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THE INCIDENCE CHROMATIC NUMBER OF TOROIDAL GRIDS

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Abstract

An incidence in a graph G is a pair (v, e) with $v \in V(G)$ and $e \in E(G)$, such that v and e are incident. Two incidences (v, e) and (w, f) are adjacent if $v = w$, or $e = f$, or the edge vw equals e or f. The incidence chromatic number of G is the smallest k for which there exists a mapping from the set of incidences of G to a set of k colors that assigns distinct colors to adjacent incidences.

In this paper, we prove that the incidence chromatic number of the toroidal grid $T_{m,n} = C_m \Box C_n$ equals 5 when $m, n \equiv 0 \pmod 5$ and 6 otherwise.

Keywords: incidence coloring, Cartesian product of cycles, toroidal grid.

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1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. An *incidence* in G is a pair (v, e) with $v \in V(G)$ and $e \in E(G)$, such that v and e are incident. We denote by $I(G)$ the set of all incidences in G. Two incidences (v, e) and (w, f) are *adjacent* if one of the following holds: (i) $v = w$, (ii) $e = f$, (iii) the edge vw equals e or f.

An *incidence k-coloring* of G is a mapping from $I(G)$ to a set of k colors such that adjacent incidences are assigned distinct colors. The *incidence chromatic number* $\chi_i(G)$ of G is the smallest k such that G admits an incidence k-coloring.

Incidence colorings were introduced by Brualdi and Massey in [2]. In that paper, the authors also conjectured that the relation $\chi_i(G) \leq \Delta(G) + 2$ holds for every graph G, where $\Delta(G)$ denotes the maximum degree of G. In [5], Guiduli disproved this *Incidence Coloring Conjecture* (ICC for short). Incidence coloring of various classes of graphs has been considered in the literature [5, 6, 7, 8, 9, 11, 12, 13, 15, 16] and the ICC conjecture was proved to hold for several classes such as trees, complete graphs and complete bipartite graphs [2], subcubic graphs [11], K_4-minor free graphs [7], graphs with maximum average degree less than $\frac{22}{9}$ [6], square, hexagonal and honeycomb meshes [8], powers of paths [9], cubic Halin graphs [13], and Halin graphs with maximum degree at least 5 [15]. The problem of determining whether a given graph has incidence chromatic number at most k or not was shown to be NP-complete by Li and Tu [10].

Incidence colorings are related to various types of vertex, edge or arc colorings. For any graph G, let $H = H(G)$ be the bipartite graph given by $V(H) = V(G) \cup E(G)$ and $E(H) = \{(v, e) : v \in V(G),\ e \in E(G),\ e \text{ and } v \text{ are incident in } G\}$. Each edge of H corresponds to an incidence of G and, therefore, any incidence coloring of G corresponds to a *strong edge coloring* (sometimes called a *distance-two edge-coloring*) of H, that is a proper coloring of the edges of H such that each color class is an induced matching in H [3].

The *subdivision* $S(G)$ of G is the graph obtained from G by inserting a vertex of degree two on every edge of G. Any incidence coloring of G then corresponds to a *distance-two vertex coloring* of the line-graph $L(S(G))$ of $S(G)$, that is a vertex coloring such that any two vertices having the same color are at distance at least 3.

Let now G^* be the digraph obtained from G by replacing each edge of G by two opposite arcs. Any incidence (v, e) of G, with $e = vw$, can then be associated with the arc vw in G^*. Therefore, any incidence coloring of G corresponds to an arc-coloring of G^* satisfying: (i) any two arcs having the same source vertex (of the form uv and uw) are assigned distinct colors, (ii) any two consecutive arcs (of the form uv and vw) are assigned distinct colors. Hence, for every color c, the subgraph of G^* induced by the c-colored arcs is a forest consisting of directed stars (whose arcs are directed towards the center). The incidence chromatic number of G therefore equals the *directed star-arboricity* of G^*, as introduced by Algor and Alon in [1].

Let G and H be graphs. The *Cartesian product* $G \Box H$ of G and H is the graph with vertex set $V(G) \times V(H)$ where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$, or $v_1 = v_2$ and $u_1u_2 \in E(G)$. Let P_n and C_n denote respectively the path and the cycle on n vertices. We will denote by $G_{m,n} = P_m \Box P_n$ the *grid* with m rows and n columns and by $T_{m,n} = C_m \Box C_n$ the *toroidal grid* with m rows and n columns.

