

## ON THE TOTAL GRAPH OF MYCIELSKI GRAPHS, CENTRAL GRAPHS AND THEIR COVERING NUMBERS

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### Abstract

The technique of counting cliques in networks is a natural problem. In this paper, we develop certain results on counting of triangles for the total graph of the Mycielski graph or central graph of star as well as complete-graph families. Moreover, we discuss the upper bounds for the number of triangles in the Mycielski and other well known transformations of graphs. Finally, it is shown that the achromatic number and edge-covering number of the transformations mentioned above are equated.

**Keywords:** total graph, central graph, middle graph, Mycielski graph, independence number, covering number, edge independence number, edge covering number, chromatic number, achromatic number.

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### 1. INTRODUCTION

We consider only finite undirected, connected graphs, without loops or multiple edges. We follow the terminology of Harary [1] or Bondy *et al.* [4]. For any graph  $G$ , let  $V(G)$ ,  $E(G)$  and  $T(G)$  denote the vertex set, the edge set, and the total graph of  $G$ , respectively. We observe that every edge of  $G$  produces at least one triangle in  $T(G)$ .

In many real-world situations, the problem of determining the number of cliques in networks is a popular problem. In fact, this relates to the technique of counting which has attracted much attention now a days [4]. This is due to the

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fact that when the number of cliques increases in networks, there will be more optimal routes for applications.

In a search for triangle-free graphs with arbitrarily large chromatic numbers, Mycielski [2] introduced the graph-transformation as follows. Let  $G$  be a graph with the vertex set  $V = \{v_i : 1 \leq i \leq n\}$ . The *Mycielski graph* of  $G$ , denoted by  $\mu(G)$ , is the graph obtained from  $G$  by adding  $n + 1$  new vertices  $V' = \{v'_i : 1 \leq i \leq n\}$  and  $u$ , then for  $1 \leq i \leq n$ , joining  $v'_i$  to the neighbours of  $v_i$  and to  $u$ .  $v_i$  and  $v'_i$  are known as *twin-vertices*, and  $V$  and  $V'$  are known as *twin-sets* in  $\mu(G)$ . The vertex  $u$  is called the *root* of  $\mu(G)$ . Clearly,  $V[\mu(G)] = V \cup V' \cup \{u\}$ . The beauty of Mycielski graph  $\mu(G)$  is that it transforms the triangle-free graph  $G$  into a triangle-free graph  $\mu(G)$ , and it produces three new triangles for every triangle of  $G$ . The *iterated Mycielski graph* is defined as follows:  $\mu^n(G) = \mu[\mu^{n-1}(G)]$  for  $n \geq 1$  and  $\mu^0(G) = G$ , [3]. For any graph  $G$ ,  $t(G)$  denotes the number of triangles in  $G$  and  $d_G(v)$  denotes the degree of a vertex  $v$  in  $G$ .

## 2. RESULTS

The following result determines the number of triangles in a total graph.

**Theorem 1.** For any  $(p, q)$  graph  $G$ ,

$$t[T(G)] = t(G) + \frac{1}{2} \sum_{i=1}^p \left[ d_G^2(v_i) + 2m_i \binom{d_G(v_i)}{3} \right],$$

where  $m_i = 1$  if  $d_G(v_i) \geq 3$ ; otherwise  $m_i = 0$ .

**Proof.** We observe that for any edge  $e = uv$  of  $G$ , there is a triangle  $\langle \{u, v, e\} \rangle$  in  $T(G)$ . Since  $G$  is of size  $q$ ,  $T(G)$  contains  $q$  distinct such triangles. For any vertex  $v$  of degree  $n$  in  $G$ , let  $e_1, e_2, \dots, e_n$  be its incident edges. We distinguish two cases.

*Case 1.* For every two distinct edges  $e_i$  and  $e_j$ , there appears a triangle  $\langle \{v, e_i, e_j\} \rangle$  in  $T(G)$ . Consequently for  $n \geq 2$ ,  $T(G)$  contains  $\binom{n}{2}$  distinct such triangles.

*Case 2.* For every three distinct edges  $e_i, e_j$  and  $e_k$  there appears a triangle  $\langle \{e_i, e_j, e_k\} \rangle$  in  $T(G)$ . Consequently for  $n \geq 3$ ,  $T(G)$  contains  $\binom{n}{3}$  distinct such triangles.

