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STAR COLORING OF SUBCUBIC GRAPHS

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Abstract

A star coloring of an undirected graph G is a coloring of the vertices of G such that (i) no two adjacent vertices receive the same color, and (ii) no path on 4 vertices is bi-colored. The star chromatic number of G, $\chi_s(G)$, is the minimum number of colors needed to star color G. In this paper, we show that if a graph G is either non-regular subcubic or cubic with girth at least 6, then $\chi_s(G) \leq 6$, and the bound can be realized in linear time.

Keywords: vertex coloring, star coloring, subcubic graphs.

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1. Introduction

All our graphs are simple, finite and undirected, and we follow West [14] for standard notations and terminology. A *vertex coloring* (or simply *coloring*) of a graph G is an assignment of colors to the vertices of G such that no two adjacent vertices receive the same color. Vertex coloring of graphs has a vast literature, and several variations of vertex coloring have been introduced and studied by many researchers. We refer to a book by Jensen and Toft [10] for an excellent survey on various graph colorings.

In a vertex coloring of G, the set of colors with the same color is called a *color class*. Obviously, the subgraph induced by the union of two color classes is a bipartite graph. In 1973, Grünbaum [9] proposed several variants of vertex coloring with restrictions on the union of two color classes. Among them, acyclic coloring and star coloring of graphs have received much attention (see [1, 3, 6,

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11, 12, 13]). The interest began when it came to be known that these coloring problems have applications in combinatorial scientific computing. In particular, the star (acyclic) coloring problems correspond to direct (indirect) schemes for recovery of the Hessian matrices. See [7] for a survey of coloring problems as they relate to sparse derivative matrices.

An *acyclic coloring* of a graph G is a coloring of G such that the union of any two color classes induces a forest, and a *star coloring* of a graph G is a coloring of G such that the union of any two color classes induces a star forest (that is, a coloring of G such that no path on four vertices is bi-colored). The *star chromatic number* (*acyclic chromatic number*) of G, denoted by $\chi_s(G)$ ($\chi_a(G)$), is the minimum number of colors required to star (acyclically) color G.

The computation of both $\chi_a(G)$ and $\chi_s(G)$ are NP-hard in general. In particular, both problems are NP-hard even when restricted to bipartite graphs [4, 5]. Albertson *et al.* [1] proved that even if the graph G is planar and bipartite, the problem of deciding whether G has a star coloring with 3 colors is NP-complete. Inapproximability results for both problems are given in [8].

The *degree* of a vertex v in G, $d_G(v)$, is the number of vertices adjacent to v. The *maximum degree* over all vertices in G is denoted by $\Delta(G)$. In 1973, Grünbaum [9] conjectured that if $\Delta(G) = r$, then $\chi_a(G) \leq r + 1$, and he proved that if $\Delta(G) = 3$, then $\chi_a(G) \leq 4$. Skulrattanakulchai [13] gave a linear time algorithm for realizing this bound. In 1979, Burstein [3] proved that if $\Delta(G) = 4$, then $\chi_a(G) \leq 5$. Recently, Kostochka and Stocker [11] showed that if $\Delta(G) = 5$, then $\chi_a(G) \leq 7$. See [11, 12] for more results on acyclic coloring, and [2] for more general notion of such restricted colorings and their bounds.

In this paper, we are interested in bounds for the star chromatic number of graphs with bounded maximum degree. Using the notion of in-coloring, Albertson *et al.* [1] proved that if a graph G has an acyclic orientation with maximum in-degree k, then $\chi_s(G) \leq k\Delta(G)+1$. As a corollary of this result, they proved that for any graph G, $\chi_s(G) \leq \Delta(G)(\Delta(G)-1)+2$, and the equality holds if some component of G is Δ-regular. Furthermore, it is known [6] that $\chi_s(G) \leq c(\Delta(G))^{\frac{3}{2}}$, for any arbitrary graph G, where c is a suitably chosen positive constant. This bound is also known to be tight within an $O((\log \Delta)^{\frac{1}{2}})$ multiplicative factor. This leads to the natural question of whether we can tightly bound the star chromatic number for graphs with bounded maximum degree. In particular, we focus on graphs G with $\Delta(G) \leq 3$. Fertin *et al.* [6] showed that if G is a cubic graph, then $\chi_s(G) \leq 9$. Later, Albertson *et al.* [1] improved this result by proving that if $\Delta(G) \leq 3$, then $\chi_s(G) \leq 7$.

We show that if a graph G is either non-regular subcubic or cubic with girth at least 6, then $\chi_s(G) \leq 6$. Also, we give a linear time algorithm to realize this bound. We show this by using a technique of Burstein [3]: We iteratively star color the vertices of the given graph G one by one. At any point, we extend the current partial star coloring of G by one more vertex after recoloring (if necessary) some previously colored vertices. The recoloring is done only for some subset of vertices each of which is at a distance at most 3 from the current vertex to be colored.

2. NOTATIONS AND TERMINOLOGY

A graph G is said to be *subcubic* if $\Delta(G) \leq 3$, and is *cubic* if $d_G(v) = 3$, for all $v \in V(G)$. The *girth* of G is the length of a shortest cycle in G. For $v \in V(G)$ and $i \geq 0$, let $N_i(\{v\}) = \{u \in V(G) : d(v,u) = i\}$ (here $d(x,y)$ is the length of a shortest path between two vertices x and y), and we simply denote it by $N_i(v)$. $N_1(v)$ is denoted by $N(v)$, and $N[v] = N(v) \cup \{v\}$. For $x, y \in V(G)$, we write $N(x) - y$ instead of $N(x) \setminus \{y\}$.