In this paper, we determine the incidence chromatic number of toroidal grids and prove that this class of graphs satisfies the ICC.

Theorem 1. *For every $m, n \geq 3$, $\chi_i(T_{m,n}) = 5$ if $m, n \equiv 0 \pmod 5$ and $\chi_i(T_{m,n}) = 6$ otherwise.*

In [8], Huang, Wang and Chung proved that $\chi_i(G_{m,n}) = 5$ for every m, n. Since every toroidal grid $T_{m,n}$ contains the grid $G_{m,n}$ as a subgraph, we get that $\chi_i(T_{m,n}) \geq 5$ for every m, n.

The paper is organized as follows. In Section 2 we give basic properties and illustrate the techniques we shall use in the proof of our main result, which is given in Section 3.

2. Preliminaries

Let G be a graph, u a vertex of G with maximum degree and v a neighbour of u. Since in any incidence coloring of G all the incidences of the form (u, e) have to get distinct colors and all of them have to get a color different from the color of (v, vu), we have:

Proposition 2. *For every graph G, $\chi_i(G) \geq \Delta(G) + 1$.*

The *square* G^2 of a graph G is given by $V(G^2) = V(G)$ and $uv \in E(G^2)$ if and only if $uv \in E(G)$ or there exists $w \in V(G)$ such that $uw, vw \in E(G)$. In other words, any two vertices within distance at most two in G are linked by an edge in G^2. Let now c be a proper vertex coloring of G^2 and μ be the mapping defined by $\mu(u, uv) = c(v)$ for every incidence (u, uv) in $I(G)$. It is not difficult to check that μ is indeed an incidence coloring of G (see Example 8 below). Therefore we have:

Proposition 3. *For every graph G, $\chi_i(G) \leq \chi(G^2)$.*

In [4], Fertin, Goddard and Raspaud proved that the chromatic number of the square of any d-dimensional grid $G_{n_1,\dots,n_d}$ is at most $2d + 1$, which thus implies the above mentioned result concerning 2-dimensional grids [8].

$$A = \boxed{\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 8 \\ 7 & 8 & 1 & 2 \end{array}}$$

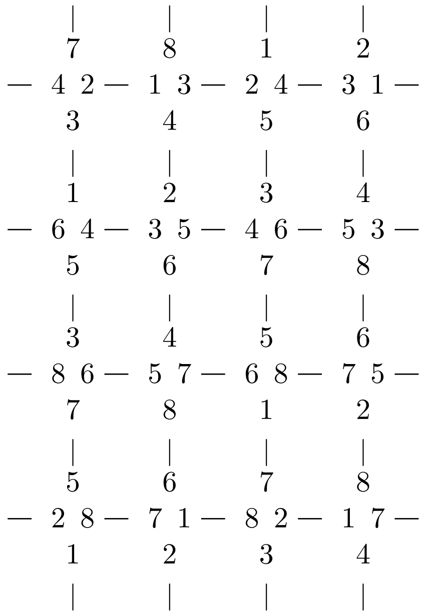

Figure 1. A pattern A and the corresponding incidence coloring of $T_{4,4}$.

In [14], we studied the chromatic number of the squares of toroidal grids and proved the following:

Theorem 4. *Let* $T_{m,n} = C_m \square C_n$. *Then* $\chi(T^2_{m,n}) \leq 7$ *except that* $\chi(T^2_{3,3}) = 9$ *and* $\chi(T^2_{3,5}) = \chi(T^2_{4,4}) = 8$.

By Proposition 3, this result provides upper bounds on the incidence chromatic number of toroidal grids.

In [14], we also proved the following:

Theorem 5. *For every* $m, n \geq 3$, $\chi(T^2_{m,n}) \geq 5$. *Moreover,* $\chi(T^2_{m,n}) = 5$ *if and only if* $m, n \equiv 0 \,(\mathrm{mod}\ 5)$.

In [16], the second author proved the following:

Theorem 6. *For a regular graph* G, $\chi_i(G) = \Delta(G) + 1$ *if and only if* $\chi(G^2) = \Delta(G) + 1$.

Since toroidal grids are 4-regular, by combining Proposition 2, Theorems 5 and 6 we get the following:

Corollary 7. *For every* $m, n \geq 3$, $\chi_i(T_{m,n}) \geq 5$. *Moreover,* $\chi_i(T_{m,n}) = 5$ *if and only if* $m, n \equiv 0 \,(\mathrm{mod}\ 5)$.

