(|L_x|, k-1)$ is even. By $n'' \geq \sum_{i=1}^{p} t_i + m + p + 1$,

$$\begin{aligned}
\beta_k(T) &= \beta_k(T') + n'' - 1 \\
&\geq \left(n' + \sum_{x\in S(T')} \min(|L_x|, k-1)\right)/2 + n'' - 1 \\
&= \left(n' + 2n'' - 2 + \sum_{x\in S(T')} \min(|L_x|, k-1)\right)/2 \\
&\geq \left(p + m - \min(m, k-1) + n + \sum_{x\in S(T)} \min(|L_x|, k-1)\right)/2 \\
&\geq \left(n + 1 + \sum_{x\in S(T)} \min(|L_x|, k-1)\right)/2 \\
&\geq \left\lceil \left(n + \sum_{x\in S(T)} \min(|L_x|, k-1)\right)/2 \right\rceil = \beta_k(T).
\end{aligned}$$

So, $\beta_k(T) = \left\lceil \left(n + \sum_{x\in S(T)} \min(|L_x|, k-1)\right)/2 \right\rceil$ if and only if we have equality througouth the previous inequality chain, that is, if and only if $\beta_k(T') = \left\lceil \left(n' + \sum_{x\in S(T')} \min(|L_x|, k-1)\right)/2 \right\rceil$ and $p + m - \min(m, k-1) = 1$. By the inductive hypothesis on T', $T' \in \mathcal{H}(k)$. Also $m \geq 1$, for otherwise $p = 1$ and we obtain a contradiction.

(i) Now, if $m \leq k-1$, then $m - \min(m, k-1) = 0$ and $p = 1$. Hence $m = k-1$ (because $d_{T''}(u) = m+1 \geq k$), and so $T'' = C(t, k-1)$ with $1 \leq t \leq k-2$ for $k \geq 3$, attached to T' by the support vertex u. Thus T is obtained from T' by using Operation $\mathcal{H}_7(k)$. It follows that $T \in \mathcal{H}(k)$.

(ii) If $m \geq k$, then $m - \min(m, k-1) = 1$ and $p = 0$. Hence $m = k$, and so $T'' = C(k)$, attached to T' by the support vertex u. Thus T is obtained from T' by using Operation $\mathcal{H}_8(k)$. It follows that $T \in \mathcal{H}(k)$.

(b) $n' + \sum_{x\in S(T')} \min(|L_x|, k-1)$ is odd. By $n'' \ge \sum_{i=1}^{p} t_i + m + p + 1$,

$$\begin{aligned}
\beta_k(T) &= \beta_k(T') + n'' - 1 \\
&\ge \left\lceil \left(n' + \textstyle\sum_{x\in S(T')} \min(|L_x|, k-1)\right)/2 \right\rceil + n'' - 1 \\
&= \left(n' + 1 + 2n'' - 2 + \textstyle\sum_{x\in S(T')} \min(|L_x|, k-1)\right)/2 \\
&\ge \left(p + m - \min(m, k-1) + 1 + n + \textstyle\sum_{x\in S(T)} \min(|L_x|, k-1)\right)/2 \\
&\ge \left(n + 2 + \textstyle\sum_{x\in S(T)} \min(|L_x|, k-1)\right)/2 \\
&> \left\lceil \left(n + \textstyle\sum_{x\in S(T)} \min(|L_x|, k-1)\right)/2 \right\rceil = \beta_k(T), \text{ a contradiction.}
\end{aligned}$$

Case 2. $d_T(u) = k$. Then let T' and T'' be the components of $T - vz$, containing z and v, respectively. Then from the two conditions in the choice of u, $d_{T''}(u) = k$, the vertices of $N_{T''}(v) \setminus \{u\}$ have degree at most k and all vertices of $V(T'') \setminus N_{T''}[v]$ have degree less than k. Hence v fulfills the conditions on the vertex w in Lemma 5. Apply the inductive hypothesis to T' since $|Y(T')| < |Y(T)|$. Moreover $|V(T'')| = n'' \ge k + 1$ and thus

$$\begin{aligned}
\beta_k(T) &= \beta_k(T') + n'' - 1 \\
&\ge \left\lceil \left(n' + \textstyle\sum_{x\in S(T')} \min(|L_x|, k-1)\right)/2 \right\rceil + n'' - 1.
\end{aligned}$$

Suppose that $|L_v| = m \ge 0$, and v has $u_1, \ldots, u_p$ as support vertices descendant from v, with $1 \le t_i = |L_{u_i}| \le k-1$ for $1 \le i \le p$ and $k \ge 2$. Also v have q vertices descendant from v which are different to leaves, support and to v (if $m = 0$). So $|S(T'' - \{v\})| = p \ge 1$, $n(T'') = n'' = \sum_{i=1}^{p} t_i + p + q + m + 1$, $q \ge 0$, and $p + q + m - \min(m, k-1) - 1 \ge 0$. We distinguish between two subcases.