Let $S = \{1,2,3,4,5,6\}$. A partial star coloring is an assignment of colors to a subset of $V(G)$ such that the colored vertices induce a graph with a star coloring. Suppose G has a partial star coloring and let v be an uncolored vertex. We say that a color $\alpha \in S$ is *available* for v if no neighbor of v is colored α. A color $\alpha \in S$ is *feasible* for v if assigning the color α to v still results in a partial star coloring. Let $\alpha, \beta \in S$. An (α,β)-*path* is a path in G with each vertex colored α or β. A vertex v is in an (α,β)- *dangerous path* ((α,β)-DP, for short) P if v is uncolored and $P - v$ is an (α,β)-path on three vertices. A vertex v is in an (α,β)-*partial dangerous path* ((α,β)-PDP, for short) P if v is colored and $P - v$ is an (α,β)-path on three vertices.

We use the following simple observation often.

Recoloring tool (RCT): Let G be a subcubic graph, and let π be a partial star coloring of G using colors from S. Let v be a vertex in G which is colored (or uncolored). Let x be a neighbor of v such that $\pi(x) = c$, and let x_1 and x_2 be the neighbors of x such that $\pi(x_1) = c_1$ and $\pi(x_2) = c_2$. If v is in both (c,c_1)- and (c,c_2)-PDP's (or DP's), then $c_1 \neq c_2$ (since π is a partial star coloring), and $|\pi((N[x] - v) \cup N(N(x) - v))| \leq 5$. Hence, there exists a color $\alpha \in S \setminus \{c, c_1, c_2\}$ such that $\alpha \notin \pi((N[x] - v) \cup N(N(x) - v))$.

In the following (unless otherwise stated), we assume that G is connected.

3. STAR COLORING OF NON-REGULAR SUBCUBIC GRAPHS

In this section, we prove that if G is a non-regular subcubic graph, then $\chi_s(G) \leq 6$. We first observe the following simple facts:

(i) A graph G is subcubic if and only if G is a subgraph of some cubic graph H.

(ii) If G is a connected non-regular subcubic graph, then every subgraph of G has a vertex of degree at most 2 (that is, G is a 2-*degenerate graph*).

Making use of the above facts, we obtain the star coloring as follows: First, we linearly order the vertices of the given connected non-regular subcubic graph G as $v_1, v_2, v_3, \ldots, v_n$ such that each v_i has at most two neighbors in the subgraph G_i induced by the vertices $\{v_i, v_{i+1}, \ldots, v_n\}$. Next, we star color the vertices of G in the reverse order starting from v_n. As a result, it suffices to show that a star coloring of G_{i+1} can be extended to a star coloring of G_i, for every $i < n$. It follows as a consequence of the lemma we prove below.

Lemma 1. *Let π be a partial star coloring of a subcubic graph G using colors in S, and let v be any uncolored vertex.*

(i) *If v has at most one colored neighbor, then there exists a color $\alpha \in S$ feasible for v.*

(ii) *If v has exactly two colored neighbors, then there exists a partial star coloring π' of G using colors in S and a color $\alpha \in S$ satisfying the following:*

- *π' has the same domain as π.*
- *$\pi'(t) \neq \pi(t)$ implies $t \in N(v)$.*
- *α is feasible for v under π'.*

Moreover, both π' and α can be found in $O(1)$ time.

Proof. (i) If at most one neighbor of v is colored, then v has five available colors, and at least three of them are obviously feasible for v.

(ii) Suppose that v has two colored neighbors, say x and y. If $\pi(x) = \pi(y)$, then since $|N(x) - v| \leq 2$ and $|N(y) - v| \leq 2$, v has five available colors, and at most four of them are not feasible, at least one of them is feasible for v. Suppose $\pi(x) \neq \pi(y)$. Assume (w.l.o.g.) that $\pi(x) = 1$ and $\pi(y) = 2$. If there exists $k \in \{3, 4, 5, 6\}$ and $k \notin \pi(N(x) - v) \cup \pi(N(y) - v)$, then k is obviously feasible for v. So, assume that $\pi(N(x) - v) \cup \pi(N(y) - v) = \{3, 4, 5, 6\}$. Again, w.l.o.g., let $\pi(N(x) - v) = \{3, 4\}$, and $\pi(N(y) - v) = \{5, 6\}$. Then, assume that v appears in all $(1, 3)$-, $(1, 4)$-, $(2, 5)$-, and $(2, 6)$-DP's (else, if v is not in one of these dangerous paths, say v is not in any of the $(1, 3)$-DP's, then 3 is feasible for v). Thus, by RCT, there exists $\alpha \in \{2, 5, 6\}$ such that $\alpha \notin \pi(N[x] - v) \cup \pi(N(N(x) - v))$. If we define π' as $\pi'(x) = \alpha$, and $\pi'(t) = \pi(t)$, for every other colored vertex t, then color 1 is feasible for v. ■

The above lemma implies the following.