Note here that this corollary is part of our main result.

Any vertex coloring of the square of a toroidal grid $T_{m,n}$ can be given as an $m \times n$ matrix whose entries correspond in an obvious way to the colors of the vertices. Such a matrix will be called an $m \times n$ *pattern* in the following.

Example 8. Figure 1 shows a 4×4 pattern A, which defines a vertex coloring of $T^2_{4,4}$, and the incidence coloring of $T_{4,4}$ induced by this pattern, according to the discussion before Proposition 3. Note for instance that the four incidences of the form (u, uv), for u being the second vertex in the third row, have color 6, which corresponds to the entry in row 3, column 2 of pattern A.

If A and B are patterns of size $m \times n$ and $m \times n'$ respectively, we shall denote by $A+B$ the pattern of size $m \times (n+n')$ obtained by "gluing" together the patterns A and B. Moreover, we shall denote by ℓA, $\ell \geq 2$, the pattern of size $m \times \ell n$ obtained by "gluing" together ℓ copies of the pattern A.

We now shortly describe the technique we shall use in the next section. The main idea is to use a pattern for coloring the square of a toroidal grid in order to get an incidence coloring of this toroidal grid. However, as shown in [14], the squares of toroidal grids are not all 6-colorable. Therefore, we shall use the notion of a *quasi-pattern* which corresponds to a vertex 6-coloring of the square of a *subgraph* of a toroidal grid obtained by deleting some edges (namely those edges that cause a conflict when transforming a vertex coloring to its corresponding incidence coloring). We can then use such a quasi-pattern in the same way as before to obtain a *partial* incidence coloring of the toroidal grid. Finally, we shall prove that such a partial incidence coloring can be extended to the whole toroidal grid without using any additional color (most of the time, several distinct extensions are available and we shall propose one of them). We shall also use the following:

Remark 9. For every $m, n \geq 3$, $p, q \geq 1$, if $\chi_i(T_{m,n}) \leq k$ then $\chi_i(T_{pm,qn}) \leq k$.

To see that, it is enough to observe that every incidence k-coloring c of $T_{m,n}$ can be extended to an incidence k-coloring $c_{p,q}$ of $T_{pm,qn}$ by "repeating" the pattern given by c, p times "vertically" and q times "horizontally".

3. PROOF OF THEOREM 1

According to Corollary 7, we only need to prove that $\chi_i(T_{m,n}) \leq 6$ for every $m, n \geq 3$. The proof is based on a series of lemmas, according to different values of m and n.

We first consider the case when $m \equiv 0 \pmod 3$. We have proved in [14] the following:

$$B = \begin{array}{|c|}\hline 3 \\ 1 \\ 2 \\ \hline\end{array} \quad C = \begin{array}{|cccc|}\hline 1 & 4 & 2 & 5 \\ 2 & 5 & 3 & 6 \\ 3 & 6 & 1 & 4 \\ \hline\end{array} \quad D = \begin{array}{|cccc|}\hline 1 & 4 & 3 & 6 \\ 2 & 5 & 1 & 4 \\ 3 & 6 & 2 & 5 \\ \hline\end{array} \quad E = \begin{array}{|cc|}\hline 2 & 5 \\ 3 & 6 \\ 1 & 4 \\ \hline\end{array}$$

$$B+C = \begin{array}{|c|cccc|}\hline 3 & 1 & 4 & 2 & 5 \\ 1 & 2 & 5 & 3 & 6 \\ 2 & 3 & 6 & 1 & 4 \\ \hline\end{array} \quad B+D+E = \begin{array}{|c|cccc|cc|}\hline 3 & 1 & 4 & 3 & 6 & 2 & 5 \\ 1 & 2 & 5 & 1 & 4 & 3 & 6 \\ 2 & 3 & 6 & 2 & 5 & 1 & 4 \\ \hline\end{array}$$

Figure 2. Patterns and quasi-patterns for Lemma 11.