Subcase 2.1. z is not a leaf, or z is a leaf of a support vertex w with $|L_w| \ge k$. Then

$$\begin{aligned}
\textstyle\sum_{x\in S(T)} \min(|L_x|, k-1) &= \textstyle\sum_{x\in S(T')} \min(|L_x|, k-1) + \min(m, k-1) + \sum_{i=1}^{p} t_i \\
&= \textstyle\sum_{x\in S(T')} \min(|L_x|, k-1) + n'' - p - q - 1 - m \\
&\quad + \min(m, k-1).
\end{aligned}$$

(a) $n' + \sum_{x\in S(T')} \min(|L_x|, k-1)$ is even. Then

$$\begin{aligned}
\beta_k(T) &= \beta_k(T') + n'' - 1 \\
&\ge \left(n' + \textstyle\sum_{x\in S(T')} \min(|L_x|, k-1)\right)/2 + n'' - 1 \\
&= \left(p + q + m - \min(m, k-1) - 1 + n + \textstyle\sum_{x\in S(T)} \min(|L_x|, k-1)\right)/2 \\
&\ge \left(n + \textstyle\sum_{x\in S(T)} \min(|L_x|, k-1)\right)/2.
\end{aligned}$$

Indeed, if $n+\sum_{x\in S(T)}\min(|L_x|,k-1)$ is even, then

$$\left(n+\sum_{x\in S(T)}\min(|L_x|,k-1)\right)/2=\left\lceil\left(n+\sum_{x\in S(T)}\min(|L_x|,k-1)\right)/2\right\rceil =\beta_k(T),$$

hence we have equality througouth the previous inequality chain, and so $p+q+m-\min(m,k-1)-1=0$. Also, if $n+\sum_{x\in S(T)}\min(|L_x|,k-1)$ is odd, then

$$\begin{aligned}\beta_k(T)&=\left\lceil\left(n+\sum_{x\in S(T)}\min(|L_x|,k-1)\right)/2\right\rceil\\&\geq\left(n+1+\sum_{x\in S(T)}\min(|L_x|,k-1)\right)/2\\&=\left\lceil\left(n+\sum_{x\in S(T)}\min(|L_x|,k-1)\right)/2\right\rceil=\beta_k(T),\end{aligned}$$

hence we have equality througouth the previous inequality chain, and so $p+q+m-\min(m,k-1)-1=1$. So, $\beta_k(T)=\left\lceil\left(n+\sum_{x\in S(T)}\min(|L_x|,k-1)\right)/2\right\rceil$ if and only if $\beta_k(T')=\left\lceil\left(n'+\sum_{x\in S(T')}\min(|L_x|,k-1)\right)/2\right\rceil$ and $0\leq p+q+m-\min(m,k-1)-1\leq 1$. By the inductive hypothesis on T', $T'\in\mathcal{H}(k)$ and $1\leq p+q\leq 2$.

(i) Now, if $m=0$, then $1\leq p+q\leq 2$, and so we distingish between two situations.

If $p+q=1$, then $p=1,q=0$, because $p\geq 1$. So $T''=C(k)$ is attached to T' by a leaf v. Thus T is obtained from T' by using Operation $\mathcal{H}_1(k)$. It follows that $T\in\mathcal{H}(k)$.

If $p+q=2$, then $p=2$ and $q=0$ because the case $p=1$ and $q=1$ is impossible. Observe that either $d_{T''}(v)=2$ and $T''=F_1(v)$ for $m=0$, or $d_{T''}(v)=1$ and $T''=F_2(v)$ for $m=0$ and $k\geq 3$, respectively, and T'' is attached to T' by a vertex v. Thus T is obtained from T' by using Operation $\mathcal{H}_{11}(k)$, or Operation $\mathcal{H}_{12}(k)$, respectively. It follows that $T\in\mathcal{H}(k)$.

(ii) If $m\geq 1$, then consider the following situations.