The above cases show that

$$\begin{aligned} t[T(G)] &\geq t(G) + q + \sum_{i=1}^p \binom{d_G(v_i)}{2} + \sum_{i=1}^p m_i \binom{d_G(v_i)}{3} \\ (1) \quad &= t(G) + \frac{1}{2} \sum_{i=1}^p \left[ d_G^2(v_i) + 2m_i \binom{d_G(v_i)}{3} \right] \end{aligned}$$

where  $m_i = 1$  if  $d_G(v_i) \geq 3$ ; otherwise  $m_i = 0$ .

On the other hand, suppose  $\langle \{v_i, v_j, v_k\} \rangle$  is a triangle in  $T(G)$ . Then the equality in (1) holds provided  $\langle \{v_i, v_j, v_k\} \rangle$  is either in  $G$  or it can be obtained from the above cases. Next, we discuss four possibilities depending on  $v_i, v_j$  and  $v_k$ .

1. If  $\{v_i, v_j, v_k\} \subseteq V(G)$ , then  $\langle \{v_i, v_j, v_k\} \rangle$  is in  $G$ .
2. If exactly one of  $v_i, v_j$  and  $v_k$ ; say  $v_i$ , is an edge  $e$  of  $G$ , then the triangle  $\langle \{e, v_j, v_k\} \rangle$  is of type discussed in our initial observation.
3. If exactly two of  $v_i, v_j$  and  $v_k$ ; say  $v_i$  and  $v_j$  are edges of  $G$ , then the triangle  $\langle \{v_i, v_j, v_k\} \rangle$  is of type discussed in Case 1.
4. If all  $v_i, v_j$  and  $v_k$  are edges of  $G$ , then the triangle  $\langle \{v_i, v_j, v_k\} \rangle$  is of type discussed in Case 2. ■

**Corollary 2.** (a)  $t[T(P_n)] = 2n - 3$  for  $n \geq 2$ .

(b)  $t[T(C_3)] = 7$  and  $t[T(C_n)] = 2n$  if  $n > 3$ .

(c)  $t[T(K_{m,n})] = \frac{mn}{6}(m^2 + n^2 + 4)$ .

(d)  $t[T(K_n)] = \frac{1}{6}(n^2 - n)(n^2 - n + 1)$ .

The Cartesian product  $G_1 \square G_2 \square \dots \square G_n$  of  $n$  graphs  $G_1, G_2, \dots, G_n$  is the graph with vertex set  $V(G_1) \times V(G_2) \times \dots \times V(G_n)$ , in which the vertex  $(a_1, a_2, \dots, a_n)$  is adjacent to the vertex  $(b_1, b_2, \dots, b_n)$  if  $a_i$  is adjacent to  $b_i$  in  $G_i$  and  $a_j = b_j$  for all  $j \neq i$ ;  $1 \leq i, j \leq n$ . Let  $\square_{i=1}^n C_{m_i} = C_{m_1} \square C_{m_2} \square \dots \square C_{m_n}$ .

**Corollary 3.**  $t[T(\square_{i=1}^n C_{m_i})] = \frac{2Mn}{3}(2n + 1)$  where  $M = m_1 m_2 \dots m_n$ ,  $m_i > 3$  for  $1 \leq i \leq n$  and  $n \geq 2$ .

**Proof.**  $\square_{i=1}^n C_{m_i}$  is a  $2n$ -regular, triangle-free graph of order  $M = m_1 m_2 \dots m_n$ . By Theorem 1,  $t[T(\square_{i=1}^n C_{m_i})] = \frac{1}{2}M(4n^2) + M\binom{2n}{3}$  and is the required result. ■

The next result determines the number of triangles in the total graph of Mycielski graphs.

**Theorem 4.** Let  $G$  be any  $(p, q)$ -graph having  $t(G)$  triangles and  $\delta(G) \geq 2$ . Then

$$t[T(\mu(G))] = 4t(G) + \frac{1}{2} \sum_{i=1}^p [3d_G^3(v_i) + d_G^2(v_i)] + \left( \frac{18q + 5p + p^3}{6} \right).$$

**Proof.** Let  $V = \{v_1, v_2, \dots, v_p\}$  be the vertex set of  $G$ . Then  $V[\mu(G)] = V \cup V' \cup \{w\}$ , where  $V' = \{v' : v \in V\}$  and  $E[\mu(G)] = E(G) \cup \{uv' : uv \in E(G)\} \cup \{v'w : v' \in V'\}$ . For any vertex  $v$  of degree  $n$  in  $G$ , let  $v_1, v_2, \dots, v_n$  be its adjacent vertices. Consequently, the vertices  $v_1, v'_1, v_2, v'_2, \dots, v_n, v'_n$  are all adjacent to  $v$  in  $\mu(G)$ . Therefore,  $d_{\mu(G)}(v) = 2n$ . Corresponding to the vertex  $v$  in  $G$ , there