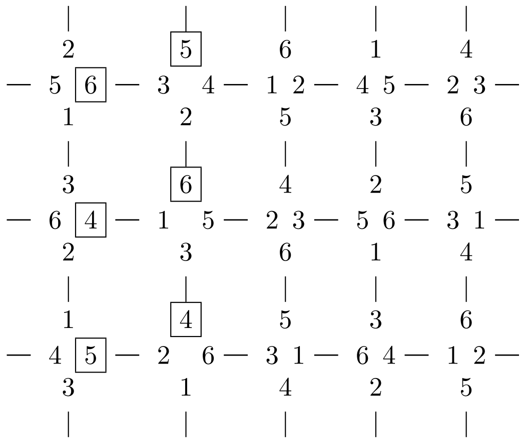

Figure 3. Incidence coloring for Lemma 11.

Proposition 10. *If $k \geq 1$, $n \geq 3$ and n is even, then $\chi(T^2_{3k,n}) \leq 6$.*

Here we prove:

Lemma 11. *If $k \geq 1$ and $n \geq 3$, then $\chi_i(T_{3k,n}) \leq 6$.*

Proof. If n is even, the result follows from Propositions 3 and 10.
We thus assume that n is odd, and we let first $k = 1$. We consider three cases.

$n = 3$. We can easily get an incidence 6-coloring by coloring the incidences of one dimension with $\{1, 2, 3\}$ and the incidences of the other dimension with $\{4, 5, 6\}$.

$n = 4\ell + 1$. Let B and C be the patterns depicted in Figure 2 and consider the quasi-pattern $B + \ell C$ (the quasi-pattern $B + C$ is depicted in Figure 2). This quasi-pattern provides a 6-coloring of $T^2_{m,n}$ if we delete all the edges linking vertices in the first column to vertices in the second column. We can use this quasi-pattern to obtain an incidence 6-coloring of $T_{m,n}$ by modifying six incidence colors, as shown in Figure 3 (modified colors are in boxes).

$n = 4\ell + 3$. Let B, D and E be the patterns depicted in Figure 2 and consider the quasi-pattern $B + \ell D + E$ (the quasi-pattern $B + D + E$ is depicted in Figure 2). As in the previous case, we can use this quasi-pattern to obtain an incidence 6-coloring of $T_{m,n}$ by modifying the same six incidence colors.

For $k \geq 2$, the result now directly follows from Remark 9. ■

We now consider the case when $m \equiv 0 \pmod 4$. For $m \equiv 0 \pmod 5$, we have proved in [14] the following:

Proposition 12. *If $k \geq 1$, $n \geq 5$ and $n \neq 7$, then $\chi(T^2_{5k,n}) \leq 6$.*

Here we prove:

Lemma 13. *If $k \geq 1$, $n \geq 3$ and $(k, n) \neq (1, 5)$, then $\chi_i(T_{4k,n}) \leq 6$.*

Proof. For $n = 5$, the result holds by Proposition 12, except for $k = 1$.

Assume now $k = 1$ and $n \neq 5$ and consider the quasi-patterns F and G depicted in Figure 4. From these patterns, we can derive a partial incidence 6-coloring of $T_{4,3}$ and $T_{4,4}$, respectively, as shown in Figure 5, where the uncolored incidences are denoted by x. It is easy to check that every such incidence has only four forbidden colors and that only incidences belonging to the same edge have to be distinct. Therefore, these partial incidence colorings can be extended to incidence 6-colorings of $T_{4,3}$ and $T_{4,4}$.

For $n \geq 6$, we shall use the quasi-pattern $H = pF + qG$ where p and q satisfy $n = 3p + 4q$ (recall that every integer except 1, 2 and 5 can be written in this form). The quasi-pattern $H = 2F + 2G$ is depicted in Figure 4. As in the previous case, this quasi-pattern provides a partial incidence 6-coloring of $T_{4,n}$ that can be extended to an incidence 6-coloring of $T_{4,n}$.

$$F = \begin{array}{|ccc|}\hline 1 & 2 & 4 \\ 1 & 2 & 4 \\ 3 & 5 & 6 \\ 3 & 5 & 6 \\ \hline\end{array} \qquad G = \begin{array}{|cccc|}\hline 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 \\ \hline\end{array}$$

$$H = 2F + 2G = \begin{array}{|ccc|ccc|cccc|cccc|}\hline 1 & 2 & 4 & 1 & 2 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 1 & 2 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\ 3 & 5 & 6 & 3 & 5 & 6 & 3 & 4 & 5 & 6 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 3 & 5 & 6 & 3 & 4 & 5 & 6 & 3 & 4 & 5 & 6 \\ \hline\end{array}$$

Figure 4. Quasi-patterns for Lemma 13.