Theorem 2. *If G is a non-regular subcubic graph, then $\chi_s(G) \leq 6$. Moreover, such a coloring can be found in $O(n)$ time, where n is the number of vertices in G.*

4. Star Coloring of Cubic Graphs

In this section, we show that if G is a cubic graph with girth at least 6, then $\chi_s(G) \leq 6$. Note that if G is a graph with girth at least 6, then for any $v \in V(G)$, (i) the subgraphs induced by $N(v)$ and $N_2(v)$ are both empty graphs, (ii) if $x, y \in N(v)$, then $N(x) \cap N(y) = \{v\}$, and (iii) if $x \in N(v)$ and $y \in N_3(v)$, then $xy \notin E(G)$. In the following, we use this fact implicitly often.

For the rest of the paper, if G is a given cubic graph with girth at least 6, then we obtain a partial star coloring π' of G from a partial star coloring π of G by changing the colors of only few vertices (using the girth assumption) which we mention them explicitly, and for every other vertex π' agrees with π. Also, if $x \in V(G)$, and if x is uncolored under π, then for convenience, we write $\pi(x) = 0$.

Before proving our main result, we derive a recoloring lemma which recolors a vertex and its local neighborhood in a partial star coloring of a given graph G.

Lemma 3. *Let G be a cubic graph with girth at least 6. Let π be a partial star coloring of G using colors in S, and let v be a vertex in G with $\pi(v) = c$. If v has two colored neighbors x and y, and an uncolored neighbor w, then there exists a partial star coloring π' of G satisfying the following:*

(i) *π' has the same domain as π.*

(ii) *$\pi'(t) \neq \pi(t)$ implies $t \in N[v] \setminus w$ or $t \in N(N(v) - w)$.*

Furthermore, $\pi'(v) \neq \pi(v)$, and if we set $\pi'(w) = c$, then any P_4-path: (w, v, v_1, v_2), where $v_1 \in N(v) - w$ and $v_2 \in N(N(v) - w)$, contains at least three colors under π'.

Moreover, π' can be found in $O(1)$ time.

Proof. We assume that $c = 1$.

Case 1. x and y have the same color. Assume (w.l.o.g.) that $\pi(x) = \pi(y) = 2$. Then $\{3, 4, 5, 6\} \subseteq \pi(N(x) - v) \cup \pi(N(y) - v)$ (else, $\pi'(v) = \alpha$, where $\alpha \in \{3, 4, 5, 6\}$, and $\alpha \notin \pi(N(x) - v) \cup \pi(N(y) - v)$). Assume (w.l.o.g.) that $\pi(N(x) - v) = \{3, 4\}$ and $\pi(N(y) - v) = \{5, 6\}$. Let x_1, x_2 be the neighbors of x, and y_1, y_2 be the neighbors of y. Assume (w.l.o.g.) that $\pi(x_1) = 3, \pi(x_2) = 4, \pi(y_1) = 5$, and $\pi(y_2) = 6$.

If $1 \notin \pi(N(N(x) - v) \cup N(N(y) - v))$, then $\pi'(x) = 1 = \pi'(y)$ and $\pi'(v) = 2$. So, assume that $1 \in \pi(N(N(x) - v) \cup N(N(y) - v))$. Let $1 \in \pi(N(N(x) - v))$. W.l.o.g, let $1 \in \pi(N(x_1) - x)$. Then $5, 6 \in \pi(N(N(x) - v))$ (else, $\pi'(x) = \alpha$, where $\alpha \in \{5, 6\}$ and $\alpha \notin \pi(N(N(x) - v))$ and $\pi'(v) = 3$ or 4). We assume that $5 \in \pi(N(x_1) - x)$ and $6 \in \pi(N(x_2) - x)$ (since the case $5, 6 \in \pi(N(x_2) - x)$ can be easily verified using similar arguments). Let x_1', x_1'' be the neighbors of x_1 such that $\pi(x_1') = 1$ and $\pi(x_1'') = 5$, and let x_2', x_2'' be the neighbors of x_2 such that $\pi(x_2') = 6$ and $\pi(x_2'') = k$, where $k \in \{0, 1, 2, 3, 5, 6\}$. We split the remaining proof into the following cases depending on k.

Case 1.1. $k = 0$. If x is not in any of the $(4,6)$-PDP's, then $\pi'(x) = 6$ and $\pi'(v) = 3$.

If x is in a $(4,6)$-PDP, then by RCT, there exists $\beta \in \{1,2,3,5\}$ such that $\beta \notin \pi((N[x_2] - x) \cup N(N(x_2) - x))$. Now, define $\pi'(x_2) = \beta$, $\pi'(x) = 4$, and $\pi'(v) = 3$.

Case 1.2. $k = 6$. Define $\pi'(x) = 6$ and $\pi'(v) = 3$.

Case 1.3. $k \in \{2,3\}$. Note that x is in a $(3,5)$-PDP (else, $\pi'(x) = 5$ and $\pi'(v) = 4$). First, assume that x is not in any of the $(3,1)$-PDP. If v is not in any of the $(2,\alpha)$-PDP's, where $\alpha \in \{5,6\}$, then define $\pi'(x) = 1$ and $\pi'(v) = \alpha$. So, assume that v is in both $(2,5)$- and $(2,6)$-PDP's. Then by RCT, there exists $\beta \in \{1,3,4\}$ such that $\beta \notin \pi((N[y] - v) \cup N(N(y) - v))$. Now, define $\pi'(x) = 1$, $\pi'(y) = \beta$, and $\pi'(v) = 2$.