exists a vertex  $v'$  in  $\mu(G)$ , which is adjacent to all the vertices  $v_1, v_2, \dots, v_n$  and  $w$ . Therefore,  $d_{\mu(G)}(v') = n + 1$ . Further,  $w$  is adjacent to all  $p$  vertices of  $V'$ . Hence,  $d_{\mu(G)}(w) = p$ . Since  $\delta(G) \geq 2$ , we have  $p \geq 3$  and  $d_{\mu(G)}(v) \geq 3$  for each vertex  $v$  in  $\mu(G)$ . Since  $G$  contains  $t(G)$  triangles,  $\mu(G)$  contains  $4t(G)$  triangles. In view of Theorem 1,

$$\begin{aligned} t(T[\mu(G)]) &= 4t(G) + \frac{1}{2} \left[ \sum_{i=1}^p (2d_G(v_i))^2 + \sum_{i=1}^p (d_G(v_i) + 1)^2 + p^2 \right] \\ &\quad + \sum_{i=1}^p \binom{2d_G(v_i)}{3} + \sum_{i=1}^p \binom{2d_G(v_i)+1}{3} + \binom{p}{3} \\ &= 4t(G) + \frac{1}{2} \left[ \sum_{i=1}^p [3d_G^3(v_i) + d_G^2(v_i)] \right] + \left( \frac{18q+5p+p^3}{6} \right). \end{aligned}$$

■

**Corollary 5.**  $t(T[\mu(C_n)]) = \left( \frac{n^3+90n}{6} \right)$  for  $n > 3$ .

**Corollary 6.**  $t(T[\mu(K_{m,n})]) = \frac{1}{6} [9mn(m^2+n^2+2)+m^3+n^3+(m+n)(6mn+5)]$ .

*Proof.* Both  $K_{1,n}$  and  $\mu(K_{1,n})$  are triangle-free graphs. We consider three cases depending on  $m$  and  $n$ :

*Case 1.*  $m = n = 1$ . Then  $t(T[\mu(K_{1,1})]) = 10$ .

*Case 2.*  $m = 1; n \geq 2, V(K_{1,n}) = \{x_1, x_2, \dots, x_{n+1}\}, E(K_{1,n}) = \{x_1x_i : 2 \leq i \leq n + 1\}$ . Then  $V[\mu(K_{1,n})] = \{x_i : 1 \leq i \leq n + 1\} \cup \{x'_i : 1 \leq i \leq n + 1\} \cup \{u\}$  and  $E[\mu(K_{1,n})] = \{x_1x_i : 2 \leq i \leq n + 1\} \cup \{x_1x'_i : 2 \leq i \leq n + 1\} \cup \{x'_1x_i : 2 \leq i \leq n + 1\} \cup \{ux'_i : 1 \leq i \leq n + 1\}$ . In  $\mu(K_{1,n})$ ,  $d_{\mu(K_{1,n})}(x_1) = 2n, d_{\mu(K_{1,n})}(x_i) = d_{\mu(K_{1,n})}(x'_i) = 2$  for  $2 \leq i \leq (n + 1)$ , and  $d_{\mu(K_{1,n})}(x'_1) = d_{\mu(K_{1,n})}(u) = n + 1$ . By Theorem 1,  $t(T[\mu(K_{1,n})]) = (5n^3 + 3n^2 + 19n + 3)/3$ .

*Case 3.*  $m, n \geq 2$ . Then  $\delta(\mu(K_{m,n})) \geq 2$ . In  $\mu(K_{m,n})$ , there are  $m+n$  vertices,  $mn$  edges,  $m$  vertices of degree of  $n$  and  $n$  vertices of degree  $m$ . By Theorem 4,  $t(T[\mu(K_{m,n})]) = \frac{1}{6} [9nm^3 + 9mn^3 + 3mn^2 + 3nm^2 + 18mn + (m+n)^3 + 5(m+n)] = \frac{1}{6} [9mn(m^2 + n^2 + 2) + m^3 + n^3 + (m+n)(6mn + 5)]$ .