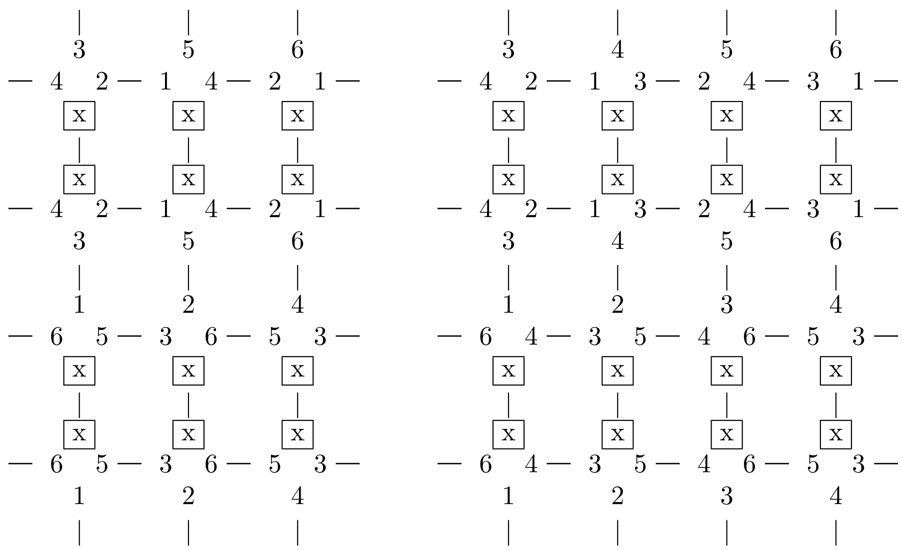

Figure 5. Partial incidence colorings for Lemma 13.

$$I = \begin{array}{|cccccc|}\hline 6 & 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 6 & 1 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ \hline \end{array} \qquad I' = \begin{array}{|cccccc|}\hline 6 & 1 & 2 & 3 & 4 & 5 \\ 6 & 1 & 2 & 3 & 4 & 5 \\ 6 & 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 6 & 1 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ \hline \end{array}$$

Figure 6. Patterns for Lemma 14.

For $k \geq 2$, the result now directly follows from Remark 9. ■

We now consider the remaining cases.

Lemma 14. *If $m, n \geq 5$, $m \neq 6, 8$ and $n \neq 7$, then $\chi_i(T_{m,n}) \leq 6$.*

Proof. Assume $m, n \geq 5$, $m \neq 6, 8$ and $n \neq 7$. By Proposition 12, we have $\chi(T^2_{5k,n}) \leq 6$ for $n \neq 7$. Hence, there exists a vertex 6-coloring of $T^2_{5k,n}$ which corresponds to some pattern M of size $5k \times n$. We claim that each row of pattern M can be repeated one or three times to get quasi-patterns that can be extended to incidence 6-colorings of the corresponding toroidal grids.

Let for instance M' be the quasi-pattern obtained from M by repeating the first row of M three times. The quasi-pattern M' has thus size $(5k+2) \times n$. The