So, x is in a $(3,1)$-PDP. Then by RCT, there exists $\beta \in \{2,4,6\}$ such that $\beta \notin \pi((N[x_1] - x) \cup N(N(x_1) - x))$. Now, define π' as follows:

If $k = 2$, then $\pi'(x_1) = \beta$, $\pi'(x) = 3$, and $\pi'(v) = 4$.

If $k = 3$ and $\beta = 2$, then $\pi'(x_1) = \beta = 2$, $\pi'(x) = 5$, and $\pi'(v) = 3$.

If $k = 3$ and $\beta \in \{4,6\}$, then $\pi'(x_1) = \beta$, and $\pi'(v) = 3$.

Case 1.4. $k = 1$. Note that x is in both $(3,5)$- and $(4,6)$-PDP's. For, if x is not in any of the $(3,5)$-PDP's, then define $\pi'(x) = 5$ and $\pi'(v) = 4$. Similar proof holds if x is not in any of the $(4,6)$-PDP's.

First, assume that x is not in any of the $(3,1)$- and $(4,1)$-PDP's. If v is not in any of the $(2,\alpha)$-PDP's, where $\alpha \in \{5,6\}$, then define $\pi'(x) = 1$ and $\pi'(v) = \alpha$. So, assume that v is in both $(2,5)$- and $(2,6)$-PDP's. Then by RCT, there exists $\beta \in \{1,3,4\}$ such that $\beta \notin \pi((N[y] - v) \cup N(N(y) - v))$. Now, define $\pi'(x) = 1$, $\pi'(y) = \beta$ and $\pi'(v) = 2$.

So, x is in at least one of $(3,1)$-, $(4,1)$-PDP's. W.l.o.g., assume that x is in a $(3,1)$-PDP. Then by RCT, there exists $\beta \in \{2,4,6\}$ such that $\beta \notin \pi((N[x_1] - x) \cup N(N(x_1) - x))$. Now, define $\pi'(x_1) = \beta$, $\pi'(x) = 3$, and $\pi'(v) = 4$.

Case 1.5. $k = 5$. Note that x is in both $(3,1)$- and $(4,6)$-PDP's. For, if x is not in any of the $(3,1)$-PDP's, then define $\pi'(x) = 1$ and $\pi'(v) = 4$. Similar proof holds if x is not in any of the $(4,6)$-PDP's.

Now, assume that x is not in any of the $(3,5)$- and $(4,5)$-PDP's. If v is not in any of the $(2,6)$-PDP's, then define $\pi'(x) = 5$ and $\pi'(v) = 6$. So, assume that v is in a $(2,6)$-PDP. Then $\{1,3,4\} \subseteq \pi(N(N(y) - v))$ (else, if $\alpha \in \{1,3,4\}$ and $\alpha \notin \pi(N(N(y) - v))$, then define $\pi'(x) = 5$, $\pi'(y) = \alpha$, and $\pi'(v) = 2$). So, there exist $c_1, c_2 \in \{1,3,4\}$ $(c_1 \neq c_2)$ such that $\pi(N(y_1) - y) = \{c_1, c_2\}$, and $c_1, c_2 \notin \pi(N(y_2) - y)$. We assume that $c_1 = 3$ and $c_2 = 4$ since the other cases can be handled using similar arguments. Assume that y is in both $(5,3)$- and $(5,4)$-PDP's (else, if y is not in any of the $(5,\alpha)$-PDP's, where $\alpha \in \{3,4\}$, then define $\pi'(y) = \alpha$, $\pi'(x) = 5$, and $\pi'(v) = 2$). Thus by RCT, there exists

$\beta \in S \setminus \{3,4,5\}$ such that $\beta \notin \pi((N[y_1]-y) \cup N(N(y_1)-y))$. Now, if $\beta = 2$, then define $\pi'(x) = 5$, $\pi'(y_1) = 2$, $\pi'(y) = 3$ and $\pi'(v) = 6$, and if $\beta \in \{1,6\}$, then define $\pi'(x) = 5$, $\pi'(y_1) = \beta$, $\pi'(y) = 5$ and $\pi'(v) = 2$.

So, x is in at least one of $(3,5)$-, $(4,5)$-PDP's. W.l.o.g., assume that x is in a $(3,5)$-PDP. Then by RCT, there exists $\beta \in \{2,4,6\}$ such that $\beta \notin \pi((N[x_1]-x) \cup N(N(x_1)-x))$. Now, define $\pi'(x_1) = \beta$, $\pi'(x) = 3$, and $\pi'(v) = 4$.

Case 2. x and y have different colors. Assume that $\pi(x) = 2$ and $\pi(y) = 3$. Then we have $\{4,5,6\} \subseteq \pi(N(x)-v) \cup \pi(N(y)-v)$ (else, $\pi'(v) = \alpha$, where $\alpha \in \{4,5,6\}$, and $\alpha \notin \pi(N(x)-v) \cup \pi(N(y)-v)$). Assume (w.l.o.g.) that $\pi(N(x)-v) = \{4,5\}$ and $6 \in \pi(N(y)-v)$. Let x_1, x_2 be the neighbors of x, and y_1, y_2 be the neighbors of y. Assume (w.l.o.g.) that $\pi(x_1) = 4, \pi(x_2) = 5, \pi(y_1) = 6$, and $\pi(y_2) = k$, where $k \in \{0,1,2,4,5,6\}$.

Case 2.1. $k = 0$. If v is not in any of the $(3,6)$-PDP's, then $\pi'(v) = 6$.