In either case, we obtained the desired result. ■

**Corollary 7.**  $t(T[\mu(K_n)]) = \frac{1}{6} (9n^4 - 19n^3 + 18n^2 - 2n)$  for  $n \geq 3$ .

*Proof.*  $V(K_n) = \{v_1, v_2, \dots, v_n\}$ . By the definition of  $\mu(K_n)$ ,  $t[\mu(K_n)] = 4\binom{n}{3}$ . By Theorem 4,  $t(T[\mu(K_n)]) = 4\binom{n}{3} + \frac{1}{2} [4n(n-1)^2 + n^3 + n^2] + 2\binom{2n-2}{3} + (n+1)\binom{n}{3} = \frac{1}{6} (9n^4 - 19n^3 + 18n^2 - 2n)$ . ■

**Definition.** The *middle graph*  $M(G)$  of a graph  $G$  is the graph whose vertex set is  $V(G) \cup E(G)$ , and two vertices of  $M(G)$  are adjacent if either they are adjacent edges of  $G$  or one is a vertex and the other is an edge of  $G$ , incident with it (see [6]).

**Theorem 8.** For any  $(p, q)$ -graph  $G$ ,

$$t[M(G)] = \frac{1}{2} \sum_{i=1}^p \left[ d_G^2(v_i) + 2m_i \binom{d_G(v_i)}{3} \right] - q,$$

where  $m_i = 1$  if  $d_G(v_i) \geq 3$ ; otherwise  $m_i = 0$ .

*Proof.* Follows similarly to that of Theorem 1. ■

The following results give the number of triangles in the middle graph of a cycle or Mycielski graph.

**Corollary 9.**  $t(M[C_n]) = n$  if  $n > 3$ .

**Corollary 10.** For any  $(p, q)$ -graph  $G$ ,

$$t[M(\mu(G))] = \frac{1}{2} \sum_{i=1}^p [3d_G^3(v_i) + d_G^2(v_i)] + \frac{p(p^2 - 1)}{6}.$$

*Proof.* Suppose  $G$  is a  $(p, q)$ -graph having  $t(G)$  triangles. From Theorems 1 and 8, we have  $t(M[G]) = t(T[G]) - q - t(G)$ . Then  $\mu(G)$  contains  $4t(G)$  triangles, and  $(3q + p)$  edges. Hence,

$$(2) \quad t(M[\mu(G)]) = t(T[\mu(G)]) - (3q + p) - 4t(G).$$

In view of Theorem 4, we have

$$(3) \quad t(T[\mu(G)]) = 4t(G) + \frac{1}{2} \sum_{i=1}^p [3d_G^3(v_i) + d_G^2(v_i)] + \left( \frac{18q + 5p + p^3}{6} \right).$$

Using (3) in (2), we get the required result. ■

**Definition.** The *central graph*  $C(G)$  of a graph  $G$  is the graph obtained by subdividing each edge of  $G$  exactly once and joining all the non-adjacent vertices of  $G$  [5]. In the following theorem, we determine the number of triangles in the total graph of a central graph.

**Theorem 11.** For any  $(p, q)$ -graph  $G$  with  $p \geq 4$ ,

$$t[T(C(G))] = \frac{m}{6}(p^4 - 3p^3 + 5p^2 - 3p + 12q),$$

where  $m = t(\overline{G})$ .

**Proof.** By the definition of  $C(G)$ ,  $V[C(G)] = V(G) \cup E(G)$  and  $deg(v_i) = p - 1$  in  $C(G)$  if  $v_i \in V(G)$ ,  $deg(v_i) = 2$  in  $C(G)$  if  $v_i \in E(G)$ . Suppose  $t[C(G)] = m$ , and let  $V(G) = \{v_1, v_2, \dots, v_p\}$ .

Define  $v_i = (b_{i1}, b_{i2}, b_{i3}, \dots, b_{ip})$  for  $1 \leq i \leq p$ . Then  $(b_{i,j})_{p \times p}$  is the adjacency matrix of  $\overline{G}$ , where  $b_{ij} = \begin{cases} 0 & \text{if } i = j \text{ or } v_i v_j \in E(G), \\ 1 & \text{if } i \neq j \text{ and } v_i v_j \notin E(G). \end{cases}$

If  $\langle \{v_i, v_j, v_k\} \rangle = K_3$  in  $G$ , then the adjacency matrix of  $\langle \{v_i, v_j, v_k\} \rangle$  is

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and its determinant is 2. Otherwise, its determinant is 0. It is easy to see that

$$t[C(G)] = \frac{1}{2} \sum_{i=j=k=1}^p \begin{vmatrix} b_{ii} & b_{ij} & b_{ik} \\ b_{ji} & b_{jj} & b_{jk} \\ b_{ki} & b_{kj} & b_{kk} \end{vmatrix} = m \text{ where } i < j < k.$$

By Theorem 1, we have  $t[T(C(G))] = m + \frac{1}{2} \left( \sum_{i=1}^p (p-1)^2 + \sum_{j=1}^q 4 \right) + \sum_{i=1}^p \binom{p-1}{3} = \frac{m}{6}(p^4 - 3p^3 + 5p^2 - 3p + 12q)$ . ■