If $p+q=1$, then $p=1,q=0$ and $0\leq m-\min(m,k-1)\leq 1$. For $m-\min(m,k-1)=0$, we have $1\leq m\leq k-1$, and so $T''=C(k-1,m)$ with $1\leq m\leq k-1$. For $m-\min(m,k-1)=1$, we have $m=k$, and so $T''=C(k-1,k)$. Hence $T''=C(k-1,m)$ with $1\leq m\leq k-1$ or $T''=C(k-1,k)$, respectively, and T'' is attached to T' by the support vertex v. Thus T is obtained from T' by using Operation $\mathcal{H}_9(k)$ or Operation $\mathcal{H}_{10}(k)$, respectively. It follows that $T\in\mathcal{H}(k)$.

If $p+q=2$, then $m-\min(m,k-1)=0$, so $m\leq k-1$ and $p=2$ and $q=0$. Observe that either $d_{T''}(v)=m+2$ and $T''=F_1(v)$ for $m>0$, or $d_{T''}(v)=m+1$ and $T''=F_2(v)$ for $m>0$ and $k\geq 3$, respectively, and T'' is attached to T' by the support vertex v. Thus T is obtained from T' by using Operation $\mathcal{H}_{11}(k)$, or Operation $\mathcal{H}_{12}(k)$, respectively. It follows that $T\in\mathcal{H}(k)$.

(b) $n' + \sum_{x\in S(T')} \min(|L_x|, k-1)$ is odd. Then

$$\begin{aligned}\beta_k(T) &= \beta_k(T') + n'' - 1\\ &\geq \Big(n' + 1 + \textstyle\sum_{x\in S(T')} \min(|L_x|, k-1)\Big)/2 + n'' - 1\\ &= \Big(p + q + m - \min(m, k-1) + n + \textstyle\sum_{x\in S(T)} \min(|L_x|, k-1)\Big)/2\\ &\geq \Big(n + 1 + \textstyle\sum_{x\in S(T)} \min(|L_x|, k-1)\Big)/2\\ &\geq \left\lceil \Big(n + \textstyle\sum_{x\in S(T)} \min(|L_x|, k-1)\Big)/2 \right\rceil = \beta_k(T).\end{aligned}$$

So, $\beta_k(T) = \left\lceil \Big(n + \sum_{x\in S(T)} \min(|L_x|, k-1)\Big)/2 \right\rceil$ if and only if we have equality througouth the previous inequality chain, that is, if and only if $\beta_k(T') = \left\lceil \Big(n' + \sum_{x\in S(T')} \min(|L_x|, k-1)\Big)/2 \right\rceil$ and $p+q+m-\min(m, k-1) = 1$. By the inductive hypothesis on T', $T' \in \mathcal{H}(k)$, $p = 1, q = 0$ and since $m - \min(m, k-1) = 0$, $0 \leq m \leq k-1$.

(i) Now, if $m = 0$, then $T'' = C(k)$, attached to T' by the leaf v. Thus T is obtained from T' by using Operation $\mathcal{H}_1(k)$. It follows that $T \in \mathcal{H}(k)$.

(ii) If $1 \leq m \leq k-1$, then $T'' = C(k-1, m)$ with $1 \leq m \leq k-1$, attached to T' by the support vertex v. Thus T is obtained from T' by using Operation $\mathcal{H}_9(k)$. It follows that $T \in \mathcal{H}(k)$.

Subcase 2.2. z is a leaf of a support vertex w with $|L_w| \leq k-1$. Then

$$\begin{aligned}\textstyle\sum_{x\in S(T)} \min(|L_x|, k-1) &= \textstyle\sum_{x\in S(T')} \min(|L_x|, k-1) + \min(m, k-1) - 1 + \sum_{i=1}^{p} t_i\\ &= \textstyle\sum_{x\in S(T')} \min(|L_x|, k-1) + n'' - p - q - 2 - m\\ &\quad + \min(m, k-1).\end{aligned}$$

(a) $n' + \sum_{x\in S(T')} \min(|L_x|, k-1)$ is even. Then

$$\begin{aligned}\beta_k(T) &= \beta_k(T') + n'' - 1\\ &\geq \Big(n' + \textstyle\sum_{x\in S(T')} \min(|L_x|, k-1)\Big)/2 + n'' - 1\\ &= \Big(p + q + m - \min(m, k-1) + n + \textstyle\sum_{x\in S(T)} \min(|L_x|, k-1)\Big)/2\\ &\geq \Big(n + 1 + \textstyle\sum_{x\in S(T)} \min(|L_x|, k-1)\Big)/2\\ &\geq \left\lceil \Big(n + \textstyle\sum_{x\in S(T)} \min(|L_x|, k-1)\Big)/2 \right\rceil = \beta_k(T).\end{aligned}$$