If v is in a $(3,6)$-PDP, then by RCT, there exists $\beta \in \{1,2,4,5\}$ such that $\beta \notin \pi((N[y]-v) \cup N(N(y)-v))$. Now, define $\pi'(y) = \beta$, and $\pi'(v) = 3$.

Case 2.2. $k = 6$. Define $\pi'(v) = 6$.

Case 2.3. $k \in \{1,2\}$. Note that v is in $(2,4)$-, $(2,5)$-, and $(3,6)$-PDP's. So by RCT, there exists $\beta \in \{1,3,6\}$ such that $\beta \notin \pi((N[x]-v) \cup N(N(x)-v))$. Now, define π' as follows:

If $k = 1$, then $\pi'(x) = \beta$, and $\pi'(v) = 2$.

If $k = 2$ and $\beta \in \{1,6\}$, then if v is not in any of the $(2,3)$-PDP's, then $\pi'(x) = \beta$, and $\pi'(v) = 2$. Otherwise if v is in a $(2,3)$-PDP, then by RCT, there exists $\gamma \in \{1,4,5\}$ such that $\gamma \notin \pi((N[y]-v) \cup N(N(y)-v))$. Now, define $\pi'(x) = \beta$, $\pi'(y) = \gamma$, and $\pi'(v) = 3$.

If $k = 2$ and $\beta = 3$, then define a partial star coloring π_1 as $\pi_1(x) = \beta = 3$, and $\pi_1(t) = \pi(t)$, for every other colored vertex t. Now, proceed as in Case 1 by assuming π_1 for G to obtain π' with the required conditions.

Case 2.4. $k \in \{4,5\}$. Let $k = 4$. Note that v is in both $(2,5)$- and $(3,6)$-PDP's. Also, v is in at least one of $(2,4)$-, $(3,4)$-PDP's. W.l.o.g., assume that v is in a $(2,4)$-PDP. Then by RCT, there exists $\beta \in \{1,3,6\}$ such that $\beta \notin \pi((N[x]-v) \cup N(N(x)-v))$. Now, define $\pi'(x) = \beta$, and $\pi'(v) = 2$.

The case $k = 5$ can be handled using similar arguments. ■

Next by using Lemma 3, we prove the following lemma which together with Lemma 1 imply our result (Theorem 5).

Lemma 4. *Let G be a cubic graph with girth at least* 6. *Let π be a partial star coloring of G using colors in S, and let v be any uncolored vertex. If v has exactly three colored neighbors, then there exists a partial star coloring π' of G using colors in S and a color $\alpha \in S$ satisfying the following:*

(i) *π' has the same domain as π.*

(ii) *$\pi'(t) \neq \pi(t)$ implies $t \in N(v)$ or $t \in N_2(v)$ or $t \in N_3(v)$.*

(iii) *α is feasible for v under π'.*

Moreover, both π' and α can be found in $O(1)$ time.

Proof. Suppose v has three colored neighbors, say x, y, z.

Case 1. All three neighbors of v have the same color. Assume (w.l.o.g.) that $\pi(x) = \pi(y) = \pi(z) = 1$. Also, assume that $\{2, 3, 4, 5, 6\} \subseteq \pi(N_2(v))$ (else, if $\alpha \in \{2, 3, 4, 5, 6\}$ and $\alpha \notin \pi(N_2(v))$, then α is obviously feasible for v). Then there exist colors $c_1, c_2 \in \{2, 3, 4, 5, 6\}$ such that $c_1 \neq c_2$ and c_1, c_2 belong to (say) $\pi(N(x) - v)$ and $c_1, c_2 \notin \pi((N(y) - v) \cup (N(z) - v))$. Assume (w.l.o.g.) that $c_1 = 2$ and $c_2 = 3$. Let x_1 and x_2 be the neighbors of x such that $\pi(x_1) = 2$ and $\pi(x_2) = 3$. We show that either 2 or 3 is feasible for v.

Now, assume that $\{4, 5, 6\} \subseteq \pi(N(N(x) - v))$ (else, if $\beta \in \{4, 5, 6\}$ and $\beta \notin \pi(N(N(x) - v))$, then define $\pi'(x) = \beta$, hence both 2 and 3 are obviously feasible for v). W.l.o.g., let x_1', x_1'' be the neighbors of x_1 such that $\pi(x_1') = 4$ and $\pi(x_1'') = 5$, and let x_2', x_2'' be the neighbors of x_2 such that $\pi(x_2') = 6$ and $\pi(x_2'') = k$, where $k \in \{0, 1, 2, 4, 5, 6\}$. We split the remaining proof into the following cases depending on k.

*Case*1.1. $k = 0$. If x is not in any of the $(3, 6)$-PDP's, then $\pi'(x) = 6$, and $\pi'(v) = 2$.

If x is in a $(3, 6)$-PDP, then by RCT, there exists $\beta \in \{1, 2, 4, 5\}$ such that $\beta \notin \pi((N[x_2] - x) \cup N(N(x_2) - x))$. Now, if we define $\pi'(x_2) = \beta$, $\pi'(x) = 3$, then 2 is feasible for v.

Case 1.2. $k \in \{1, 2\}$. We assume that x is in both $(2, 4)$- and $(2, 5)$-PDP's (else, if x is not in any of the $(2, \alpha)$-PDP's, where $\alpha \in \{4, 5\}$, then define $\pi'(x) = \alpha$, hence 3 is obviously feasible for v). So by RCT, there exists $\beta \in \{1, 3, 6\}$ such that $\beta \notin \pi((N[x_1] - x) \cup N((N(x_1) - x)))$.