**Corollary 12.**  $t[T(C(K_n))] = \frac{1}{6}(n^4 - 3n^3 + 11n^2 - 9n)$  for  $n \geq 4$ .

**Proof.** Since  $C(G)$  is triangle-free, by substituting  $p = n \geq 4$ ,  $q = n(n-1)/2$  and  $m = 0$  we get  $t[T(C(K_n))] = \frac{1}{6}(n^4 - 3n^3 + 11n^2 - 9n)$ . ■

**Corollary 13.**  $t[T(C(K_{m,n}))] = t[T(K_{m+n})] + \frac{mn}{2}(m+n-6)$  for  $m, n \geq 3$ .

**Proof.** For  $m, n \geq 3$ , in  $C(K_{m,n})$ , there are  $(mn + m + n)$  vertices in which  $mn$  vertices are of degree two, and  $(m + n)$  vertices are of degree  $(m + n - 1)$ . Moreover  $C(K_{m,n})$  contains  $\binom{m}{3} + \binom{n}{3}$  triangles. Therefore from Theorem 1,  $t[T(C(K_{m,n}))] = \binom{m}{3} + \binom{n}{3} + \frac{1}{2} \left( (m+n)(m+n-1)^2 + 4mn \right) + (m+n) \binom{m+n-1}{3} = \frac{1}{6} \left( ([m+n]^2 - [m+n])^2 + ([m+n]^2 - [m+n]) - 3mn(m+n-6) \right)$ .

By Corollary 5,  $t[T(C(K_{m,n}))] = t[T(K_{m+n})] + \frac{mn}{2}(m+n-6)$ . ■

### 3. COVERINGS AND ACHROMATIC NUMBERS ON CERTAIN FAMILIES OF GRAPHS

We now study certain coverings, and achromatic numbers, connected with total graph of Mycielski graphs on star-graph or complete graph families. It is well-known that for any graph  $G$  of order  $n$ ,  $\alpha(G) + \beta(G) = n$ , where  $\alpha(G)$  and  $\beta(G)$  denote the independence number and covering number of  $G$ , respectively. In addition, if  $\delta(G) > 0$ , then  $\alpha'(G) + \beta'(G) = n$ , where  $\alpha'(G)$  and  $\beta'(G)$  denote

the edge-independence number and edge-covering number of  $G$ , respectively [4]. The *chromatic number*  $\chi(G)$ , of a graph  $G$  is the minimum  $k$  for which  $G$  is  $k$ -colourable. The *achromatic number*  $\chi_c(G)$  of a graph  $G$  is the largest number of colours needed to legally colour the vertices of  $G$  so that the adjacent vertices get different colours, and for every pair of distinct colours  $c_1, c_2$ , there exists at least one edge whose end vertices are coloured by  $c_1, c_2$  [7, 3]. In the following theorem, we determine the above mentioned parameters for the star-graph families.

**Theorem 14.** For a star-graph  $K_{1,n}$  with  $n \geq 2$ , we have

(a)  $\alpha' [T(\mu(K_{1,n}))] = \alpha[\mu(K_{1,n})] + \alpha(K_{1,n}) + \alpha'[\mu(K_{1,n})] - \alpha'(K_{1,n})$ ,

(b)  $\beta' [T(\mu(K_{1,n}))] = \beta'[\mu(K_{1,n})] + \beta'(K_{1,n}) + \beta[\mu(K_{1,n})] - \beta(K_{1,n})$ .

Furthermore, (a) = (b).

**Proof.** Let  $V(K_{1,n}) = \{x_i : 1 \leq i \leq n + 1\}$  for  $n \geq 2$  and  $E(K_{1,n}) = \{e_j = x_1x_{j+1} : 1 \leq j \leq n\}$ . It is easy to see that for  $K_{1,n}$ , we have  $\alpha = n$  and  $\beta = 1$ . By König's theorem when applied to  $K_{1,n}$ , we have  $\alpha' = \beta = 1$  and  $\beta' = \alpha = n$ , (see Figure 1(a)).

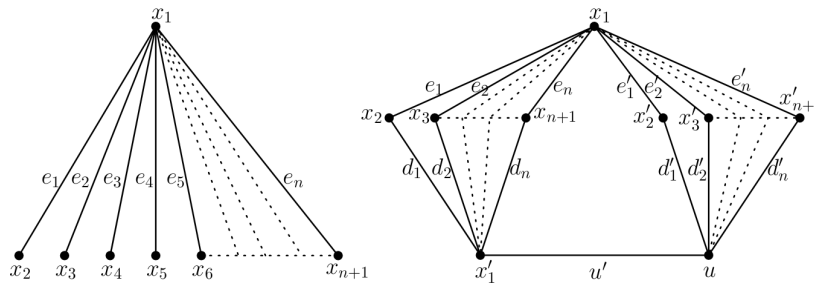


Figure 1. (a) Star graph  $K_{1,n}$ .