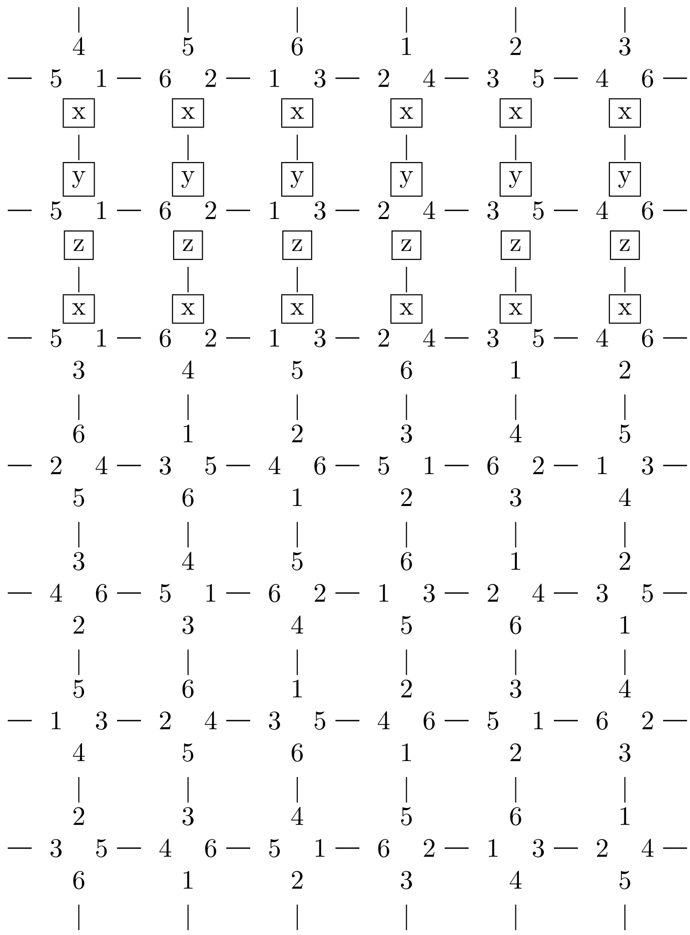

Figure 7. A partial incidence coloring of $T_{7,6}$.

quasi-pattern M' induces a partial incidence coloring of $T_{5k+2,n}$ in which the only uncolored incidences are those lying on the edges linking vertices in the first row to vertices in the second row and on the edges linking vertices in the second row to vertices in the third row.

We illustrate this in Figure 6 with a pattern I of size 5×6 (this pattern induces a vertex 6-coloring of $T^2_{5,6}$) and its associated pattern I' of size 7×6. The partial incidence coloring of $T_{7,6}$ obtained from I' is then given in Figure 7, where uncolored incidences are denoted by x, y and z.

Observe now that in each column, the two incidences denoted by x have three forbidden colors in common and each of them has four forbidden colors in total. Therefore, we can assign them the same color. Now, in each column, the

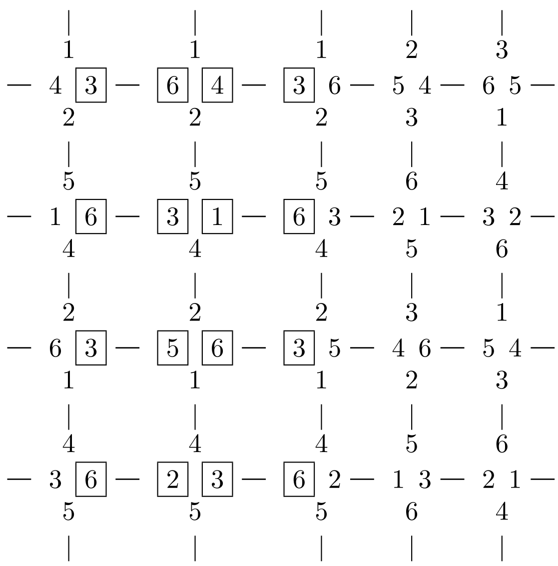

Figure 8. An incidence 6-coloring of $T_{4,5}$.

$$J = \begin{array}{|ccccccc|}
\hline
3 & 5 & 6 & 3 & 4 & 5 & 6 \\
1 & 2 & 4 & 1 & 2 & 3 & 4 \\
1 & 2 & 4 & 1 & 2 & 3 & 4 \\
3 & 5 & 6 & 3 & 4 & 5 & 6 \\
3 & 5 & 6 & 3 & 4 & 5 & 6 \\
1 & 2 & 4 & 1 & 2 & 3 & 4 \\
1 & 2 & 4 & 1 & 2 & 3 & 4 \\
\hline
\end{array}$$

Figure 9. A quasi-pattern for Lemma 15.

incidences denoted by y and z have four forbidden colors in common (the color assigned to x is one of them) and each of them has five forbidden colors in total. They can be thus colored with distinct colors. Doing that, we extend the partial incidence coloring of $T_{7,6}$ to an incidence 6-coloring of $T_{7,6}$.

The same technique can be used for obtaining an incidence 6-coloring of $T_{5k+2,n}$ since all the columns are "independent" in the quasi-pattern M', with respect to uncolored incidences.