If $\beta \in \{1, 6\}$, then define $\pi'(x_1) = \beta$, and $\pi'(x) = 4$ or 5. Hence both 2 and 3 are feasible for v.

If $k = 1$ and $\beta = 3$, then define $\pi'(x_1) = \beta$, and $\pi'(x) = 2$. Hence 3 is feasible for v.

If $k = 2$ and $\beta = 3$, then define $\pi'(x_1) = \beta$. Hence 2 is feasible for v.

Case 1.3. $k \in \{4, 5\}$. Let $k = 4$. Assume that x is in a $2, 5-$PDP (else, define $\pi'(x) = 5$, and hence 3 is feasible for v). Note that $2 \in \pi(N(x_1'') - x_1)$. Also, assume that $\{1, 3, 6\} \subseteq \pi(N(N(x_1) - x))$ (else, define $\pi'(x_1) = \alpha$, where $\alpha \in \{1, 3, 6\}$ and $\alpha \notin \pi(N(N(x_1) - x))$, and $\pi'(x) = 2$, so, 3 is feasible for v). We assume that $\pi(N(x_1') - x_1) = \{1, 3\}$ and $\pi(N(x_1'') - x_1) = \{2, 6\}$ (since the other possibilities can be easily verified in a similar manner).

Now assume that x_1 is in both $(4,1)$- and $(4,3)$-PDP's (else, if x_1 is not in any of the $(4,\alpha)$-PDP's, where $\alpha \in \{1,3\}$, then define $\pi'(x_1) = \alpha$, and $\pi'(x) = 2$, hence 3 is feasible for v). So by RCT, there exists $\beta \in \{2,5,6\}$ such that $\beta \notin \pi((N[x_1'] - x_1) \cup N(N(x_1') - x_1))$. Now, if we define $\pi'(x_1') = \beta$, $\pi'(x_1) = 4$, and $\pi'(x) = 2$, then 3 is feasible for v.

Case 1.4. The case $k = 5$ can be handled using similar arguments.

Case 1.5. $k = 6$. If we define $\pi'(x) = 6$, then 2 is obviously feasible for v.

Case 2. Two neighbors of v have the same color, the third has a different color from these two. Assume (w.l.o.g.) that $\pi(x) = \pi(y) = 1$ and $\pi(z) = 2$. Also, assume that $\{3,4,5,6\} \subseteq \pi(N_2(v))$ (else, if $\beta \in \{3,4,5,6\}$ and $\beta \notin \pi(N_2(v))$, then β is feasible for v).

If $2 \notin \pi(N(x) - v) \cup \pi(N(y) - v)$, then if z has at most one colored neighbor, then it is easy to see that there exists a partial star coloring π_1 of G such that $\pi_1(z) \neq 2$, since $|\pi(N[z] - v) \cup \pi(N(N(z) - v))| \leq 4$. Else, by applying Lemma 3 to z, there exists a partial star coloring π_1 of G such that $\pi_1(z) \neq 2$. If $\pi_1(z) \neq 1$, then define π' as π_1, and the color 2 is feasible for v (since if $\pi_1(v) = 2$, then the path (v, z, z_1, z_2), where $z_1 \in N(z) - v$ and $z_2 \in N(N(z) - v)$, contains at least three colors in G under π_1). If $\pi_1(z) = 1$, then proceed as in Case 1 by assuming the partial star coloring π_1 for G to get a partial star coloring π' of G and a feasible color for v.

So, assume that $2 \in \pi(N(x) - v) \cup \pi(N(y) - v)$. W.l.o.g., let x_1 be a neighbor of x such that $\pi(x_1) = 2$. Let z_1, z_2 be the neighbors of z. By assumptions, there exists a color $k \in \{3,4,5,6\}$ such that $k \in \pi(N(z) - v)$ and $k \notin \pi(N(x) - v) \cup \pi(N(y) - v)$, say $k = 6$. Let $\pi(z_1) = 6$, and let $\pi(z_2) = r$, $r \in \{0,1,3,4,5,6\}$. We split the remaining proof into the following cases depending on r.

Case 2.1. $r = 0$. If v is not in any of the $(2,6)$-DP's, then 6 is feasible for v.

If v is in a $(2,6)$-DP, then there exists $\beta \in \{3,4,5\}$ such that $\beta \notin \pi((N[z] - \{v, z_2\}) \cup (N(z_1) - z))$. Now, if we define $\pi'(z) = \beta$, then 6 is feasible for v.

Case 2.2 $r = 6$. It is easy to see that 6 is feasible for v. So, assume $r \neq 6$. Also, assume that v is in a $(2,6)$-DP (else, 6 is obviously feasible for v). Note that $2 \in \pi(N(N(z) - v))$. W.l.o.g., let $2 \in \pi(N(z_1) - z)$.