(b) Mycielski graph of  $K_{1,n}$ .

By the definition of Mycielski graph,  $|V[\mu(K_{1,n})]| = 2n + 3$  and  $|E[\mu(K_{1,n})]| = 4n + 1$ , (see Figure 1(b)). Clearly,  $\{x_1, x'_1, u\}$  is a covering of  $\mu(K_{1,n})$  and hence  $\beta[\mu(K_{1,n})] \leq 3$ . For any covering  $S$  of  $\mu(K_{1,n})$ , we have  $\sum_{v_i \in S} \deg_{\mu(K_{1,n})}(v_i) \geq |E[\mu(K_{1,n})]| = 4n + 1$ . Since  $\Delta(\mu(K_{1,n})) = 2n$ , it follows that  $|S| \geq 3$ . Consequently,  $\beta[\mu(K_{1,n})] = 3$  and  $\alpha[\mu(K_{1,n})] = 2n$ .

Next, we see that  $\{x_1x_i : 2 \leq i \leq n\} \cup \{x_1x'_i : 2 \leq i \leq n\} \cup \{x_{n+1}x'_1, x'_{n+1}u\}$  covers all the vertices of  $\mu(K_{1,n})$ , and has  $2n$  edges. So,  $\beta'[\mu(K_{1,n})] \leq 2n$ . But  $\alpha[\mu(K_{1,n})] = 2n$ . So, we need at least  $2n$  edges to cover all the vertices of  $\mu(K_{1,n})$ . Therefore,  $\beta'[\mu(K_{1,n})] = 2n$  and  $\alpha'[\mu(K_{1,n})] = 3$ .

By the definition of  $T[\mu(K_{1,n})]$ ,  $|V(T[\mu(K_{1,n})])| = 6n + 4$  (see Figure 2). It requires at least  $3n + 2$  edges to cover the vertices of  $T[\mu(K_{1,n})]$ . Therefore,  $\beta'[T[\mu(K_{1,n})]] \geq 3n + 2$ . However,  $L = \{e_i e'_i : 1 \leq i \leq n\} \cup \{x_{i+1}d_i : 2 \leq i \leq$

$n\} \cup \{x'_{i+1}d'_i : 1 \leq i \leq n\} \cup \{x_1x_2, d_1x'_1, uu'\}$ , where  $e_i = x_1x_{i+1}$ ,  $e'_i = x_1x'_{i+1}$  for  $1 \leq i \leq n$ , and  $d_i = x'_1x_{i+1}$ , and  $d'_i = ux'_{i+1}$  for  $1 \leq i \leq n$ , in  $\mu(K_{1,n})$ . Clearly,  $L$  covers all the vertices of  $T[\mu(K_{1,n})]$  and contains  $3n + 2$  edges only. Therefore,  $\beta' [T [\mu(K_{1,n})]] = 3n + 2$  and  $\alpha' [T [\mu(K_{1,n})]] = 3n + 2$ . This gives the required results. ■

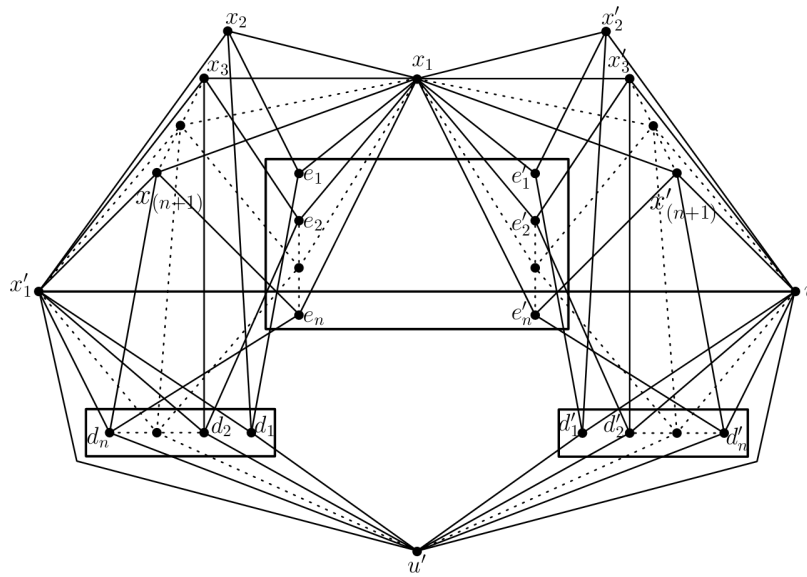


Figure 2. Total graph of Mycielski graph of  $K_{1,n}$ .