If we repeat three times several distinct rows of pattern M, each repeated row will produce a chain of four uncolored incidences, as before, and any two such chains in the same column will be "independent", since they will be separated by an edge whose incidences are both colored. Hence, we will be able to extend the corresponding quasi-pattern to an incidence 6-coloring of the toroidal grid,

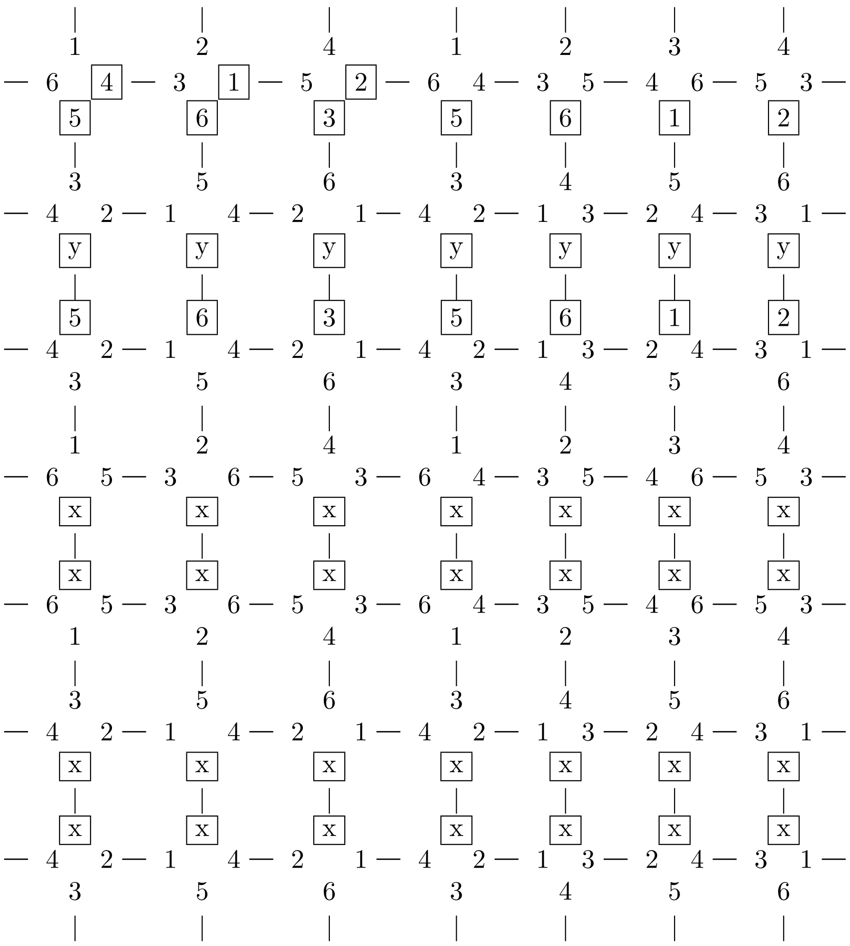

Figure 10. A partial incidence 6-coloring of $T_{7,7}$.

by assigning available colors to each chain as we did above.

Starting from a pattern M of size $5k \times n$, we can thus obtain quasi-patterns of size $(5k+2) \times n$, $(5k+4) \times n$, $(5k+6) \times n$ and $(5k+8) \times n$, by repeating respectively one, two, three or four lines from M. Using these quasi-patterns, we can produce incidence 6-colorings of the toroidal grid $T_{m,n}$, $m, n \geq 5$, $n \neq 7$, for every m except $m = 6$, 8. ∎

The only remaining cases are $m = 4$, $n = 5$ and $m = n = 7$. Then we have:

Lemma 15. *$\chi_i(T_{4,5}) \leq 6$ and $\chi_i(T_{7,7}) \leq 6$.*

Proof. Let $m = 4$ and $n = 5$. Consider the pattern C of size 3×4 depicted in Figure 2. As in the proof of Lemma 14, we can repeat the first row of C three times to get a quasi-pattern C' that can be extended to an incidence 6-coloring of $T_{5,4}$. We then exchange m and n to get an incidence 6-coloring of $T_{4,5}$, depicted in Figure 8 (the colors assigned to uncolored incidences are drawn in boxes).

Let now $m = n = 7$ and consider the quasi-pattern J depicted in Figure 9. This quasi-pattern provides the partial incidence coloring of $T_{7,7}$ given in Figure 10, where incidences with modified colors are in boxes and uncolored incidences are denoted by x and y. Observe now that the incidences denoted by y have five forbidden colors while the incidences denoted by x have four forbidden colors. Therefore, this partial coloring can be extended to an incidence 6-coloring of $T_{7,7}$. ■

By Corollary 7 and Lemmas 11–15, Theorem 1 follows directly.

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