Case 2.3. $r = 1$. Then $\{3,4,5\} \subseteq \pi(N(N(z) - v))$ (else, define $\pi'(z) = \beta$, where $\beta \in \{3,4,5\}$ and $\beta \notin \pi(N(N(z) - v))$; hence, 6 is feasible for v). So, there exist $c_1, c_2 \in \{3,4,5\}$ ($c_1 \neq c_2$) such that $\pi(N(z_2) - z) = \{c_1, c_2\}$, and $c_1, c_2 \notin \pi(N(z_1) - z)$. Assume that z is in both $(1, c_1)$- and $(1, c_2)$-PDP's (else, if z is not in any of the $(1,\alpha)$-PDP's, where $\alpha \in \{c_1, c_2\}$ then define $\pi'(z) = \alpha$, and so 6 is feasible for v). Thus by RCT, there exists $\beta \in S \setminus \{1, c_1, c_2\}$ such that $\beta \notin \pi((N[z_2] - z) \cup N(N(z_2) - z))$. Now, define a partial star coloring π_1 as follows: $\pi_1(z_2) = \beta$, $\pi_1(z) = 1$, and $\pi_1(t) = \pi(t)$, for every other colored vertex t.

Then proceed as in Case 1 by assuming π_1 for G to get a partial star coloring π' of G and a feasible color for v.

Case 2.4. $r \in \{3,4,5\}$. If $1 \notin \pi(N(N(z)-v))$, then define a partial star coloring π_1 as $\pi_1(z) = 1$, and $\pi_1(t) = \pi(t)$, for every other colored vertex t. Then proceed as in Case 1 by assuming π_1 for G to get a partial star coloring π' of G and a feasible color for v.

So, assume that $1 \in \pi(N(N(z)-v))$. Also, recall that $2 \in \pi(N(z_1)-z)$. Let $r = 3$. Then, $\{4,5\} \subseteq \pi(N(N(z)-v))$ (else, define $\pi'(z) = \beta$, where $\beta \in \{4,5\}$ and $\beta \notin \pi(N(N(z)-v))$, and hence 6 is feasible for v).

Assume that $1 \in \pi(N(z_1)-z)$. Then $\pi(N(z_2)-z) = \{4,5\}$. Assume that z is in both $(3,4)$- and $(3,5)$-PDP's (else, if z is not in any of the $(3,\alpha)$-PDP's, where $\alpha \in \{4,5\}$ then define $\pi'(z) = \alpha$, and 6 is feasible for v). So by RCT, there exists $\beta \in \{1,2,6\}$ such that $\beta \notin \pi((N[z_2]-z) \cup N(N(z_2)-z))$. Now, define π' as follows: $\pi'(z_2) = \beta$, and $\pi'(z) = 3$. Hence, 6 is feasible for v.

Assume that $1 \in \pi(N(z_2)-z)$. So, (w .l.o.g.) $4 \in \pi(N(z_1)-z)$ and $5 \in \pi(N(z_2)-z)$. If z is not in any of the $(3,1)$-PDP's, then define a partial star coloring π_1 as $\pi_1(z) = 1$, and $\pi_1(t) = \pi(t)$, for every other colored vertex t. Then proceed as in Case 1 by assuming π_1 for G to get a partial star coloring π' of G and a feasible color for v. So, we conclude that z is in a $(3,1)$-PDP. Also, z is in a $(3,5)$-PDP (else, define $\pi'(z) = 5$, and 6 is feasible for v). Thus by RCT, there exists $\beta \in \{2,4,6\}$ such that $\beta \notin \pi((N[z_2]-z) \cup N(N(z_2)-z))$. Now, define π' as follows: $\pi'(z_2) = \beta$, and $\pi'(z) = 3$. Hence 6 is feasible for v.

Case 2.5. The case $r \in \{4,5\}$ can be handled using similar arguments.

Case 3. All three neighbors of v have the distinct colors. Assume (w.l.o.g.) that $\pi(x) = 1, \pi(y) = 2$ and $\pi(z) = 3$. Also, assume that $\{4,5,6\} \subseteq \pi(N_2(v))$ (else, if $\beta \in \{4,5,6\}$ and $\beta \notin \pi(N_2(v))$, then β is feasible for v).

If $1 \notin \pi(N(y)-v) \cup \pi(N(z)-v)$, then if x has at most one colored neighbor, then it is easy to see that there exists a partial star coloring π_1 of G such that $\pi_1(x) \neq 1$, since $|\pi(N[x]-v) \cup \pi(N(N(x)-v))| \leq 4$. Else, by applying Lemma 3 to x, there exists a partial star coloring π_1 of G such that $\pi_1(x) \neq 1$. If $\pi_1(x) \neq 2,3$, then define π' as π_1, and the color 1 is feasible for v (since if $\pi_1(v) = 1$, then the path (v,x,x_1,x_2), where $x_1 \in N(x)-v$ and $x_2 \in N(N(x)-v)$, contains at least three colors in G under π_1). If $\pi_1(x) = 2$ or 3, then proceed as in Case 2 by assuming the partial star coloring π_1 for G to get a partial star coloring π' of G and a feasible color for v. Similar arguments hold if $2 \notin \pi(N(x)-v) \cup \pi(N(z)-v)$ or $3 \notin \pi(N(x)-v) \cup \pi(N(y)-v)$. So, suppose that $1 \in \pi(N(y)-v) \cup \pi(N(z)-v)$, $2 \in \pi(N(x)-v) \cup \pi(N(z)-v)$ and $3 \in \pi(N(x)-v) \cup \pi(N(y)-v)$. Hence, all the colors in $(N(x)-v) \cup (N(y)-v) \cup (N(z)-v)$ are distinct. We split the remaining proof into two cases.

Case 3.1. The colors $1, 2, 3$ appear in distinct sets in $\{\pi(N(x)-v), \pi(N(y)-v), \pi(N(z)-v)\}$.