- Theorem 15.** (a)  $\beta[T(K_{1,n})] = \beta'[T(K_{1,n})] = \beta'[M(K_{1,n})] = \alpha[M(K_{1,n})] = n + 1$ .  
 (b)  $\beta[C(K_{1,n})] = \beta'[C(K_{1,n})] = \beta[\mu(K_n)] = \beta'[\mu(K_n)] = n + 1$ , for all  $n \geq 2$ .  
 (c)  $\chi_c(C(K_{1,n})) = \chi_c(M(K_{1,n})) = \chi(M(K_{1,n})) = \chi(T(K_{1,n})) = n + 1$ .

**Proof.** By referring Figure 3(a),  $\{x_2, x_3, \dots, x_{n+1}\}$  is an independent set in  $T(K_{1,n})$ . So,  $\alpha[T(K_{1,n})] \geq n$ . Since  $x_1$  is adjacent with all the vertices in  $T(K_{1,n})$ , any independent set in  $T(K_{1,n})$  with at least two vertices can not contain  $x_1$ . Figure 3(a) shows that any independent set can have at most one vertex  $e_i$  from  $\{e_1, e_2, \dots, e_n\}$ . Suppose  $\{v_1, v_2, \dots, v_{n+1}\}$  is an independent set in  $T(K_{1,n})$ . Then some  $v_i$  must be in  $\{e_1, e_2, \dots, e_n\}$ ; say  $v_i = e_j$  for some  $j$ . This is a contradiction to the fact that  $e_j$  is adjacent with  $x_{j+1}$  in  $T(K_{1,n})$ . This shows that  $\alpha[T(K_{1,n})] = n$  and  $\beta[T(K_{1,n})] = n + 1$ . Similarly, we can show that  $\beta'[T(K_{1,n})] = \beta'[M(K_{1,n})] = \alpha[M(K_{1,n})] = \beta[C(K_{1,n})] = \beta'[C(K_{1,n})] = n + 1$  for  $n \geq 2$ .

Next, we prove that  $\beta[\mu(K_n)] = \beta'[\mu(K_n)] = n + 1$ . Let  $\{v_1, v_2, \dots, v_n\}$  be the vertices of  $K_n$ . By the definition of Mycielski graph,  $|V[\mu(K_n)]| = 2n + 1$ .



This shows that  $\beta'[\mu(K_n)] \geq n + 1$ . We observe that  $(n + 1)$  edges of  $\{v_i v'_{i+1} : 1 \leq i \leq n - 1\} \cup \{v_n v'_1 \cup v'_1 u\}$  covers all the vertices of  $\mu(K_n)$ . Therefore,  $\beta'[\mu(K_n)] = n + 1$ .

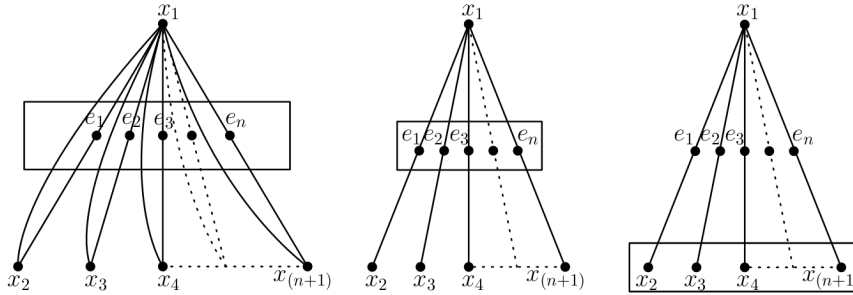


Figure 3. (a) Total graph of  $K_{1,n}$ . (b) Middle graph of  $K_{1,n}$ . (c) Central graph of  $K_{1,n}$ .

