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Note

TWO SHORT PROOFS ON TOTAL DOMINATION

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Abstract

A set of vertices of a graph G is a total dominating set if each vertex of G is adjacent to a vertex in the set. The total domination number of a graph $\gamma_t(G)$ is the minimum size of a total dominating set. We provide a short proof of the result that $\gamma_t(G) \leq \frac{2}{3}n$ for connected graphs with $n \geq 3$ and a short characterization of the extremal graphs.

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A set of vertices of a graph G is a *total dominating set* if each vertex of G is adjacent to a vertex in the set. (See [3] for background.) The *total domination number* of a graph $\gamma_t(G)$ is the minimum size of a total dominating set. The definition immediately implies that a total dominating set is a dominating set with no isolated vertices. The total domination number is defined exactly for graphs without isolated vertices.

The following basic upper bound is due to Cockayne, Dawes, and Hedetniemi [2]. We present a shorter proof.

Theorem 1. *Let G be a connected graph with $n \geq 3$. Then $\gamma_t(G) \leq \frac{2}{3}n$.*

Proof. Let T be a spanning tree of G and v be a leaf of T. Label each vertex of T with its distance from v mod 3. This produces three sets that partition the vertices of G. Then some set contains at least one third of the vertices of G, and the union S of the other two contains at most two thirds of the vertices. Each internal vertex of T is adjacent to a vertex in each of the other sets. Replace any isolated leaves in S with their neighbors. Then S is a total dominating set. ∎

The graphs for which $\gamma_t(G) = \lfloor \frac{2}{3}n \rfloor$ have been characterized by [1]. We present a short proof for when $\gamma_t(G) = \frac{2}{3}n$. The *depth of a vertex* v in a tree T is the minimum distance between v and a leaf of T. A *brush* is a graph formed by starting with some graph G and identifying a leaf of a copy of P_3 with each vertex of G.

Theorem 2. *Let G be a connected graph with $n \geq 3$. Then $\gamma_t(G) = \frac{2}{3}n$ exactly when G is C_3, C_6, or a brush.*

Proof. It is easily seen that the stated graphs are extremal, since in a brush each depth 1 vertex and a neighbor must be in the total dominating set. Let $\gamma_t(G) = \frac{2}{3}n$, so $n = 3k$. The result is obvious for $n = 3$. Let $n \geq 6$. Let T be a spanning tree of G, so $\frac{2}{3}n \geq \gamma_t(T) \geq \gamma_t(G) = \frac{2}{3}n$, so $\gamma_t(T) = \frac{2}{3}n$. Note that no star except $K_{1,2}$ can be extremal since $\gamma_t(K_{1,s}) = 2 \leq \frac{2}{3}n$. Hence T has a minimum total dominating set S containing no leaves since any leaf could be replaced by a corresponding nonleaf distance two away if necessary.

Suppose that two leaves v_1 and v_2 of T have a common neighbor u. If $T - v_1$ has a smaller total dominating set S', then $u \in S'$, so S' is also a total dominating set for T. Hence $\gamma_t(T - v_1) = |S|$, but this contradicts the upper bound, so some leaf of T has a neighbor of degree 2.

If T has leaves v_1 and v_2 with neighbors u_1 and u_2 with a common neighbor w, then u_1, u_2, and w are contained in S. Then deleting v_1 and u_1 from T only allows deleting u_1 from S, similarly contradicting the upper bound.

Suppose that deleting all depth 1 vertices of degree 2 and their neighbors produces a forest F. Then each isolated vertex and every leaf of each component of F are already dominated. Then each component of F has fewer than two-thirds of its vertices in S. Thus T cannot achieve the upper bound, so F does not exist. Thus T is a brush.

Since T was arbitrary, any spanning tree of G is a brush. Adding edges between depth 2 vertices does not change γ_t. But adding any other edge produces a spanning tree that is not a brush unless $T = P_6$ and $G = C_6$. ■

A similar approach can be used to prove the characterization of the extremal graphs when $n = 3k + 2$, but the case $n = 3k + 1$ is more complicated.

References

[1] R.C. Brigham, J.R. Carrington and R.P. Vitray, *Connected graphs with maximum total domination number*, J. Combin. Comput. Combin. Math. **34** (2000) 81–96.

[2] E.J. Cockayne, R.M. Dawes and S.T. Hedetniemi, *Total domination in graphs*, Networks **10** (1980) 211–219.
doi:10.1002/net.3230100304

[3] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, Inc, 1998).

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