So, $\beta_k(T) = \left\lceil \Big(n + \sum_{x\in S(T)} \min(|L_x|, k-1)\Big)/2 \right\rceil$ if and only if we have equality througouth the previous inequality chain, that is, if and only if $\beta_k(T') =$

$\left\lceil \left(n' + \sum_{x\in S(T')} \min(|L_x|, k-1)\right)/2\right\rceil$ and $p+q+m-\min(m,k-1)=1$. By the inductive hypothesis on T', $T' \in \mathcal{H}(k)$, $p=1$, $q=0$ and since $m-\min(m,k-1)=0$, $0 \le m \le k-1$.

(i) Now, if $m=0$, then $T''=C(k)$, attached to T' by the leaf v. Thus T is obtained from T' by using Operation $\mathcal{H}_8(k)$. It follows that $T \in \mathcal{H}(k)$.

(ii) If $1 \le m \le k-1$, then $T''=C(k-1,m)$ with $1 \le m \le k-1$, attached to T' by the support vertex v. Thus T is obtained from T' by using Operation $\mathcal{H}_{13}(k)$. It follows that $T \in \mathcal{H}(k)$.

(b) $n' + \sum_{x\in S(T')} \min(|L_x|, k-1)$ is odd. Then

$$\begin{aligned}\beta_k(T) &\ge \left\lceil \left(n' + \textstyle\sum_{x\in S(T')} \min(|L_x|, k-1)\right)/2\right\rceil + n'' - 1\\ &= \left(p+q+m-\min(m,k-1)+1+n+\textstyle\sum_{x\in S(T)} \min(|L_x|, k-1)\right)/2\\ &\ge \left(n+2+\textstyle\sum_{x\in S(T)} \min(|L_x|, k-1)\right)/2\\ &> \left\lceil \left(n+\textstyle\sum_{x\in S(T)} \min(|L_x|, k-1)\right)/2\right\rceil = \beta_k(T), \text{ a contradiction.}\end{aligned}$$

■

In this paper, we established a new lower bound on the k-independance number for any graph. Then we provided a constructive characterization of trees attaining this lower bound. It would be interesting to further investigate a characterization of other classes of graphs (bipartite graphs) attaining this lower bound.

Acknowledgements

We would like to thank the anonymous referees for some comments and suggestions on the former version of this manuscript.

References

[1] S.T. Hedetniemi, *Hereditary properties of graphs*, J. Combin. Theory (B) **14** (1973) 94–99.
doi:10.1016/S0095-8956(73)80009-7

[2] M. Blidia, M. Chellali, O. Favaron and N. Meddah, *On k-independence in graphs with emphasis on trees*, Discrete Math. **307** (2007) 2209–2216.
doi:10.1016/j.disc.2006.11.007

[3] M. Borowiecki and D. Michalak, *Generalized independence and domination in graphs*, Discrete Math. **191** (1998) 51–56.
doi:10.1016/S0012-365X(98)00092-2

[4] R.L. Brooks, *On coloring the nodes of a network*, Math. Proc. Cambridge Philos. Soc. **37** (1941) 194–197.
doi:10.1017/S030500410002168X

[5] Y. Caro and Z. Tuza, *Improved lower bounds on k-independence*, J. Graph Theory **15** (1991) 99–107.
doi:10.1002/jgt.3190150110

[6] O. Favaron, *k-domination and k-independence in graphs*, Ars Combin. **25** (1988) 159–167.

[7] O. Favaron, *On a conjecture of Fink and Jacobson concerning k-domination and k-dependence*, J. Combin. Theory (B) **39** (1985) 101–102.
doi:10.1016/0095-8956(85)90040-1

[8] J.F. Fink and M.S. Jacobson, *n-domination in graphs*, in: Graph Theory with Applications to Algorithms and Computer Science, John Wiley and Sons, New York (1985) 283–300.

[9] J.F. Fink and M.S. Jacobson, *n-domination, n-dependence and forbidden subgraphs*, in: Graph Theory with Applications to Algorithms and Computer Science, John Wiley and Sons, New York (1985) 301–311.

[10] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, Inc., New York, 1998).

[11] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Domination in Graphs: Advanced Topics (Marcel Dekker, Inc., New York, 1998).

[12] L. Volkmann, *A characterization of bipartite graphs with independence number half their order*, Australas. J. Combin. **41** (2008) 219–222.

Received 16 May 2011
Revised 17 May 2012
Accepted 31 May 2012