Since  $\{v'_1, v'_2, \dots, v'_n\}$  are the independent vertices of  $\mu(K_n)$ ,  $\alpha[\mu(K_n)] \geq n$ . Suppose  $S = \{u_1, u_2, \dots, u_{n+1}\}$  is an independent set in  $\mu(K_n)$ . Clearly, no two vertices of  $S$  can be in  $V(K_n)$  and  $u \notin S$ . Therefore exactly one  $u_i$  of  $S$  is in  $V(K_n)$ . But this  $u_i$  is adjacent to all other  $u'_j$  for  $i \neq j$ . This contradicts the independence of  $S$ . Hence  $\alpha[\mu(K_n)] = n$  and  $\beta[\mu(K_n)] = n + 1$ . This completes the proof of (a) and (b). (c) directly follows from [7]. ■

#### 4. UPPER BOUNDS

**Definition.** A sequence of real numbers  $(p_1, p_2, \dots, p_n)$  is said to be *majorised* by another such sequence  $(q_1, q_2, \dots, q_n)$  if  $p_i \leq q_i$  for  $1 \leq i \leq n$ . Let  $G$  and  $H$  be two graphs of the same order. Then  $G$  is degree-majorised by  $H$  if the nondecreasing degree sequence of  $G$  is majorised by that of  $H$  [4].

**Theorem 16.** Suppose  $H$  is degree-majorised by  $G$  with  $t(H) \leq t(G)$ . Then

- (a)  $t[T(H)] \leq t[T(G)]$ .
- (b)  $t[\mu^n(H)] \leq t[\mu^n(G)]$ .
- (c)  $t(T[\mu^n(H)]) \leq t(T[\mu^n(G)])$ .

In particular,  $t(T[\mu^n(C(K_{1,n}))) \leq t(T[\mu^{n+1}(K_n)])$  for  $n \geq 2$ .

**Proof.** Let  $V(H) = \{v_i : 1 \leq i \leq n\}$  and  $V(G) = \{u_i : 1 \leq i \leq n\}$ . Since  $H$  is degree-majorised by  $G$ ,  $\sum_{i=1}^n d_H^2(v_i) \leq \sum_{i=1}^n d_G^2(u_i)$  and  $\sum_{i=1}^n m_i \binom{d_H(v_i)}{3} \leq \sum_{i=1}^n m_i \binom{cd_G(u_i)}{3}$ . By Theorem 1,  $t[T(H)] \leq t[T(G)]$ . Since  $H$  is degree-majorised by  $G$  with  $t(H) \leq t(G)$ , by the definition of the Mycielski graph,  $\mu(H)$  is degree-majorised by  $\mu(G)$  with  $t[\mu(H)] \leq t[\mu(G)]$ ,  $\mu^n(H)$  is degree-majorised

by  $\mu^n(G)$  and  $t[\mu^n(H)] \leq t[\mu^n(G)]$ . Since  $\mu^n(H)$  is degree-majorised by  $\mu^n(G)$  with  $t[\mu^n(H)] \leq t[\mu^n(G)]$ , we get  $t(T[\mu^n(H)]) \leq t(T[\mu^n(G)])$ .

Finally, for any  $n \geq 2$ , each of  $C(K_{1,n})$  and  $\mu(K_n)$  is of order  $2n + 1$ , and  $\mu(K_n)$  contains  $n$  vertices of degree  $2n - 2$ ,  $n$  vertices of degree  $n$  and a vertex of degree  $n$ . Whereas,  $C(K_{1,n})$  contains  $n$  vertices of degree  $n$ ,  $n$  vertices of degree 2 and a vertex of degree  $n$ . Therefore,  $C(K_{1,n})$  is degree-majorised by  $\mu(K_n)$  and  $t(T[\mu^n(C(K_{1,n}))]) \leq t(T[\mu^{n+1}(K_n)])$ . ■

**Theorem 17.** I. Let  $G$  be any graph of order  $n$ . Then

- (a)  $t[T(G)] \leq t[T(K_n)]$ ,
- (b)  $t[\mu^n(G)] \leq t[\mu^n(K_n)]$ ,
- (c)  $t[T[\mu^n(G)]] \leq t[T[\mu^n(K_n)]]$ .

II. If  $G$  is a triangle-free graph of order  $n \geq 2$ , then for some positive integers  $a$  and  $b$  with  $a + b = n$ , we have

- (d)  $t[T(G)] \leq t[T(K_{a,b})]$ ,
- (e)  $t[\mu^n(G)] \leq t[\mu^n(K_{a,b})]$ ,
- (f)  $t[T[\mu^n(G)]] \leq t[T[\mu^n(K_{a,b})]]$ .

**Proof.** Obviously, every graph  $G$  of order  $n$  is degree-majorised by  $K_n$  with  $t(G) \leq t(K_n)$ . By Theorem 16, (a), (b) and (c) follow. Since  $G$  is a triangle-free graph,  $G$  is degree-majorised by some complete bipartite graph  $K_{a,b}$ , [4]. Consequently, (d), (e) and (f) hold. ■

## Conclusion

It would be interesting to determine the results for general graphs other than star-graph or complete graph families.

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