Suppose (w.l.o.g.) that $\{2,4\} \subseteq \pi(N(x)-v)$, $\{3,5\} \subseteq \pi(N(y)-v)$, and $\{1,6\} \subseteq \pi(N(z)-v)$. Let x_1, x_2 be the neighbors of x such that $\pi(x_1) = 2$ and $\pi(x_2) = 4$.

We assume that v is in a $(1,4)$-DP (else, 4 is obviously feasible for v). Also, if either 5 or 6 does not belong to $\pi(N(N(x)-v))$, then define $\pi'(x) = 5$ or 6, and hence 4 is feasible for v. So, $\{5,6\} \subseteq \pi(N(N(x)-v))$. If $3 \notin \pi(N(N(x)-v))$, then define a partial star coloring π_1 as $\pi_1(x) = 3$, and $\pi_1(t) = \pi(t)$, for every other vertex t. Then proceed as in Case 2 by assuming the partial star coloring π_1 for G to get a partial star coloring π' of G and a feasible color for v.

So, we conclude that $\{3,5,6\} \subseteq \pi(N(N(x)-v))$. Then there exist $c_1, c_2 \in \{3,5,6\}$ $(c_1 \neq c_2)$ such that $\{c_1, c_2\} \subseteq \pi(N(x_1)-x)$, and $c_1, c_2 \notin \pi(N(x_2)-x)$. Assume that x is in both $(2,c_1)$- and $(2,c_2)$-PDP's (else, if x is not in any of the $(2,\alpha)$-PDP's, where $\alpha \in \{c_1, c_2\}$ then define $\pi'(x) = \alpha$, hence, 4 is feasible for v, since v is not in any of the $(\alpha,4)$-DP's). So, by RCT, there exists $\beta \in S \backslash \{2, c_1, c_2\}$ such that $\beta \notin \pi(N[x_1]-x) \cup \pi(N(N(x_1)-x))$. Now, define a partial star coloring π_1 as follows: $\pi_1(x_1) = \beta$, $\pi_1(x) = 2$, and $\pi_1(t) = \pi(t)$, for every other colored vertex t. Then proceed as in Case 2 by assuming the partial star coloring π_1 for G to get a partial star coloring π' of G and a feasible color for v.

Case 3.2. Two of the colors in $\{1,2,3\}$ appear in some $\pi(N(t)-v)$, where $t \in \{x,y,z\}$. Then there exist $c_1, c_2 \in \{4,5,6\}$ and $c_1 \neq c_2$ such that $\pi(N(r)-v) = \{c_1, c_2\}$, for some $r \in \{x,y,z\}$. Assume (w.l.o.g.) that $c_1 = 4, c_2 = 5$, and $r = x$. Note that $4, 5 \notin \pi(N(y)-v) \cup \pi(N(z)-v)$. We assume that v is in both $(1,4)$- and $(1,5)$-DP's (else, 4 or 5 is feasible for v). So, by RCT, there exists $\beta \in \{2,3,6\}$ such that $\beta \notin \pi(N[x]-v) \cup \pi(N(N(x)-v))$.

Now, if $\beta \in \{2,3\}$, then define a partial star coloring π_1 as $\pi_1(x) = \beta$, and $\pi_1(t) = \pi(t)$, for every other vertex t, and proceed as in Case 2 by assuming the partial star coloring π_1 for G to get a partial star coloring π' of G and a feasible color for v.

If $\beta = 6$, then define $\pi'(x) = 6$, and hence 4 or 5 is feasible for v. ■

From Lemmas 1 and 4, we have the following.

Theorem 5. *If G is a cubic graph with girth at least 6, then $\chi_s(G) \leq 6$. Moreover, such a coloring can be found in $O(n)$ time, where n is the number of vertices in G.*

5. Concluding Remarks

In this paper, we have shown that if G is a graph which is non-regular subcubic or cubic with girth at least 6, then $\chi_s(G) \leq 6$. We do not know of any graph G

explicitly which is non-regular subcubic or cubic with girth at least 6 such that $\chi_s(G) = 6$. However, we have the following observations:

(1) If G is a cubic graph, then $G - e$ is a non-regular subcubic graph, for any edge $e \in E(G)$, and it is easy to verify that $\chi_s(G) \leq \chi_s(G - e) + 1$, for any edge $e \in E(G)$. So, if for every non-regular subcubic graph G, $\chi_s(G) \leq 5$, then for every cubic graph G, we have $\chi_s(G) \leq 6$. Hence, by Theorem 2, it follows that at least one of the following always holds:

  (i) There exists a non-regular subcubic graph G with $\chi_s(G) = 6$.

  (ii) For every cubic graph G, we have $\chi_s(G) \leq 6$.

(2) There exists a cubic graph G with girth 4 such that $\chi_s(G) = 6$. For, consider the Möbius ladder M_8 obtained by adding edges between antipodal vertices of an 8-cycle. It has been shown in [1] that $\chi_s(M_8) = 6$.

(3) There exists a cubic graph G with girth 5 such that $\chi_s(G) = 5$. For example, consider the Petersen graph P. Then, it is easy to see that $\chi_s(P) = 5$.

The girth assumption in Lemma 4 played a crucial role while recoloring a vertex in the neighborhood of an uncolored vertex. But, we believe that the girth assumption in Lemma 4 (and in Theorem 5) can be dropped, and hence we propose the following.

Conjecture 6. *If G is a graph with $\Delta(G) \leq 3$, then $\chi_s(G) \leq 6$.*

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